

Bayes Rule



Rev. Thomas Bayes
(1702-1761)

- The product rule gives us two ways to factor a joint probability:

$$P(A, B) = P(A | B)P(B) = P(B | A)P(A)$$

- Therefore, $P(A | B) = \frac{P(B | A)P(A)}{P(B)}$
- Why is this useful?
 - Key tool for probabilistic inference: can get *diagnostic probability* from *causal probability*
 - E.g., $P(\text{Cavity} | \text{Toothache})$ from $P(\text{Toothache} | \text{Cavity})$
 - Can update our beliefs based on evidence

Bayes Rule example

- Marie is getting married tomorrow, at an outdoor ceremony in the desert. In recent years, it has rained only 5 days each year ($5/365 = 0.014$). Unfortunately, the weatherman has predicted rain for tomorrow. When it actually rains, the weatherman correctly forecasts rain 90% of the time. When it doesn't rain, he incorrectly forecasts rain 10% of the time. What is the probability that it will rain on Marie's wedding?

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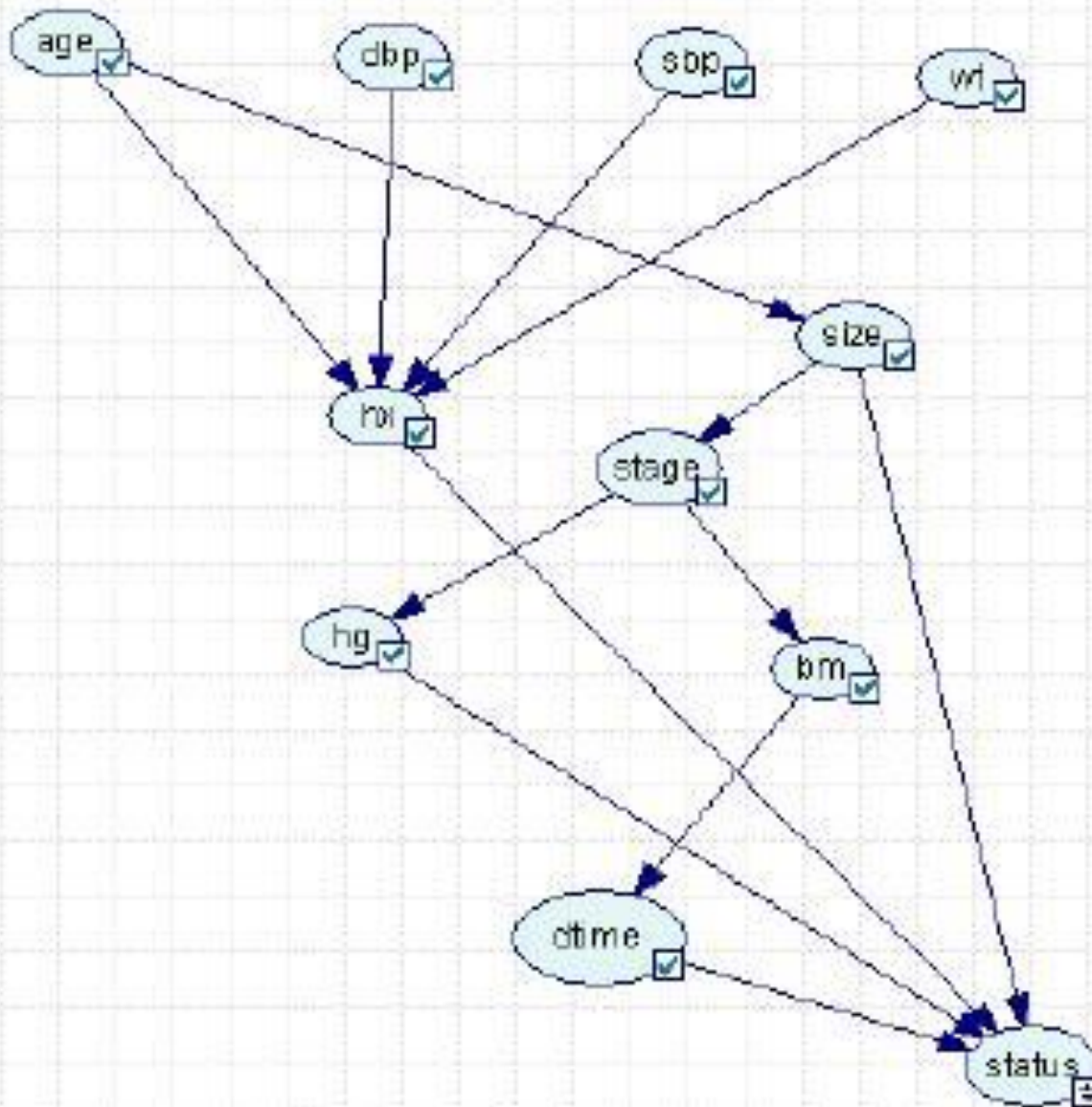
$$\begin{aligned} P(\text{Rain} \mid \text{Predict}) &= \frac{P(\text{Predict} \mid \text{Rain})P(\text{Rain})}{P(\text{Predict})} \\ &= \frac{P(\text{Predict} \mid \text{Rain})P(\text{Rain})}{P(\text{Predict} \mid \text{Rain})P(\text{Rain}) + P(\text{Predict} \mid \neg\text{Rain})P(\neg\text{Rain})} \\ &= \frac{0.9 \times 0.014}{0.9 \times 0.014 + 0.1 \times 0.986} = \frac{0.0126}{0.0126 + 0.0986} = 0.111 \end{aligned}$$

Bayes rule: Another example

- 1% of women at age forty who participate in routine screening have breast cancer. 80% of women with breast cancer will get positive mammographies. 9.6% of women without breast cancer will also get positive mammographies. A woman in this age group had a positive mammography in a routine screening. What is the probability that she actually has breast cancer?

$$\begin{aligned}P(\text{Cancer} \mid \text{Positive}) &= \frac{P(\text{Positive} \mid \text{Cancer})P(\text{Cancer})}{P(\text{Positive})} \\ &= \frac{P(\text{Positive} \mid \text{Cancer})P(\text{Cancer})}{P(\text{Positive} \mid \text{Cancer})P(\text{Cancer}) + P(\text{Positive} \mid \neg\text{Cancer})P(\neg\text{Cancer})} \\ &= \frac{0.8 \times 0.01}{0.8 \times 0.01 + 0.096 \times 0.99} = \frac{0.008}{0.008 + 0.095} = 0.0776\end{aligned}$$

Actual Example



Probabilities

- File rede_genie_dne.dne

Probabilistic inference

- Suppose the agent has to make a decision about the value of an unobserved *query variable* X given some observed *evidence variable(s)* $E = e$
 - Partially observable, stochastic, episodic environment
 - Examples: $X = \{\text{spam, not spam}\}$, $e = \text{email message}$
 $X = \{\text{zebra, giraffe, hippo}\}$, $e = \text{image features}$



Dear Sir.

First, I must solicit your confidence in this transaction, this is by virtue of its nature as being utterly confidential and top secret. ...

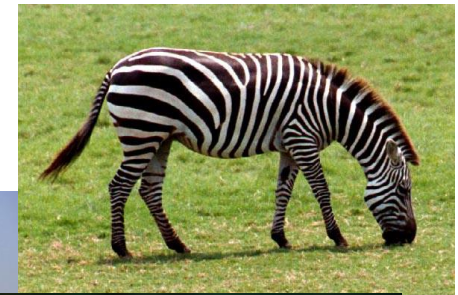


TO BE REMOVED FROM FUTURE MAILINGS, SIMPLY REPLY TO THIS MESSAGE AND PUT "REMOVE" IN THE SUBJECT.

99 MILLION EMAIL ADDRESSES FOR ONLY \$99



Ok, I know this is blatantly OT but I'm beginning to go insane. Had an old Dell Dimension XPS sitting in the corner and decided to put it to use, I know it was working pre being stuck in the corner, but when I plugged it in, hit the power nothing happened.



Bayesian decision theory

- Let x be the value predicted by the agent and x^* be the true value of X .
- The agent has a **loss function**, which is 0 if $x = x^*$ and 1 otherwise (0/1 loss)
- Expected loss:

$$\sum_x L(x^*, x)P(x | e)$$

- What is the estimate of X that minimizes the expected loss?
 - The one that has the greatest posterior probability $P(x|e)$
 - This is called the **Maximum a Posteriori (MAP)** decision

MAP decision

- Value x of X that has the highest posterior probability given the evidence $E = e$:

$$x^* = \arg \max_x P(X = x | E = e) = \frac{P(E = e | X = x)P(X = x)}{P(E = e)}$$

$$\propto \arg \max_x P(E = e | X = x)P(X = x)$$

$$\underbrace{P(x | e)}_{\text{posterior}} \propto \underbrace{P(e | x)}_{\text{likelihood}} \underbrace{P(x)}_{\text{prior}}$$

- Maximum likelihood (ML) decision:

$$x^* = \arg \max_x P(e | x)$$

Naïve Bayes model

- Suppose we have many different types of observations (symptoms, features) E_1, \dots, E_n that we want to use to obtain evidence about an underlying hypothesis X
- MAP decision involves estimating

$$P(X | E_1, \dots, E_n) \propto P(E_1, \dots, E_n | X)P(X)$$

- If each feature E_i can take on k values, how many entries are in the (conditional) joint probability table $P(E_1, \dots, E_n | X = x)$?

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- MAP decision involves estimating

$$P(X | E_1, \dots, E_n) \propto P(E_1, \dots, E_n | X)P(X)$$

- We can make the simplifying assumption that the different features are conditionally independent *given the hypothesis*:

$$P(E_1, \dots, E_n | X) = \prod_{i=1}^n P(E_i | X)$$

- If each feature can take on k values, what is the complexity of storing the resulting distributions?

Naïve Bayes model

- Posterior:

$$\begin{aligned} P(X = x | E_1 = e_1, \dots, E_n = e_n) \\ \propto P(X = x) P(E_1 = e_1, \dots, E_n = e_n | X = x) \\ = P(X = x) \prod_{i=1}^n P(E_i = e_i | X = x) \end{aligned}$$

- MAP decision:

$$x^* = \arg \max_x \underbrace{P(x | e)}_{\text{posterior}} \propto \underbrace{P(x)}_{\text{prior}} \underbrace{\prod_{i=1}^n P(e_i | x)}_{\text{likelihood}}$$

Case study: Spam filter

- **MAP decision:** to minimize the probability of error, we should classify a message as spam if

$$P(\text{spam} \mid \text{message}) > P(\neg\text{spam} \mid \text{message})$$



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- **MAP decision:** to minimize the probability of error, we should classify a message as spam if
$$P(\text{spam} \mid \text{message}) > P(\neg\text{spam} \mid \text{message})$$
- We have $P(\text{spam} \mid \text{message}) \propto P(\text{message} \mid \text{spam})P(\text{spam})$
and $\neg P(\text{spam} \mid \text{message}) \propto P(\text{message} \mid \neg\text{spam})P(\neg\text{spam})$
- To enable classification, we need to be able to estimate the **likelihoods** $P(\text{message} \mid \text{spam})$ and $P(\text{message} \mid \neg\text{spam})$ and **priors** $P(\text{spam})$ and $P(\neg\text{spam})$

Naïve Bayes Representation

- Goal: estimate likelihoods $P(\text{message} \mid \text{spam})$ and $P(\text{message} \mid \neg\text{spam})$ and priors $P(\text{spam})$ and $P(\neg\text{spam})$
- Likelihood: **bag of words** representation
 - The message is a sequence of words (w_1, \dots, w_n)
 - The order of the words in the message is not important
 - Each word is conditionally independent of the others given message class (spam or not spam)



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Bag of words illustration

2007-01-23: State of the Union Address

George W. Bush (2001-)

abandon accountable affordable afghanistan africa aided ally anbar armed army baghdad bless challenges chamber chaos
choices civilians coalition commanders commitment confident confront congressman constitution corps debates deduction
deficit deliver democratic deploy dikembe diplomacy disruptions earmarks economy einstein elections eliminates
expand extremists failing faithful families freedom fuel funding god haven ideology immigration impose
insurgents iran **iraq** islam julie lebanon love madam marine math medicare moderation neighborhoods nuclear offensive
palestinian payroll province pursuing **qaeda** radical regimes resolve retreat rieman sacrifices science sectarian senate
september shia stays strength students succeed sunni tax territories **terrorists** threats uphold victory
violence violent **war** washington weapons wesley

US Presidential Speeches Tag Cloud

<http://chir.ag/projects/preztags/>

Bag of words illustration

2007-01-23: State of the Union Address

George W. Bush (2001-)

abandon

choices c

deficit c

expand

insurgen

palestini

septemb

violenc

1962-10-22: Soviet Missiles in Cuba

John F. Kennedy (1961-63)

abandon achieving adversaries aggression agricultural appropriate armaments **arms** assessments atlantic ballistic berlin
buildup burdens cargo college commitment communist constitution consumers cooperation crisis **cuba** dangers
declined **defensive** deficit depended disarmament divisions domination doubled **economic** education
elimination emergence endangered equals **europa** expand exports fact false family forum **freedom** fulfill gromyko
halt hazards **hemisphere** hospitals ideals **independent** industries inflation labor latin limiting minister **missiles**
modernization neglect **nuclear** oas obligation observer **offensive** peril pledged predicted purchasing quarantine quote
recession rejection republics retaliatory safeguard sites solution **soviet** space spur stability standby **strength**
surveillance **tax** territory treaty undertakings unemployment **war** warhead **weapons** welfare western widen withdraw

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Bag of words illustration

2007-01-23: State of the Union Address

George W. Bush (2001-)

1962-10-22: Soviet Missiles in Cuba

John F. Kennedy (1961-63)

1941-12-08: Request for a Declaration of War

Franklin D. Roosevelt (1933-45)

abandoning acknowledge aggression aggressors airplanes armaments **armed** army assault assembly authorizations bombing
britain british cheerfully claiming constitution curtail december defeats defending delays **democratic** dictators disclose
economic empire endanger **facts** false forgotten fortunes france **freedom** fulfilled fullness fundamental gangsters
german germany **god** guam harbor hawaii **hemisphere** hint **hitler** hostilities immune improving indies innumerable
invasion **islands** isolate **japanese** labor metals midst midway **navy** nazis obligation offensive
officially **pacific** partisanship patriotism pearl peril perpetrated perpetual philippine preservation privilege reject
repaired **resisting** retain revealing rumors seas soldiers speaks speedy **stamina** **strength** sunday sunk supremacy tanks taxes
treachery true tyranny undertaken victory **war** wartime washington

US Presidential Speeches Tag Cloud

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Naïve Bayes Representation

- Goal: estimate likelihoods $P(\text{message} \mid \text{spam})$ and $P(\text{message} \mid \neg\text{spam})$ and priors $P(\text{spam})$ and $P(\neg\text{spam})$
- Likelihood: **bag of words** representation
 - The message is a sequence of words (w_1, \dots, w_n)
 - The order of the words in the message is not important
 - Each word is conditionally independent of the others given message class (spam or not spam)

$$P(\text{message} \mid \text{spam}) = P(w_1, \dots, w_n \mid \text{spam}) = \prod_{i=1}^n P(w_i \mid \text{spam})$$

- Thus, the problem is reduced to estimating marginal likelihoods of individual words $P(w_i \mid \text{spam})$ and $P(w_i \mid \neg\text{spam})$

Summary: Decision rule

- General MAP rule for Naïve Bayes:

$$x^* = \arg \max_x \underbrace{P(x | e)}_{\text{posterior}} \propto \underbrace{P(x)}_{\text{prior}} \underbrace{\prod_{i=1}^n P(e_i | x)}_{\text{likelihood}}$$

$$P(\textit{spam} | w_1, \dots, w_n) \propto P(\textit{spam}) \prod_{i=1}^n P(w_i | \textit{spam})$$

- Thus, the filter should classify the message as spam if

$$P(\textit{spam}) \prod_{i=1}^n P(w_i | \textit{spam}) > P(\neg \textit{spam}) \prod_{i=1}^n P(w_i | \neg \textit{spam})$$

Parameter estimation

- Model parameters: feature likelihoods $P(\text{word} \mid \text{spam})$ and $P(\text{word} \mid \neg\text{spam})$ and priors $P(\text{spam})$ and $P(\neg\text{spam})$
 - How do we obtain the values of these parameters?

prior

spam:	0.33
\neg spam:	0.67

$P(\text{word} \mid \text{spam})$

the :	0.0156
to :	0.0153
and :	0.0115
of :	0.0095
you :	0.0093
a :	0.0086
with:	0.0080
from:	0.0075
...	

$P(\text{word} \mid \neg\text{spam})$

the :	0.0210
to :	0.0133
of :	0.0119
2002:	0.0110
with:	0.0108
from:	0.0107
and :	0.0105
a :	0.0100
...	

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- Model parameters: feature likelihoods $P(\text{word} \mid \text{spam})$ and $P(\text{word} \mid \neg\text{spam})$ and priors $P(\text{spam})$ and $P(\neg\text{spam})$
 - How do we obtain the values of these parameters?
 - Need *training set* of labeled samples from both classes

$$P(\text{word} \mid \text{spam}) = \frac{\text{\# of word occurrences in spam messages}}{\text{total \# of words in spam messages}}$$

- This is the *maximum likelihood* (ML) estimate, or estimate that maximizes the likelihood of the training data:

$$\prod_{d=1}^D \prod_{i=1}^{n_d} P(w_{d,i} \mid \text{class}_{d,i})$$

d : index of training document, i : index of a word

Parameter estimation

- Parameter estimate:

$$P(\text{word} \mid \text{spam}) = \frac{\text{\# of word occurrences in spam messages}}{\text{total \# of words in spam messages}}$$

- Parameter smoothing: dealing with words that were never seen or seen too few times
 - **Laplacian smoothing:** pretend you have seen every vocabulary word one more time than you actually did

$$P(\text{word} \mid \text{spam}) = \frac{\text{\# of word occurrences in spam messages} + 1}{\text{total \# of words in spam messages} + V}$$

(V: total number of unique words)

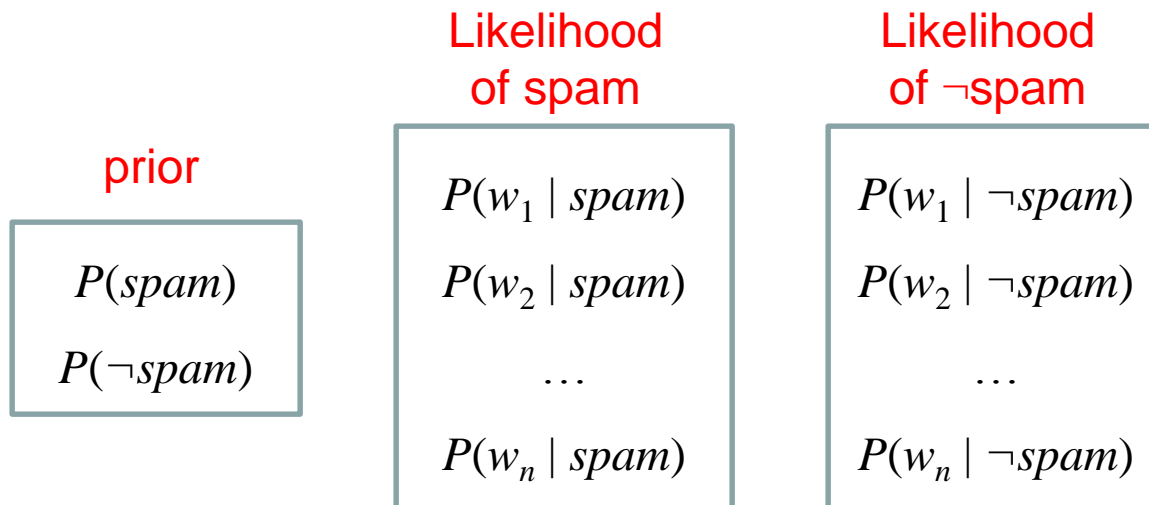
Summary of model and parameters

- Naïve Bayes model:

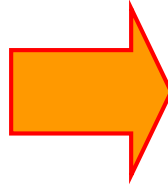
$$P(\textit{spam} | \textit{message}) \propto P(\textit{spam}) \prod_{i=1}^n P(w_i | \textit{spam})$$

$$P(\neg \textit{spam} | \textit{message}) \propto P(\neg \textit{spam}) \prod_{i=1}^n P(w_i | \neg \textit{spam})$$

- Model parameters:

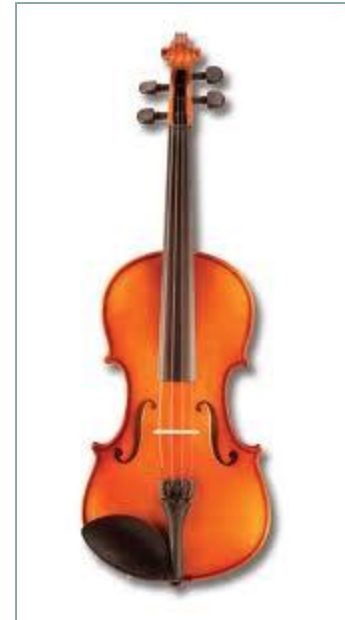


Bag-of-words models for images



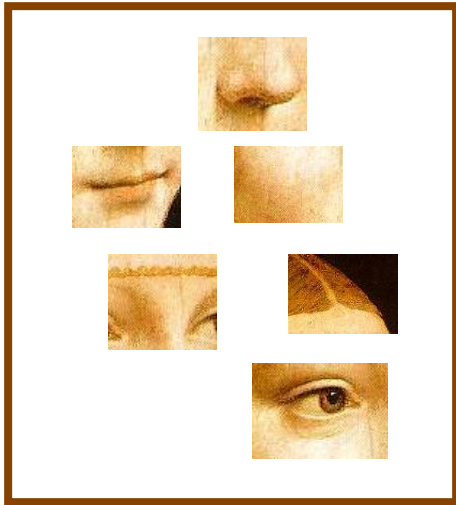
Bag-of-word models for images

1. Extract image features



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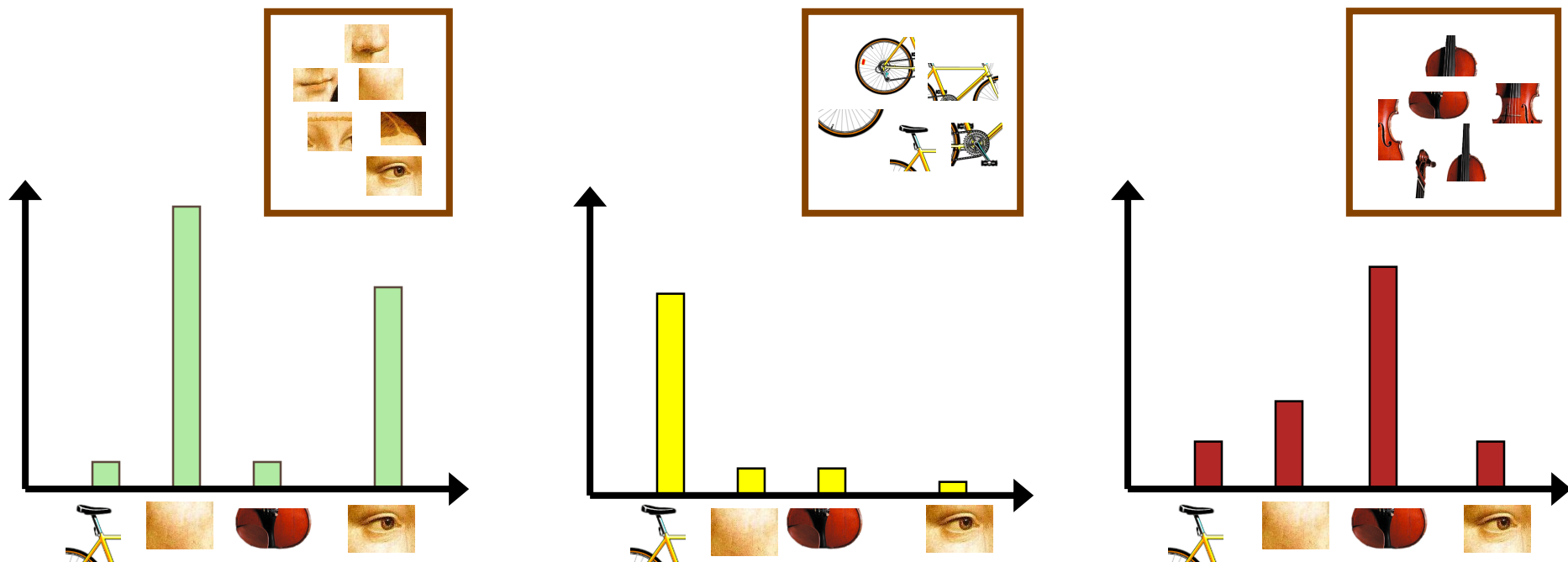
Bag-of-word models for images

1. Extract image features
2. Learn “visual vocabulary”



Bag-of-words models for images

1. Extract image features
2. Learn “visual vocabulary”
3. Map image features to visual words



Bayesian decision making: Summary

- Suppose the agent has to make decisions about the value of an unobserved *query variable* X based on the values of an observed *evidence variable* E
- **Inference problem:** given some evidence $E = e$, what is $P(X | e)$?
- **Learning problem:** estimate the parameters of the probabilistic model $P(X | E)$ given a *training sample* $\{(x_1, e_1), \dots, (x_n, e_n)\}$