Support Vector Machines

Introduction to Data Mining, 2nd Edition

by

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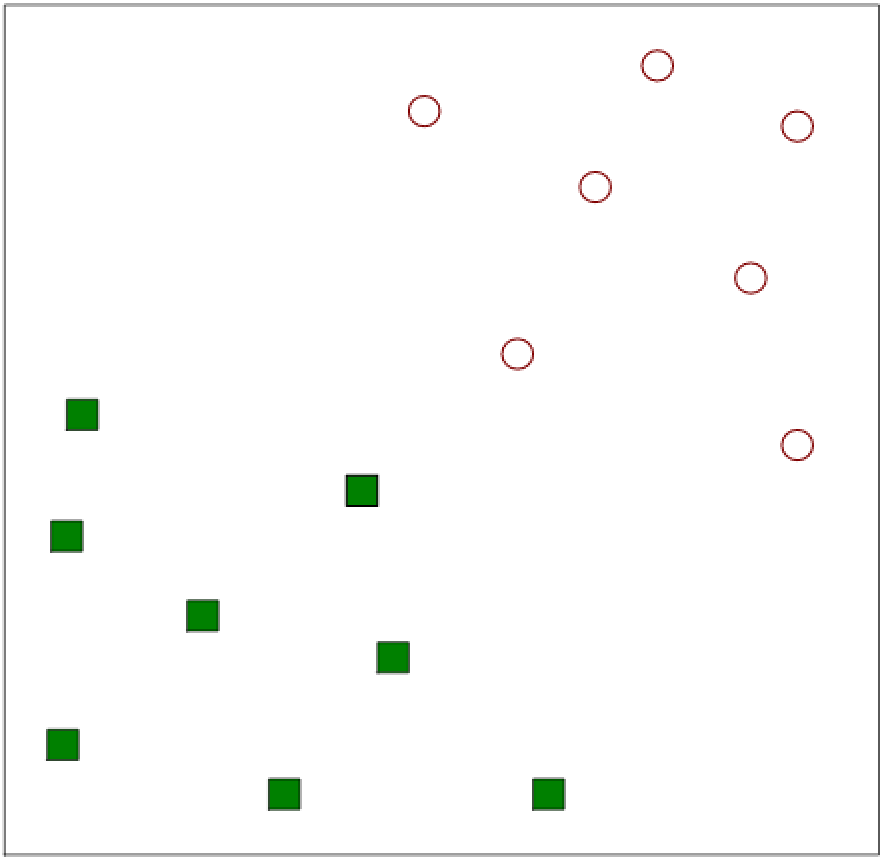
Support Vector Machines



- **Find a linear hyperplane (decision boundary) that will separate the data**

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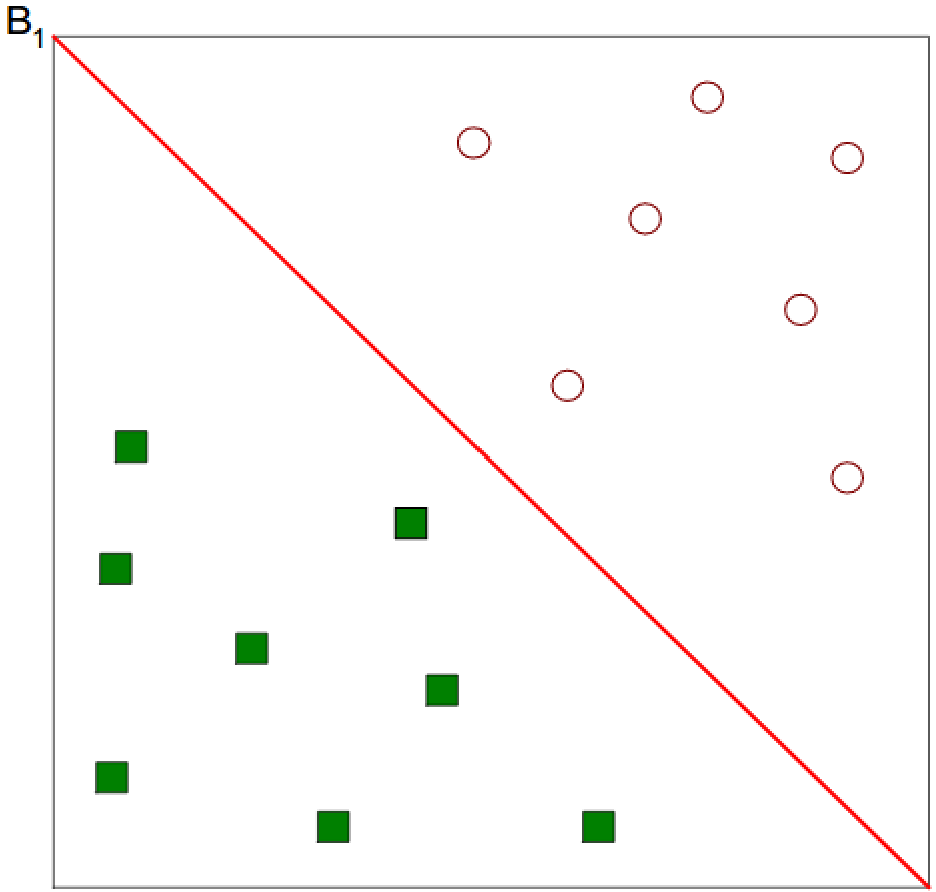

- One Possible Solution

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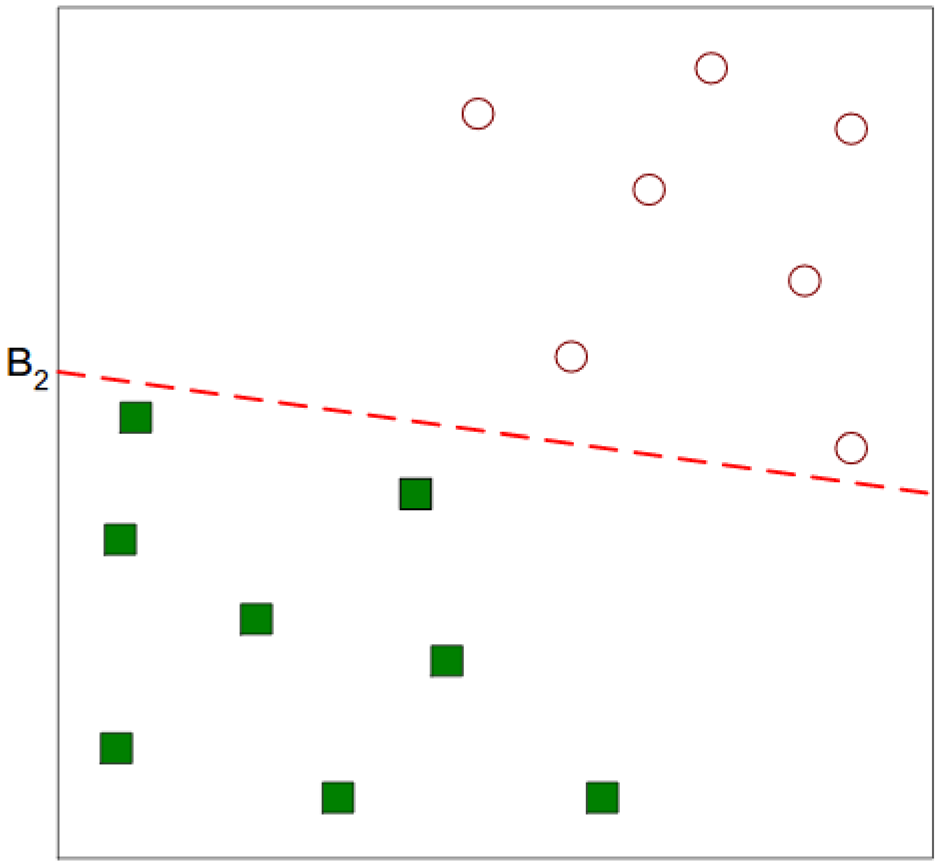

- **Another possible solution**

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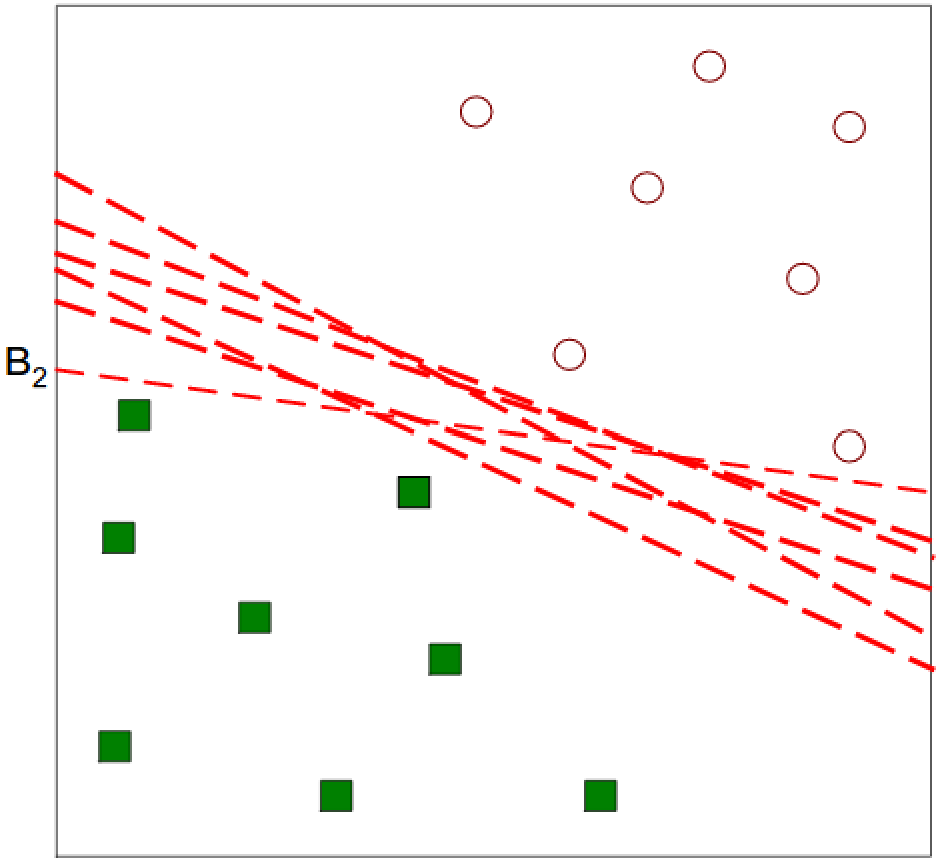

- **Other possible solutions**

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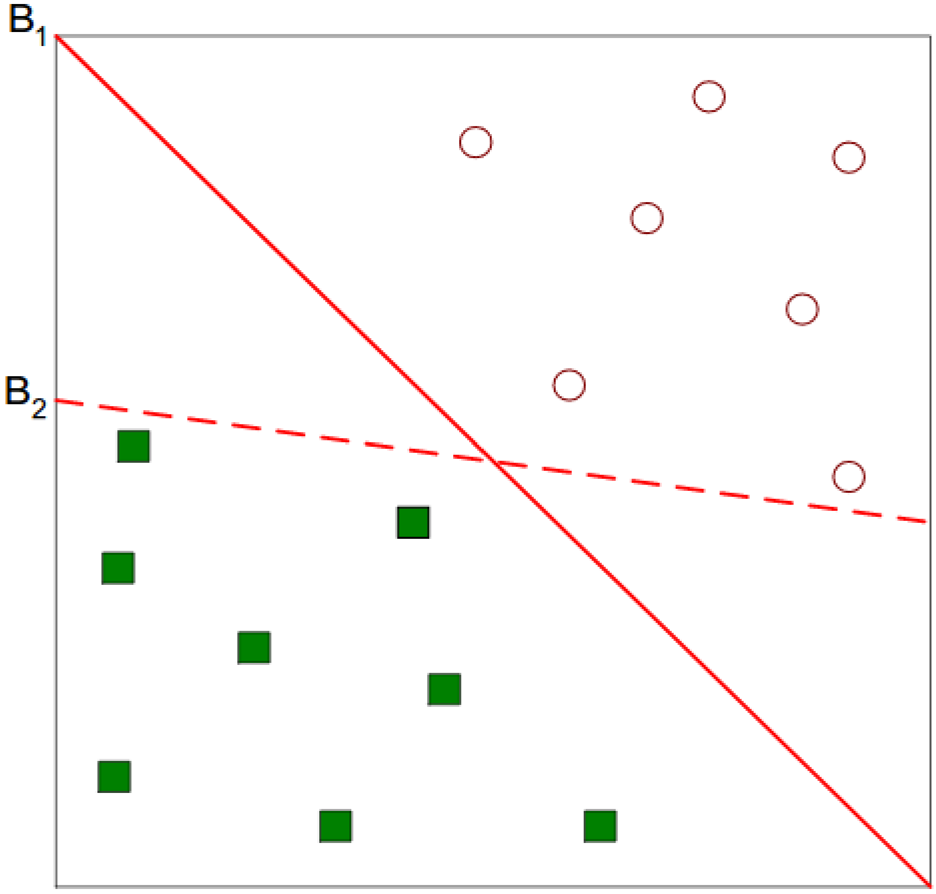

- **Which one is better? B1 or B2?**
- **How do you define better?**

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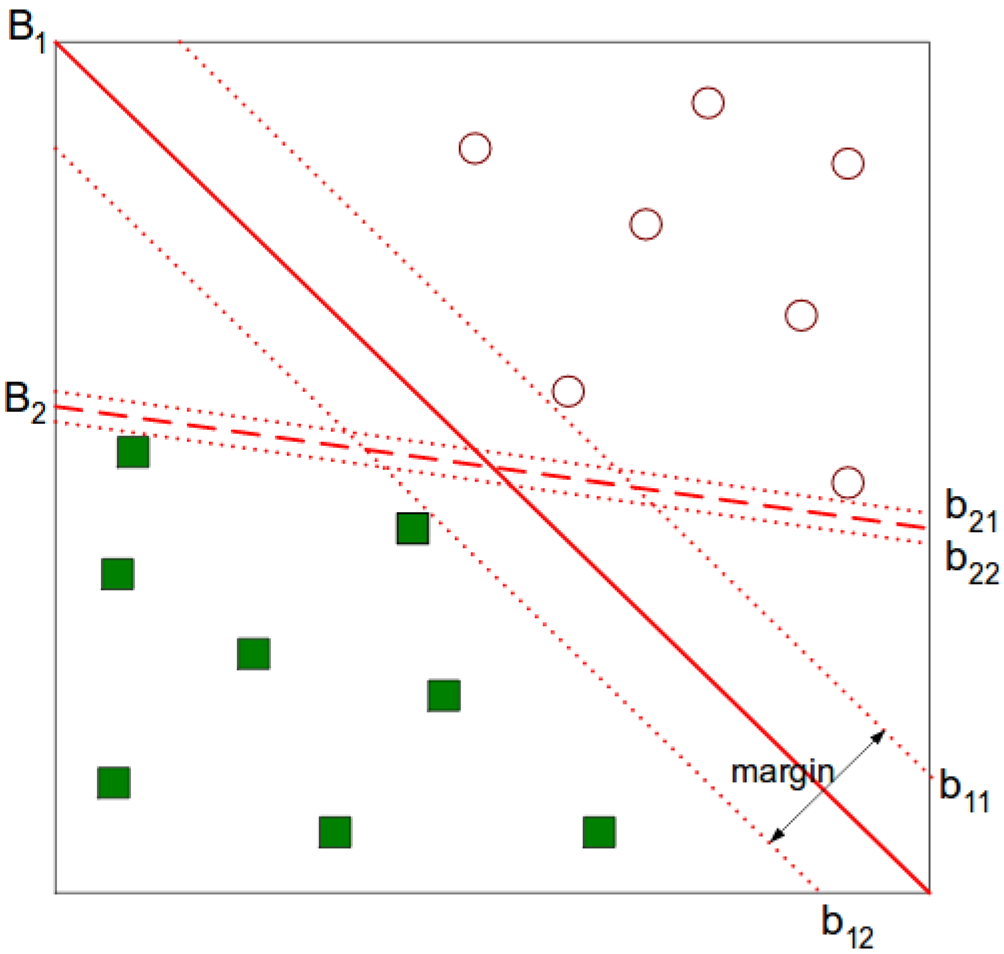

- **Find hyperplane maximizes the margin => B1 is better than B2**

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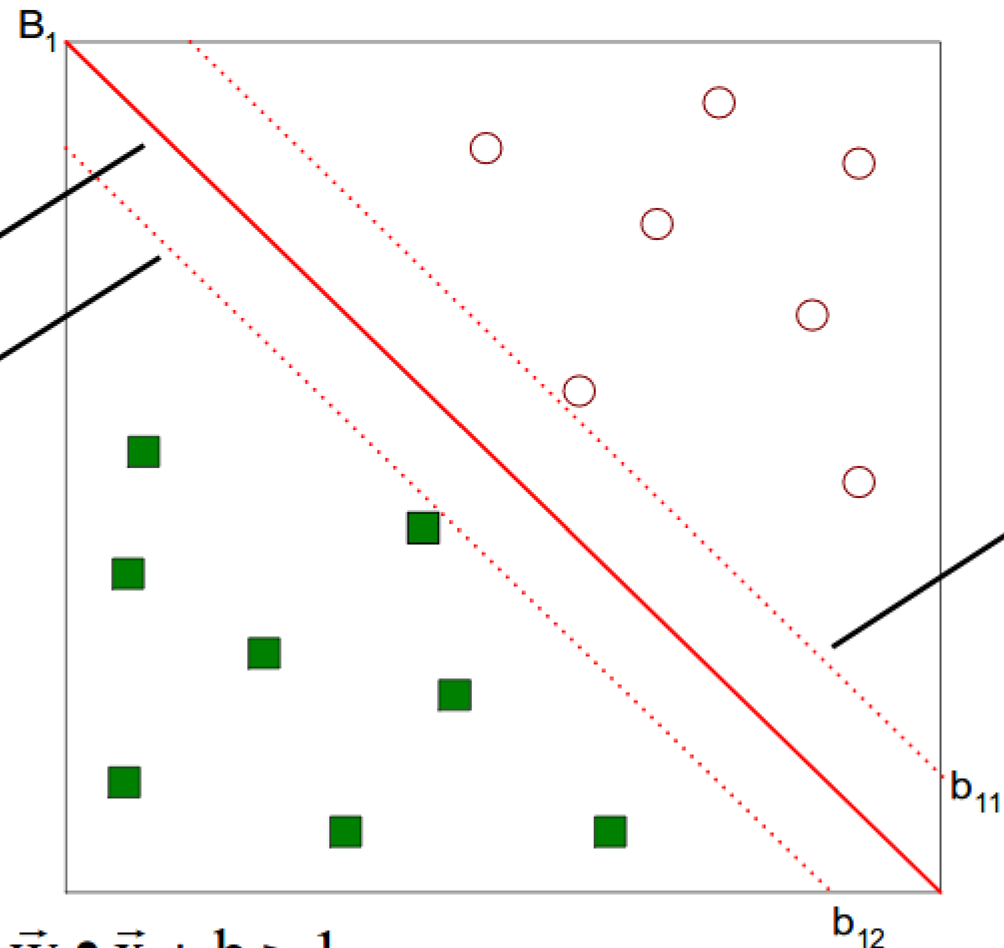

$$\vec{w} \bullet \vec{x} + b = 0$$

$$\vec{w} \bullet \vec{x} + b = -1$$

$$\vec{w} \bullet \vec{x} + b = +1$$

$$f(\vec{x}) = \begin{cases} 1 & \text{if } \vec{\mathrm{w}} \bullet \vec{\mathrm{x}} + \mathrm{b} \geq 1 \\ -1 & \text{if } \vec{\mathrm{w}} \bullet \vec{\mathrm{x}} + \mathrm{b} \leq -1 \end{cases}$$

$$\text{Margin} = \frac{2}{\|\vec{w}\|}$$

Linear SVM

- **Linear model:**

$$f(\vec{x}) = \begin{cases} 1 & \text{if } \vec{\mathrm{w}} \bullet \vec{\mathrm{x}} + \mathrm{b} \geq 1 \\ -1 & \text{if } \vec{\mathrm{w}} \bullet \vec{\mathrm{x}} + \mathrm{b} \leq -1 \end{cases}$$

- **Learning the model is equivalent to determining the values of** $\vec{w}$ and b
  - **How to find** $\vec{w}$ and b **from training data?**

Learning Linear SVM

- **Objective is to maximize:** $\text{Margin} = \frac{2}{\|\vec{w}\|}$
  - **Which is equivalent to minimizing:** $L(\vec{w}) = \frac{\|\vec{w}\|^2}{2}$
  - **Subject to the following constraints:**

$$y_i = \begin{cases} 1 & \text{if } \vec{\mathrm{w}} \bullet \vec{\mathrm{x}}_i + \mathrm{b} \geq 1 \\ -1 & \text{if } \vec{\mathrm{w}} \bullet \vec{\mathrm{x}}_i + \mathrm{b} \leq -1 \end{cases}$$

or

$$y_i(w \cdot x_i + b) \geq 1, \quad i = 1, 2, \ldots, N$$

- **This is a constrained optimization problem**
  - **Solve it using Lagrange multiplier method**

Learning Linear SVM

- **Decision boundary depends only on support vectors**
  - **If you have data set with same support vectors, decision boundary will not change**
  - **How to classify using SVM once w and *b* are found? Given a test record, x_i**

$$f(\vec{x}_i) = \begin{cases} 1 & \text{if } \vec{w} \bullet \vec{x}_i + b \geq 1 \\ -1 & \text{if } \vec{w} \bullet \vec{x}_i + b \leq -1 \end{cases}$$

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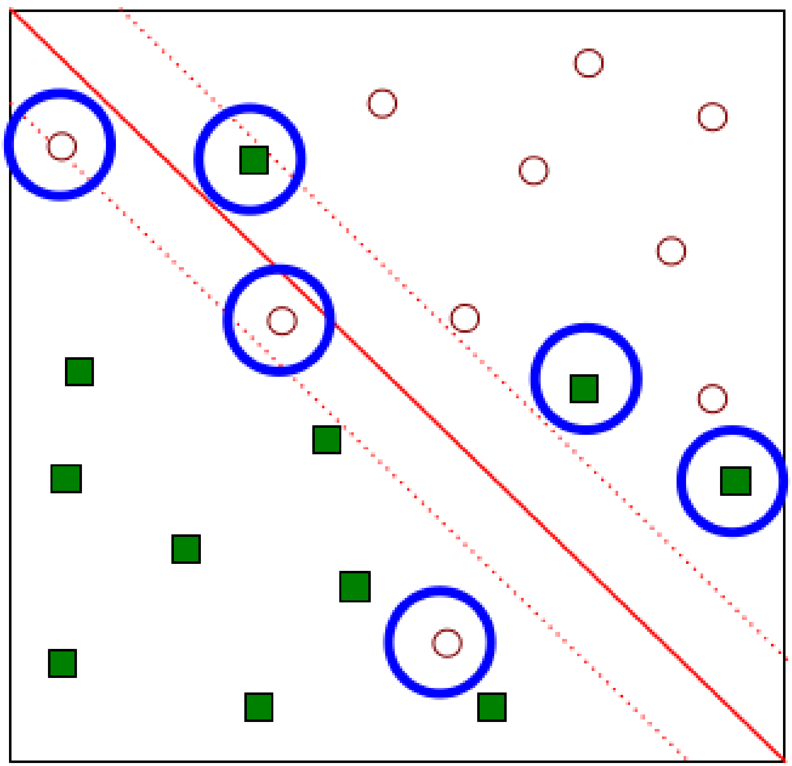
- **What if the problem is not linearly separable?**



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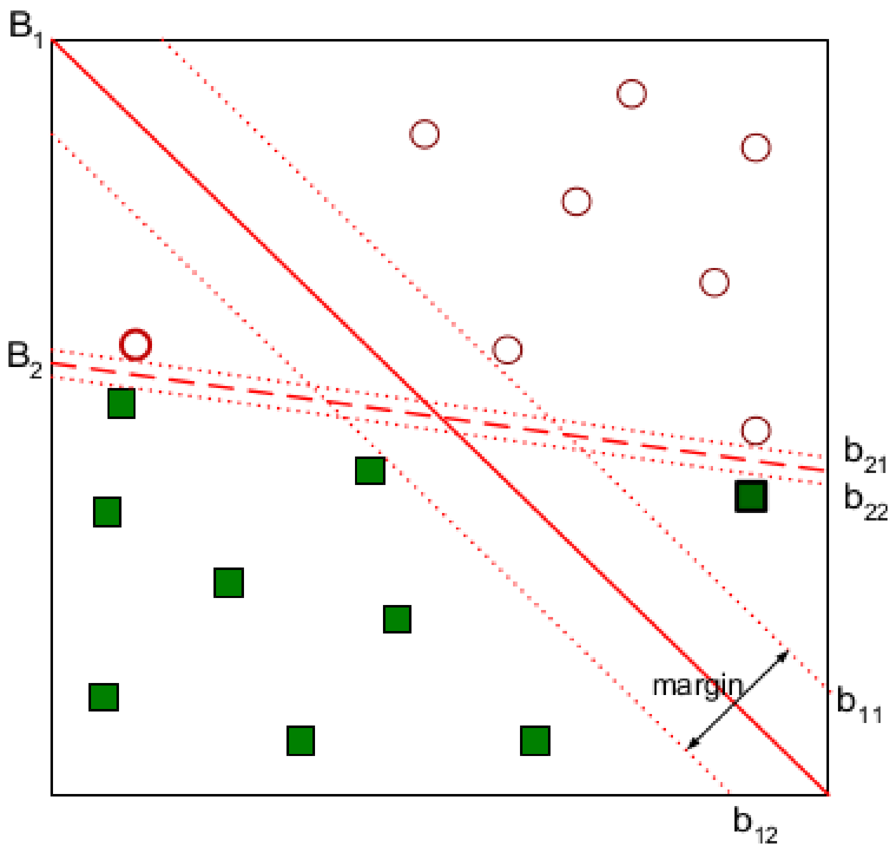
- What if the problem is not linearly separable?
  - Introduce slack variables
    - Need to minimize: $$L(w) = \frac{\|\vec{w}\|^2}{2} + C\left(\sum_{i=1}^{N} \xi_i^k\right)$$
    - Subject to:

$$y_i = \begin{cases} 1 & \text{if } \vec{w} \bullet \vec{x}_i + b \geq 1 - \xi_i \\ -1 & \text{if } \vec{w} \bullet \vec{x}_i + b \leq -1 + \xi_i \end{cases}$$

    - If k is 1 or 2, this leads to same objective function as linear SVM but with different constraints (see textbook)

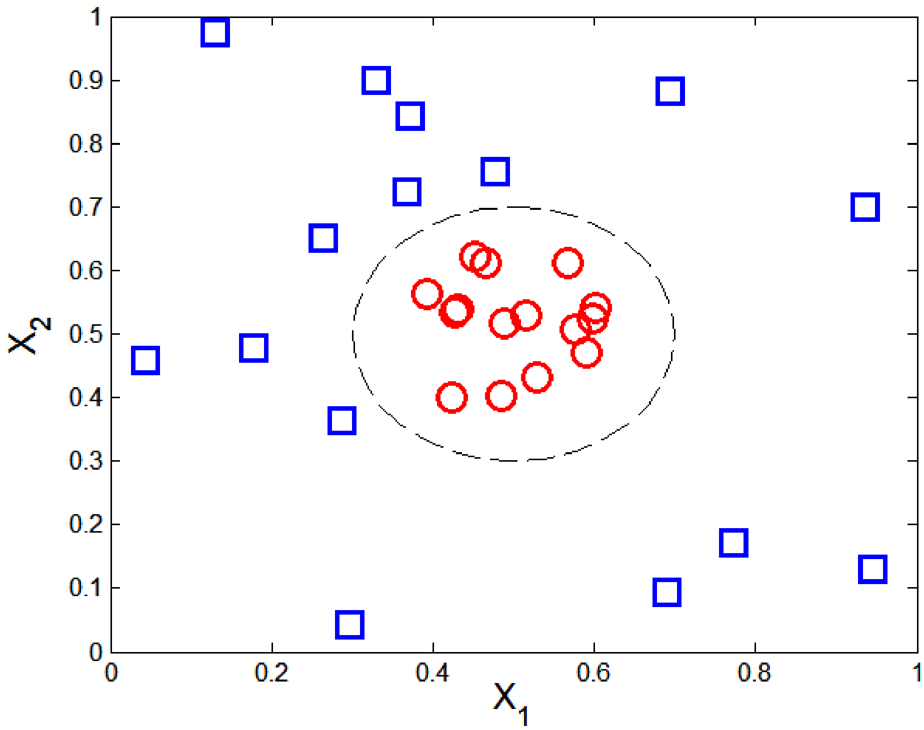
Support Vector Machines



- **Find the hyperplane that optimizes both factors**

Nonlinear Support Vector Machines

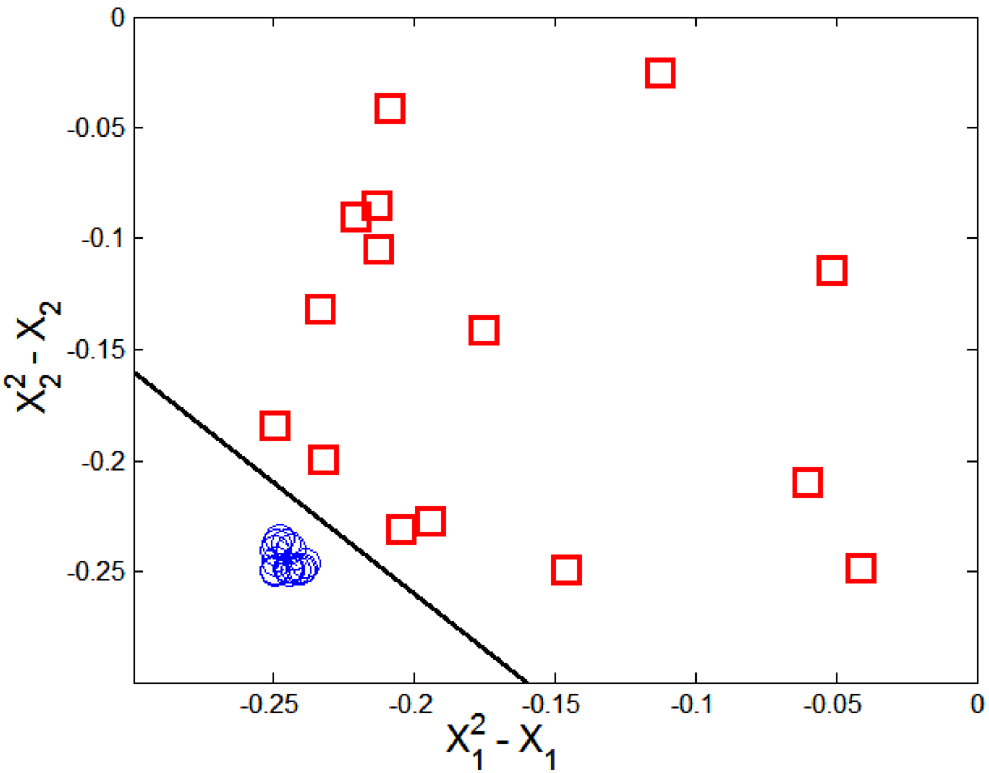
- **What if decision boundary is not linear?**



$$y(x_1, x_2) = \begin{cases} 1 & \text{if } \sqrt{(x_1 - 0.5)^2 + (x_2 - 0.5)^2} > 0.2 \\ -1 & \text{otherwise} \end{cases}$$

Nonlinear Support Vector Machines

- **Trick: Transform data into higher dimensional space**



$$x_1^2 - x_1 + x_2^2 - x_2 = -0.46.$$

$$\Phi : (x_1, x_2) \longrightarrow (x_1^2, x_2^2, \sqrt{2}x_1, \sqrt{2}x_2, 1).$$

$$w_4 x_1^2 + w_3 x_2^2 + w_2 \sqrt{2} x_1 + w_1 \sqrt{2} x_2 + w_0 = 0.$$

Decision boundary:

$$\vec{w} \cdot \Phi\left(\vec{x}\right) + b = 0$$

Learning Nonlinear SVM

- **Optimization problem:**

$$\min_{w} \frac{\|\mathbf{w}\|^2}{2}$$

$$\textit{subject to} \quad y_i(w \cdot \Phi(x_i) + b) \geq 1, \ \forall\{(x_i, y_i)\}$$

- **Which leads to the same set of equations (but involve Φ(x) instead of x)**

$$L_D = \sum_{i=1}^{n} \lambda_i - \frac{1}{2}\sum_{i,j} \lambda_i \lambda_j y_i y_j \Phi(\mathbf{x}_i) \cdot \Phi(\mathbf{x}_j)$$

$$\mathbf{w} = \sum_i \lambda_i y_i \Phi(\mathbf{x}_i)$$

$$\lambda_i \{y_i(\sum_j \lambda_j y_j \Phi(\mathbf{x}_j) \cdot \Phi(\mathbf{x}_i) + b) - 1\} = 0,$$

$$f(\mathbf{z}) = sign(\mathbf{w} \cdot \Phi(\mathbf{z}) + b) = sign(\sum_{i=1}^{n} \lambda_i y_i \Phi(\mathbf{x}_i) \cdot \Phi(\mathbf{z}) + b).$$

Learning NonLinear SVM

- **Issues:**
  - **What type of mapping function Φ should be used?**
  - **How to do the computation in high dimensional space?**
    - **Most computations involve dot product $\Phi(x_i) \bullet \Phi(x_j)$**
    - **Curse of dimensionality?**

Learning Nonlinear SVM

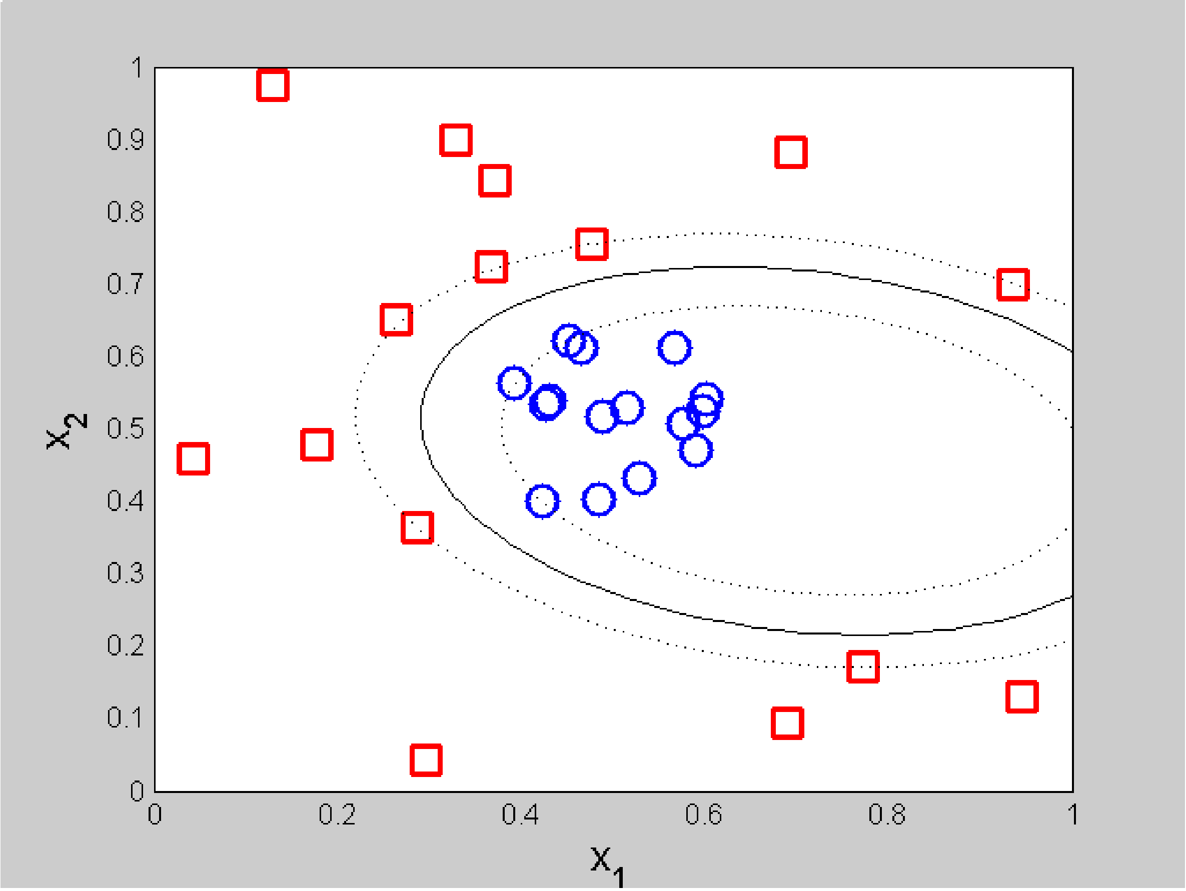
- **Kernel Trick:**
  - $\Phi(x_i) \bullet \Phi(x_j) = K(x_i, x_j)$
  - $K(x_i, x_j)$ is a kernel function (expressed in terms of the coordinates in the original space)
    - Exampl

$$K(\mathbf{x}, \mathbf{y}) = (\mathbf{x} \cdot \mathbf{y} + 1)^p$$

$$K(\mathbf{x}, \mathbf{y}) = e^{-\|\mathbf{x}-\mathbf{y}\|^2/(2\sigma^2)}$$

$$K(\mathbf{x}, \mathbf{y}) = \tanh(k\mathbf{x} \cdot \mathbf{y} - \delta)$$

Example of Nonlinear SVM



SVM with polynomial degree 2 kernel

Learning Nonlinear SVM

- **Advantages of using kernel:**
  - **Don't have to know the mapping function Φ**
  - **Computing dot product $\Phi(x_i) \bullet \Phi(x_j)$ in the original space avoids curse of dimensionality**

- **Not all functions can be kernels**
  - **Must make sure there is a corresponding Φ in some high-dimensional space**
  - **Mercer's theorem (see textbook)**

Characteristics of SVM

- Since the learning problem is formulated as a convex optimization problem, efficient algorithms are available to find the global minima of the objective function (many of the other methods use greedy approaches and find locally optimal solutions)
- Overfitting is addressed by maximizing the margin of the decision boundary, but the user still needs to provide the type of kernel function and cost function
- Difficult to handle missing values
- Robust to noise
- High computational complexity for building the model