Source:
http://www.cs.illinois.edu/%7Eslazebni/fall13/lec13_bayesian_inference.pptx

Bayes Rule

Rev. Thomas Bayes
(1702-1761)

- The product rule gives us two ways to factor a joint probability:

$$P(A,B) = P(A \mid B)P(B) = P(B \mid A)P(A)$$

- Therefore, $P(A \mid B) = \dfrac{P(B \mid A)P(A)}{P(B)}$

- Why is this useful?
  - Key tool for probabilistic inference: can get *diagnostic probability* from *causal probability*
    - E.g., P(Cavity | Toothache) from P(Toothache | Cavity)
  - Can update our beliefs based on evidence

Bayes Rule example

- Marie is getting married tomorrow, at an outdoor ceremony in the desert. In recent years, it has rained only 5 days each year (5/365 = 0.014). Unfortunately, the weatherman has predicted rain for tomorrow. When it actually rains, the weatherman correctly forecasts rain 90% of the time. When it doesn't rain, he incorrectly forecasts rain 10% of the time. What is the probability that it will rain on Marie's wedding?

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$$P(\text{Rain} \mid \text{Predict}) = \frac{P(\text{Predict} \mid \text{Rain})P(\text{Rain})}{P(\text{Predict})}$$

$$= \frac{P(\text{Predict} \mid \text{Rain})P(\text{Rain})}{P(\text{Predict} \mid \text{Rain})P(\text{Rain}) + P(\text{Predict} \mid \neg\text{Rain})P(\neg\text{Rain})}$$

$$= \frac{0.9 \times 0.014}{0.9 \times 0.014 + 0.1 \times 0.986} = \frac{0.0126}{0.0126 + 0.0986} = 0.111$$

Bayes rule: Another example

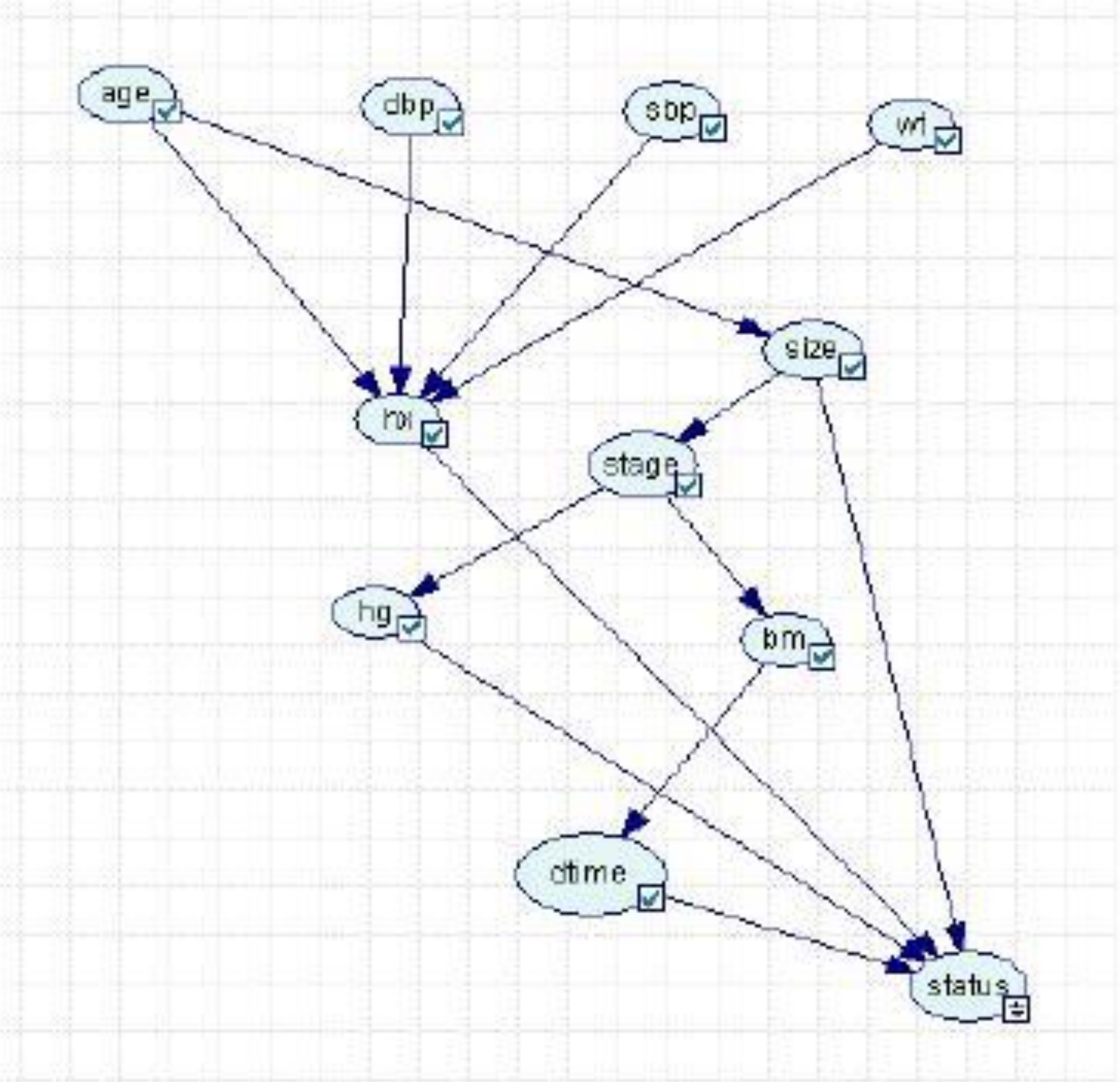
- 1% of women at age forty who participate in routine screening have breast cancer. 80% of women with breast cancer will get positive mammographies. 9.6% of women without breast cancer will also get positive mammographies. A woman in this age group had a positive mammography in a routine screening. What is the probability that she actually has breast cancer?

$$P(\text{Cancer} \mid \text{Positive}) = \frac{P(\text{Positive} \mid \text{Cancer}) P(\text{Cancer})}{P(\text{Positive})}$$

$$= \frac{P(\text{Positive} \mid \text{Cancer}) P(\text{Cancer})}{P(\text{Positive} \mid \text{Cancer}) P(\text{Cancer}) + P(\text{Positive} \mid \neg\text{Cancer}) P(\neg\text{Cancer})}$$

$$= \frac{0.8 \times 0.01}{0.8 \times 0.01 + 0.096 \times 0.99} = \frac{0.008}{0.008 + 0.095} = 0.0776$$

Actual Example



Probabilities

- File rede_genie_dne.dne

Probabilistic inference

- Suppose the agent has to make a decision about the value of an unobserved *query variable* X given some observed *evidence variable(s)* E = e
  - Partially observable, stochastic, episodic environment
  - Examples: X = {spam, not spam}, e = email message
    X = {zebra, giraffe, hippo}, e = image features

✗

Dear Sir.

First, I must solicit your confidence in this transaction, this is by virtue of its nature as being utterly confidencial and top secret. …

✗

TO BE REMOVED FROM FUTURE MAILINGS, SIMPLY REPLY TO THIS MESSAGE AND PUT "REMOVE" IN THE SUBJECT.

99 MILLION EMAIL ADDRESSES FOR ONLY $99

✓

Ok, Iknow this is blatantly OT but I'm beginning to go insane. Had an old Dell Dimension XPS sitting in the corner and decided to put it to use, I know it was working pre being stuck in the corner, but when I plugged it in, hit the power nothing happened.

Bayesian decision theory

- Let x be the value predicted by the agent and x* be the true value of X.
- The agent has a **loss function**, which is 0 if x = x* and 1 otherwise (0/1 loss)
- Expected loss:

$$\sum_{x} L(x^*, x) P(x \mid e)$$

- What is the estimate of X that minimizes the expected loss?
  - The one that has the greatest posterior probability P(x|e)
  - This is called the Maximum a Posteriori (MAP) decision

MAP decision

- Value x of X that has the highest posterior probability given the evidence E = e:

$$x^* = \arg\max_x P(X = x \mid E = e) = \frac{P(E = e \mid X = x)P(X = x)}{P(E = e)}$$

$$\propto \arg\max_x P(E = e \mid X = x)P(X = x)$$

$$\underbrace{P(x \mid e)}_{\text{posterior}} \propto \underbrace{P(e \mid x)}_{\text{likelihood}}\underbrace{P(x)}_{\text{prior}}$$

- Maximum likelihood (ML) decision:

$$x^* = \arg\max_x P(e \mid x)$$

Naïve Bayes model

- Suppose we have many different types of observations (symptoms, features) $E_1, \ldots, E_n$ that we want to use to obtain evidence about an underlying hypothesis X
- MAP decision involves estimating

$$P(X \mid E_1, \ldots, E_n) \propto P(E_1, \ldots, E_n \mid X) P(X)$$

  – If each feature E_i can take on k values, how many entries are in the (conditional) joint probability table $P(E_1, \ldots, E_n \mid X = x)$?

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- We can make the simplifying assumption that the different features are conditionally independent *given the hypothesis*:

$$P(E_1, \ldots, E_n \mid X) = \prod_{i=1}^{n} P(E_i \mid X)$$

  – If each feature can take on *k* values, what is the complexity of storing the resulting distributions?

Naïve Bayes model

- Posterior:

$$P(X = x \mid E_1 = e_1, \ldots, E_n = e_n)$$

$$\propto P(X = x) P(E_1 = e_1, \ldots, E_n = e_n \mid X = x)$$

$$= P(X = x) \prod_{i=1}^{n} P(E_i = e_i \mid X = x)$$

- MAP decision:

$$x^* = \arg\max_x \underbrace{P(x \mid e)}_{\text{posterior}} \propto \underbrace{P(x)}_{\text{prior}} \underbrace{\prod_{i=1}^{n} P(e_i \mid x)}_{\text{likelihood}}$$

Case study: Spam filter

- **MAP decision:** to minimize the probability of error, we should classify a message as spam if

$$P(\text{spam} \mid \text{message}) > P(\neg\text{spam} \mid \text{message})$$

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$$P(\text{spam} \mid \text{message}) > P(\neg\text{spam} \mid \text{message})$$

- We have $P(\text{spam} \mid \text{message}) \propto P(\text{message} \mid \text{spam})P(\text{spam})$ and $\neg P(\text{spam} \mid \text{message}) \propto P(\text{message} \mid \neg\text{spam})P(\neg\text{spam})$

- To enable classification, we need to be able to estimate the likelihoods $P(\text{message} \mid \text{spam})$ and $P(\text{message} \mid \neg\text{spam})$ and priors $P(\text{spam})$ and $P(\neg\text{spam})$

Naïve Bayes Representation

- Goal: estimate likelihoods P(message | spam) and P(message | ¬spam) and priors P(spam) and P(¬spam)
- Likelihood: ***bag of words*** representation
  - The message is a sequence of words $(w_1, \ldots, w_n)$
  - The order of the words in the message is not important
  - Each word is conditionally independent of the others given message class (spam or not spam)

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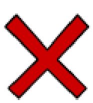

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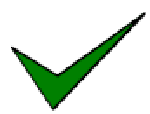

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Bag of words illustration

US Presidential Speeches Tag Cloud
http://chir.ag/projects/preztags/

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$$P(message \mid spam) = P(w_1, \ldots, w_n \mid spam) = \prod_{i=1}^{n} P(w_i \mid spam)$$

  - Thus, the problem is reduced to estimating marginal likelihoods of individual words $P(w_i \mid spam)$ and $P(w_i \mid \neg spam)$

Summary: Decision rule

- General MAP rule for Naïve Bayes:

$$x^* = \arg\max_x \underbrace{P(x \mid e)}_{\text{posterior}} \propto \underbrace{P(x)}_{\text{prior}} \underbrace{\prod_{i=1}^{n} P(e_i \mid x)}_{\text{likelihood}}$$

$$P(spam \mid w_1, \ldots, w_n) \propto P(spam) \prod_{i=1}^{n} P(w_i \mid spam)$$

- Thus, the filter should classify the message as spam if

$$P(spam) \prod_{i=1}^{n} P(w_i \mid spam) \quad > \quad P(\neg spam) \prod_{i=1}^{n} P(w_i \mid \neg spam)$$

Parameter estimation

- Model parameters: feature likelihoods P(word | spam) and P(word | ¬spam) and priors P(spam) and P(¬spam)
  - How do we obtain the values of these parameters?

prior

spam:	0.33
¬spam:	0.67

P(word | spam)

the :	0.0156
to :	0.0153
and :	0.0115
of :	0.0095
you :	0.0093
a :	0.0086
with:	0.0080
from:	0.0075
...	

P(word | ¬spam)

the :	0.0210
to :	0.0133
of :	0.0119
2002:	0.0110
with:	0.0108
from:	0.0107
and :	0.0105
a :	0.0100
...	

Parameter estimation

- Model parameters: feature likelihoods P(word | spam) and P(word | ¬spam) and priors P(spam) and P(¬spam)
  - How do we obtain the values of these parameters?
  - Need *training set* of labeled samples from both classes

$$P(\text{word} \mid \text{spam}) = \frac{\text{\# of word occurrences in spam messages}}{\text{total \# of words in spam messages}}$$

  - This is the *maximum likelihood* (ML) estimate, or estimate that maximizes the likelihood of the training data:

$$\prod_{d=1}^{D} \prod_{i=1}^{n_d} P(w_{d,i} \mid class_{d,i})$$

d: index of training document, *i*: index of a word

Parameter estimation

- Parameter estimate:

$$P(\text{word} \mid \text{spam}) = \frac{\text{\# of word occurrences in spam messages}}{\text{total \# of words in spam messages}}$$

- Parameter smoothing: dealing with words that were never seen or seen too few times
  - **Laplacian smoothing:** pretend you have seen every vocabulary word one more time than you actually did

$$P(\text{word} \mid \text{spam}) = \frac{\text{\# of word occurrences in spam messages} + 1}{\text{total \# of words in spam messages} + V}$$

(V: total number of unique words)

Summary of model and parameters

- Naïve Bayes model:

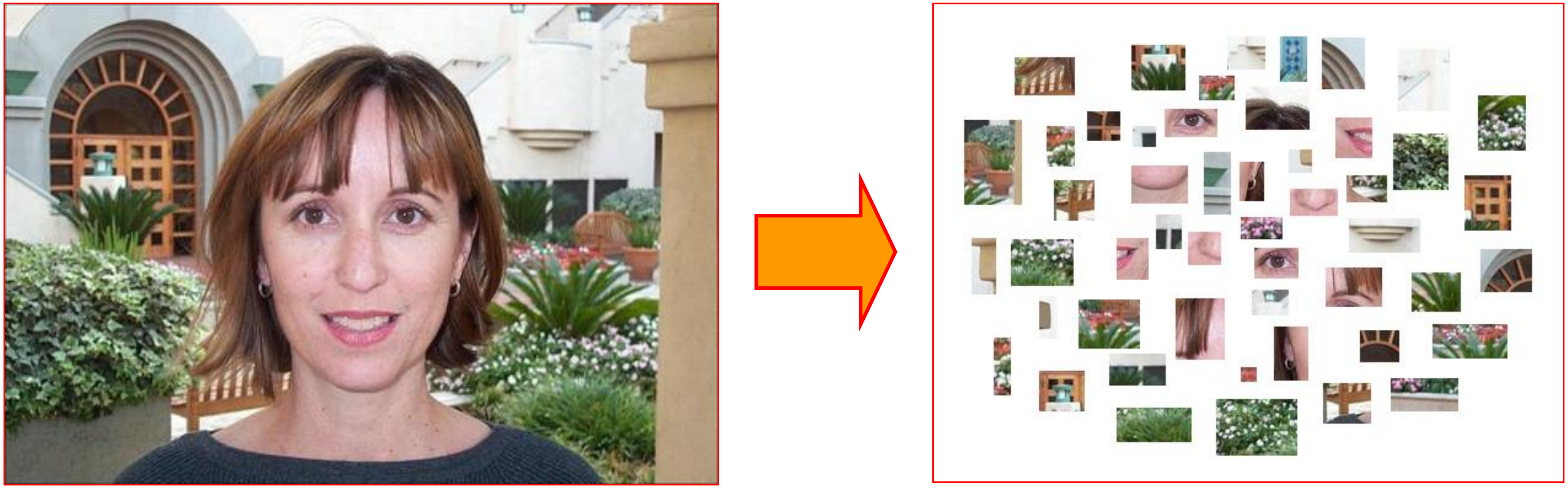
$$P(spam \mid message) \propto P(spam) \prod_{i=1}^{n} P(w_i \mid spam)$$

$$P(\neg spam \mid message) \propto P(\neg spam) \prod_{i=1}^{n} P(w_i \mid \neg spam)$$

- Model parameters:

prior	Likelihood of spam	Likelihood of ¬spam
$P(spam)$	$P(w_1 \mid spam)$	$P(w_1 \mid \neg spam)$
$P(\neg spam)$	$P(w_2 \mid spam)$	$P(w_2 \mid \neg spam)$
	$\ldots$	$\ldots$
	$P(w_n \mid spam)$	$P(w_n \mid \neg spam)$

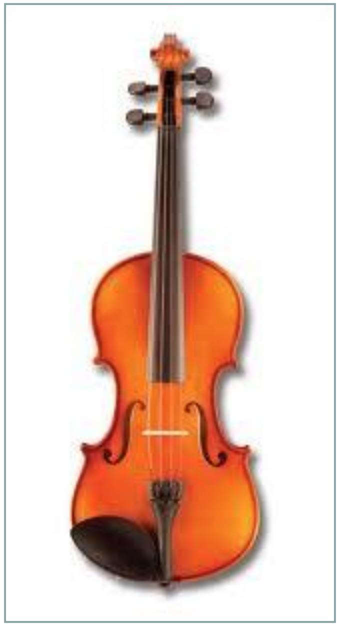
Bag-of-word models for images



Bag-of-word models for images

1. Extract image features





Bag-of-word models for images

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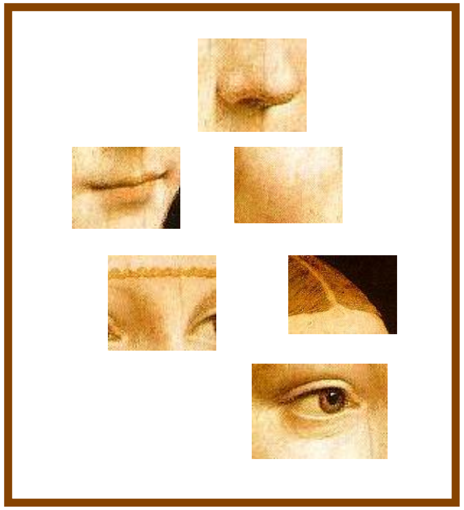
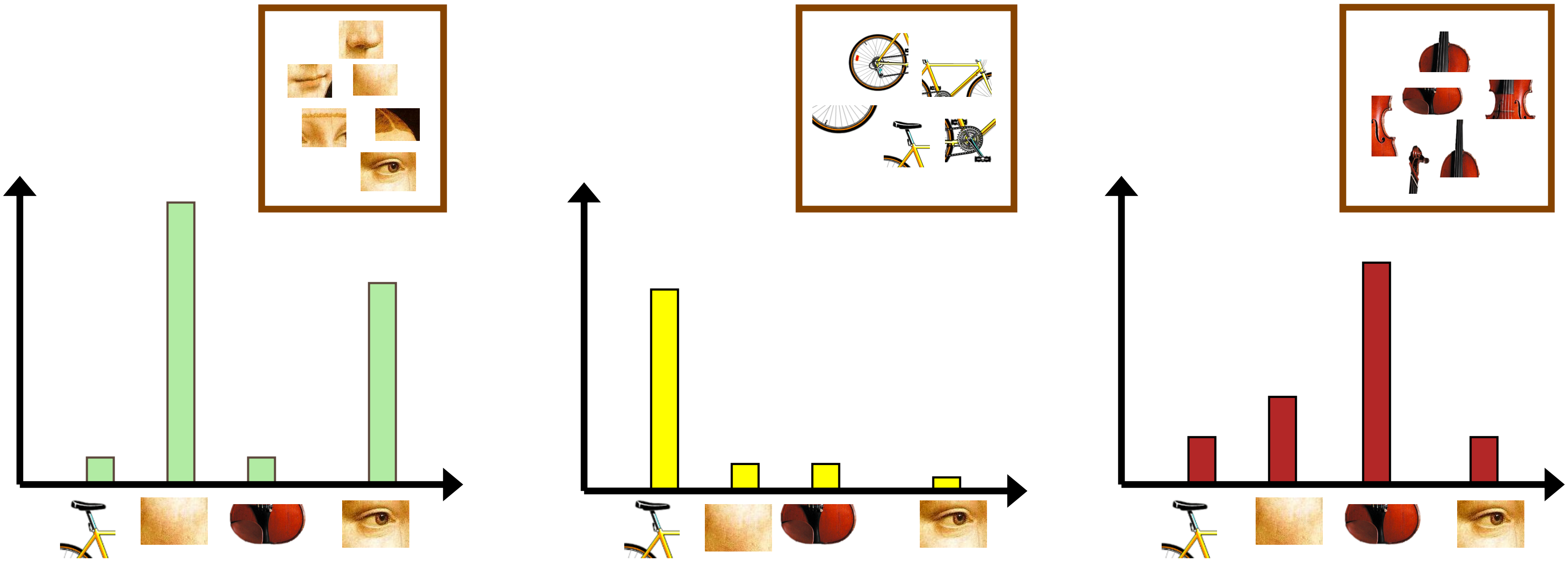


Bag-of-word models for images

1. Extract image features
2. Learn “visual vocabulary”

Bag-of-word models for images

1. Extract image features
2. Learn “visual vocabulary”
3. Map image features to visual words



Bayesian decision making: Summary

- Suppose the agent has to make decisions about the value of an unobserved *query variable* X based on the values of an observed *evidence variable* E
- **Inference problem:** given some evidence E = e, what is P(X | e)?
- **Learning problem:** estimate the parameters of the probabilistic model P(X | E) given a *training sample* $\{(x_1,e_1), \ldots, (x_n,e_n)\}$