VC 15/16 – TP6
Frequency Space

Mestrado em Ciência de Computadores

Mestrado Integrado em Engenharia de Redes e Sistemas Informáticos

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Outline

- Fourier Transform
- Frequency Space
- Spatial Convolution

Acknowledgements: Most of this course is based on the excellent courses offered by Prof. Shree Nayar at Columbia University, USA and by Prof. Srinivasa Narasimhan at CMU, USA. Please acknowledge the original source when reusing these slides for academic purposes.

Topic: Fourier Transform

- Fourier Transform
- Frequency Space
- Spatial Convolution

How to Represent Signals?

- Option 1: Taylor series represents any function using polynomials.

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \ldots.$$

- Polynomials are not the best - unstable and not very physically meaningful.

- Easier to talk about "signals" in terms of its "frequencies" (how fast/often signals change, etc).

Jean Baptiste Joseph Fourier (1768-1830)

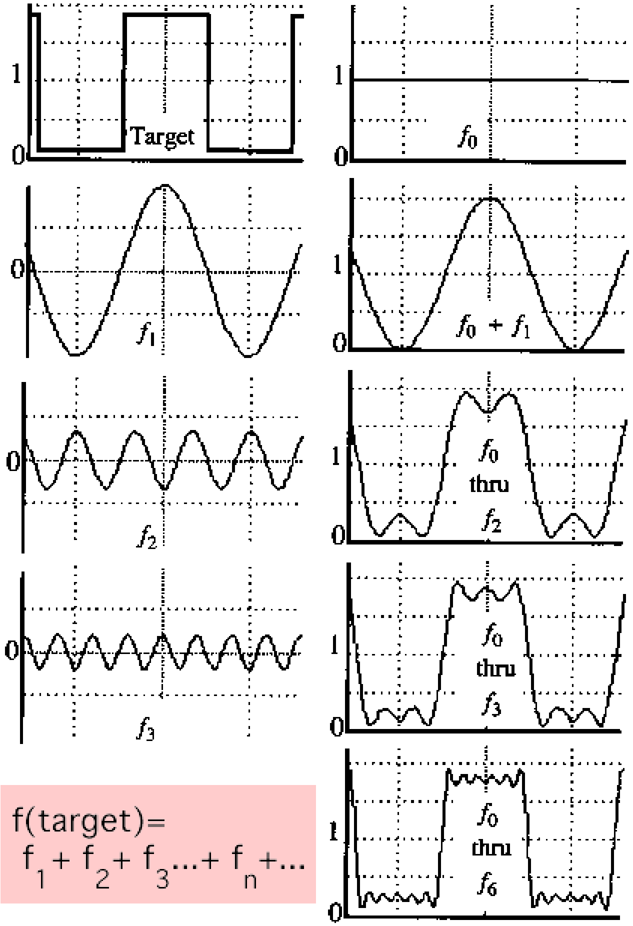
- Had a crazy idea (1807):
- **Any** periodic function can be rewritten as a weighted sum of Sines and Cosines of different frequencies.
- Don’t believe it?
  - Neither did Lagrange, Laplace, Poisson and other big wigs
  - Not translated into English until 1878!
- But it’s true!
  - called Fourier Series
  - Possibly the greatest tool used in Engineering

A Sum of Sinusoids

- Our building block:

$$A\sin(\omega x + \phi)$$

- Add enough of them to get any signal *f(x)* you want!
- How many degrees of freedom?
- What does each control?
- Which one encodes the coarse vs. fine structure of the signal?



f(target)= $f_1 + f_2 + f_3 ... + f_n + ...$

Fourier Transform

- We want to understand the frequency ω of our signal. So, let's reparametrize the signal by ω instead of *x*:

$$f(x) \longrightarrow \boxed{\text{Fourier Transform}} \longrightarrow F(\omega)$$

- For every ω from 0 to inf, ***F(ω)*** holds the amplitude *A* and phase ϕ of the corresponding sine
  - How can *F* hold both? Complex number trick!

$$A\sin(\omega x + \phi)$$

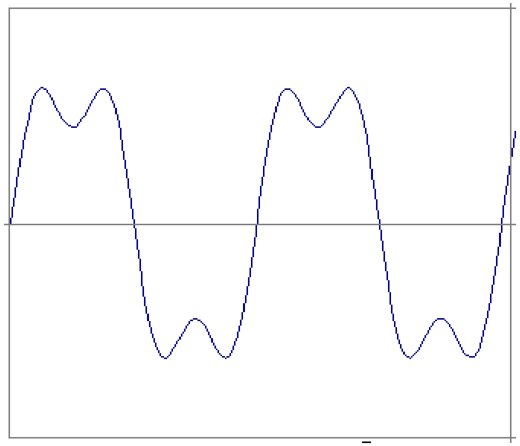
$$F(\omega) = R(\omega) + iI(\omega)$$

$$A = \pm\sqrt{R(\omega)^2 + I(\omega)^2}$$

$$\phi = \tan^{-1}\frac{I(\omega)}{R(\omega)}$$

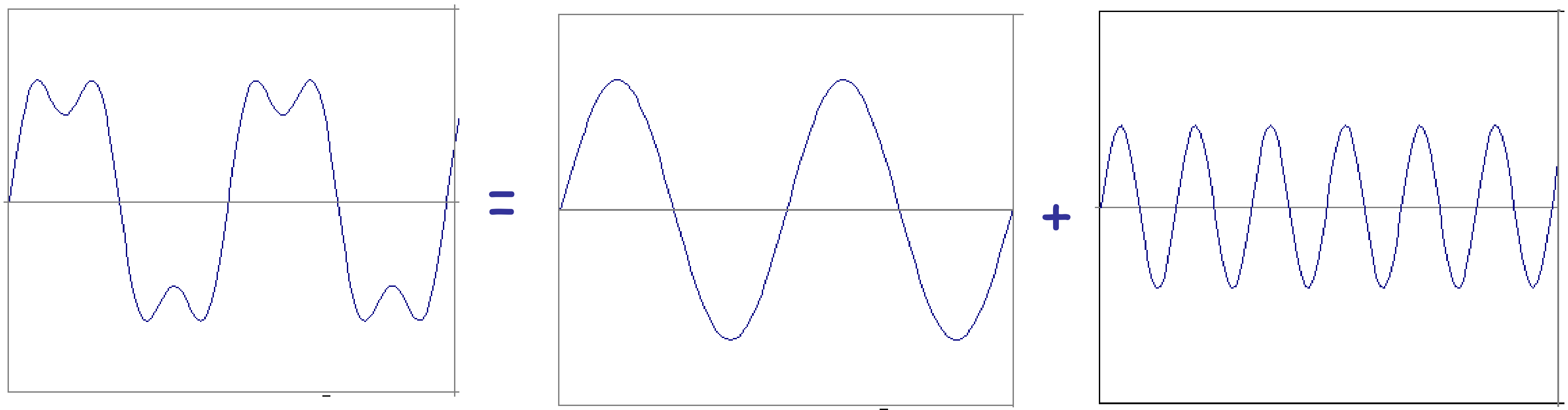
Time and Frequency

- example : $g(t) = \sin(2pf\ t) + (1/3)\sin(2p(3f)\ t)$



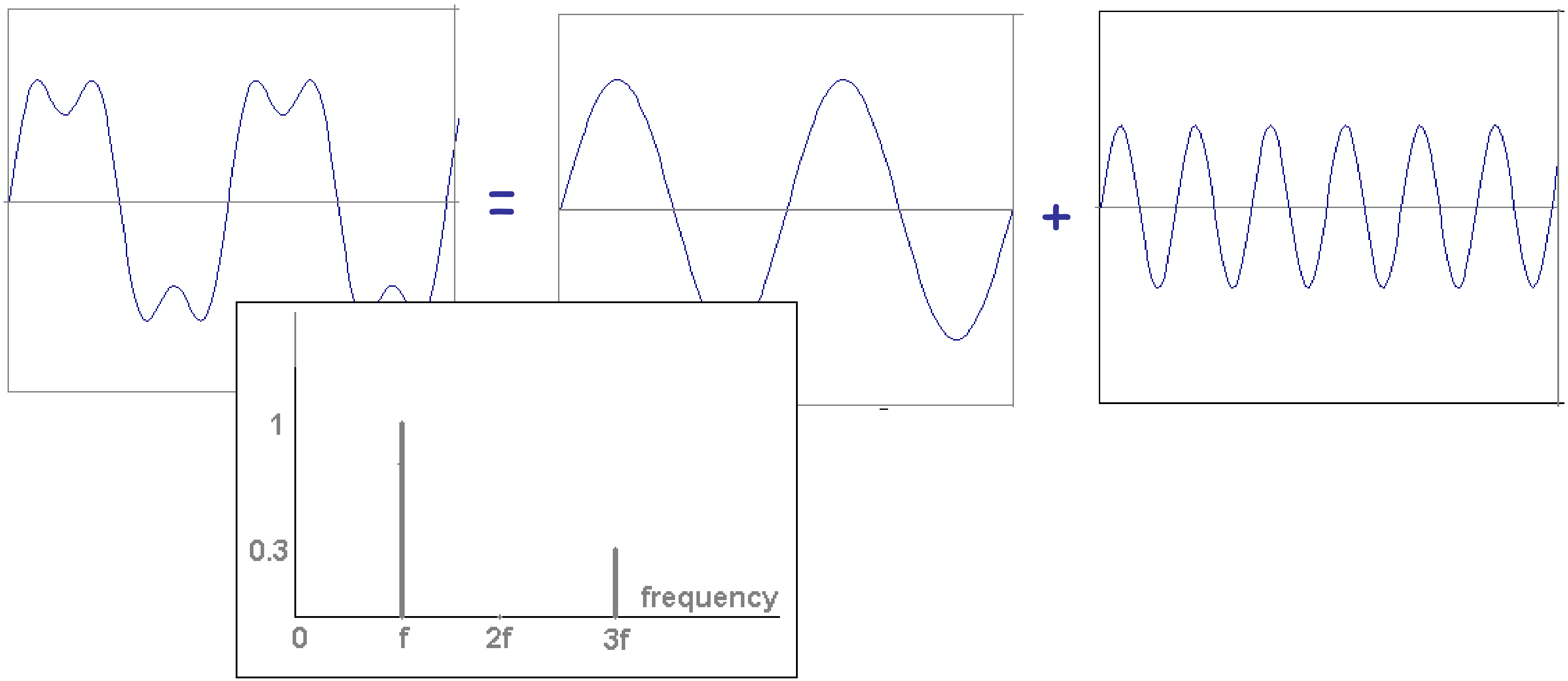
Time and Frequency

- example : $g(t) = \sin(2pf\, t) + (1/3)\sin(2p(3f)\, t)$



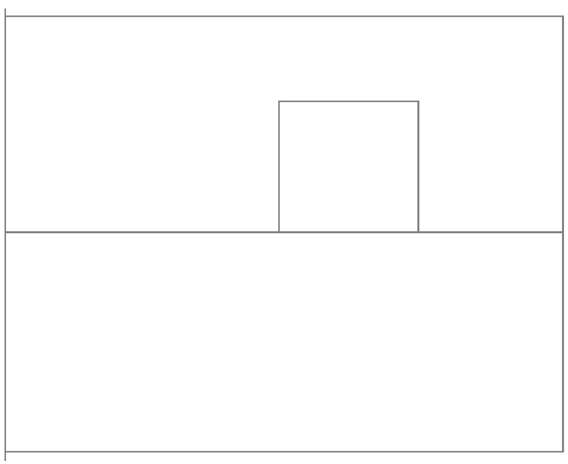
Frequency Spectra

- example : $g(t) = \sin(2pf\ t) + (1/3)\sin(2p(3f)\ t)$

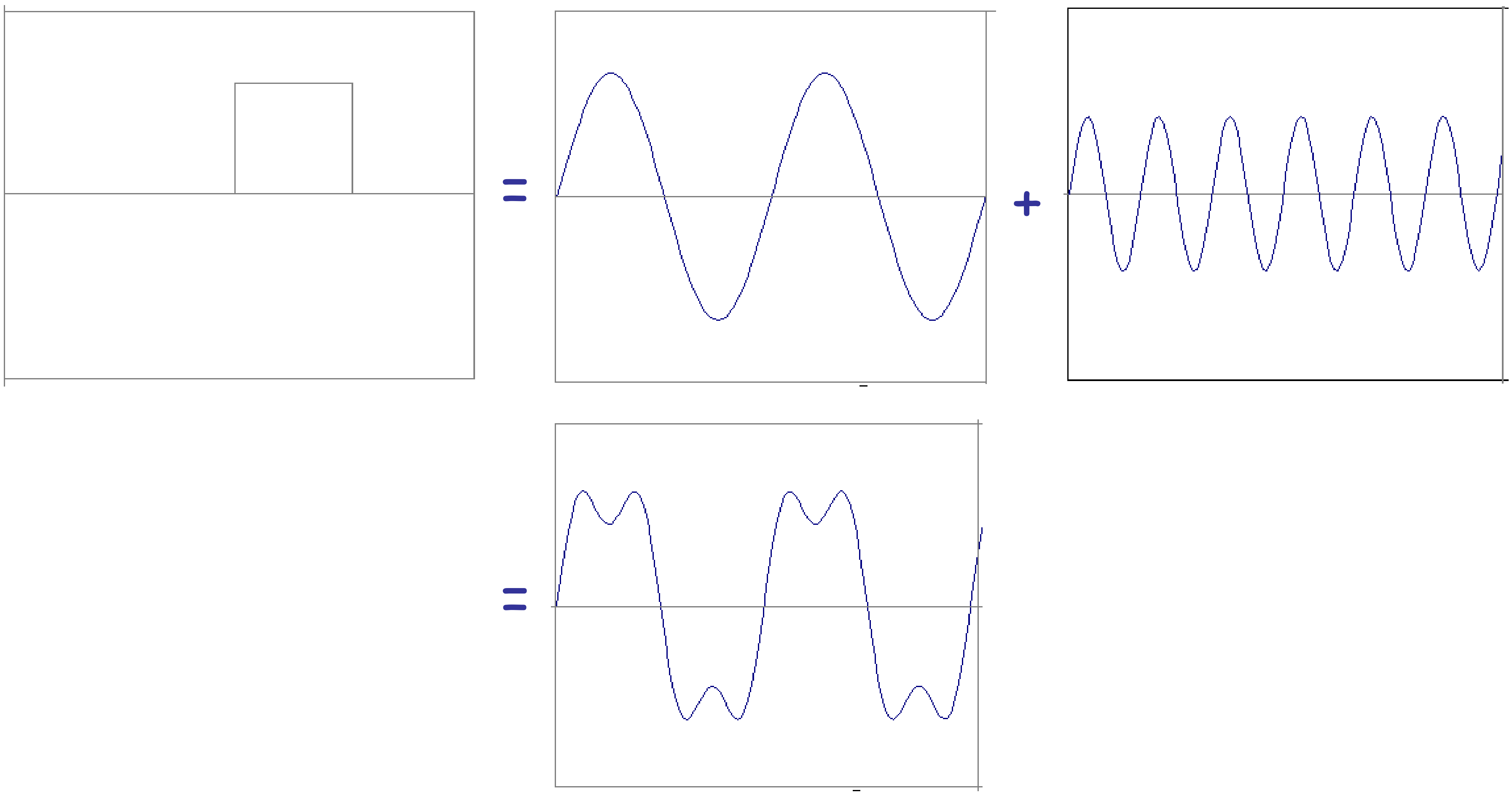


Frequency Spectra

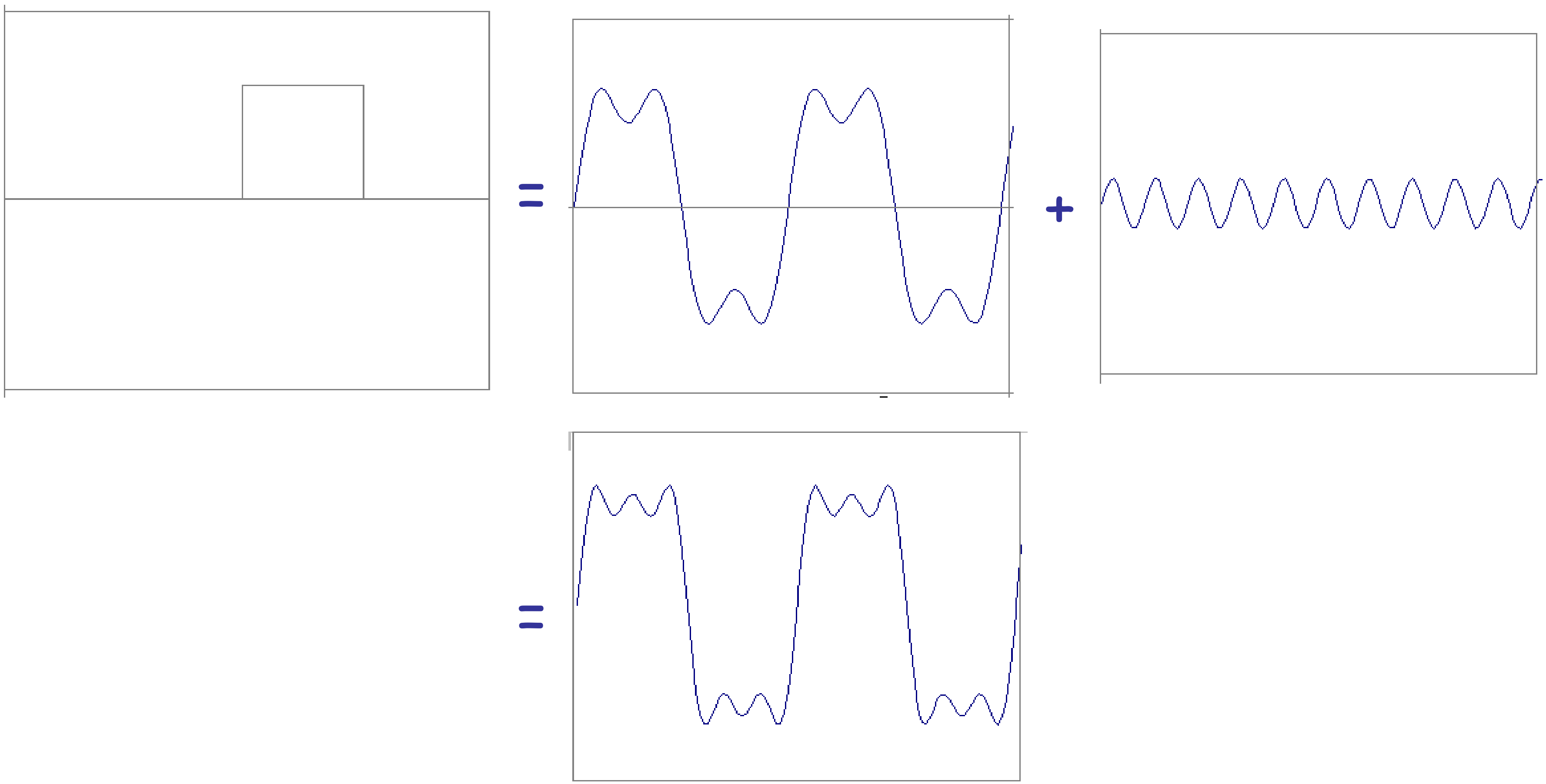
- Usually, frequency is more interesting than the phase



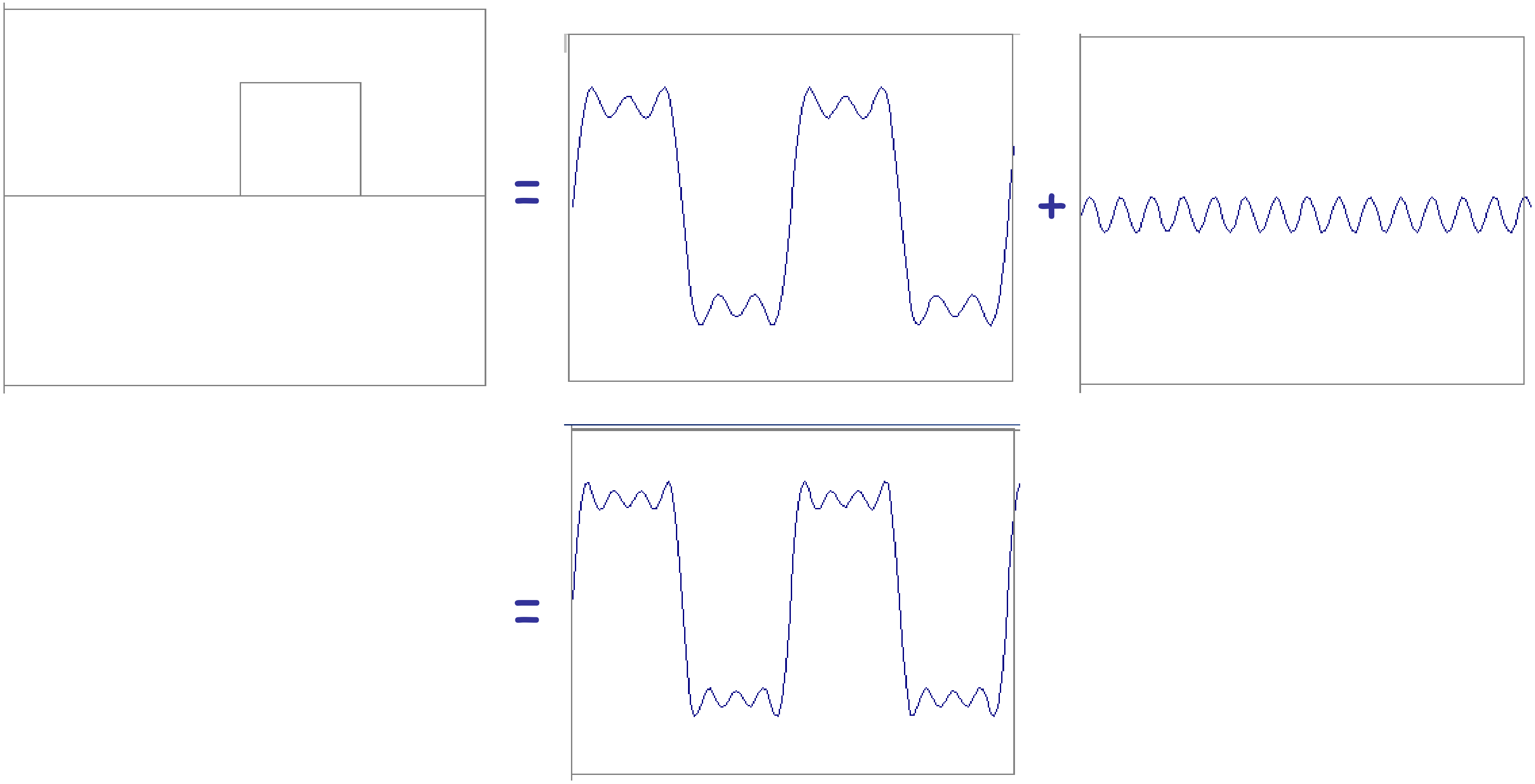
Frequency Spectra



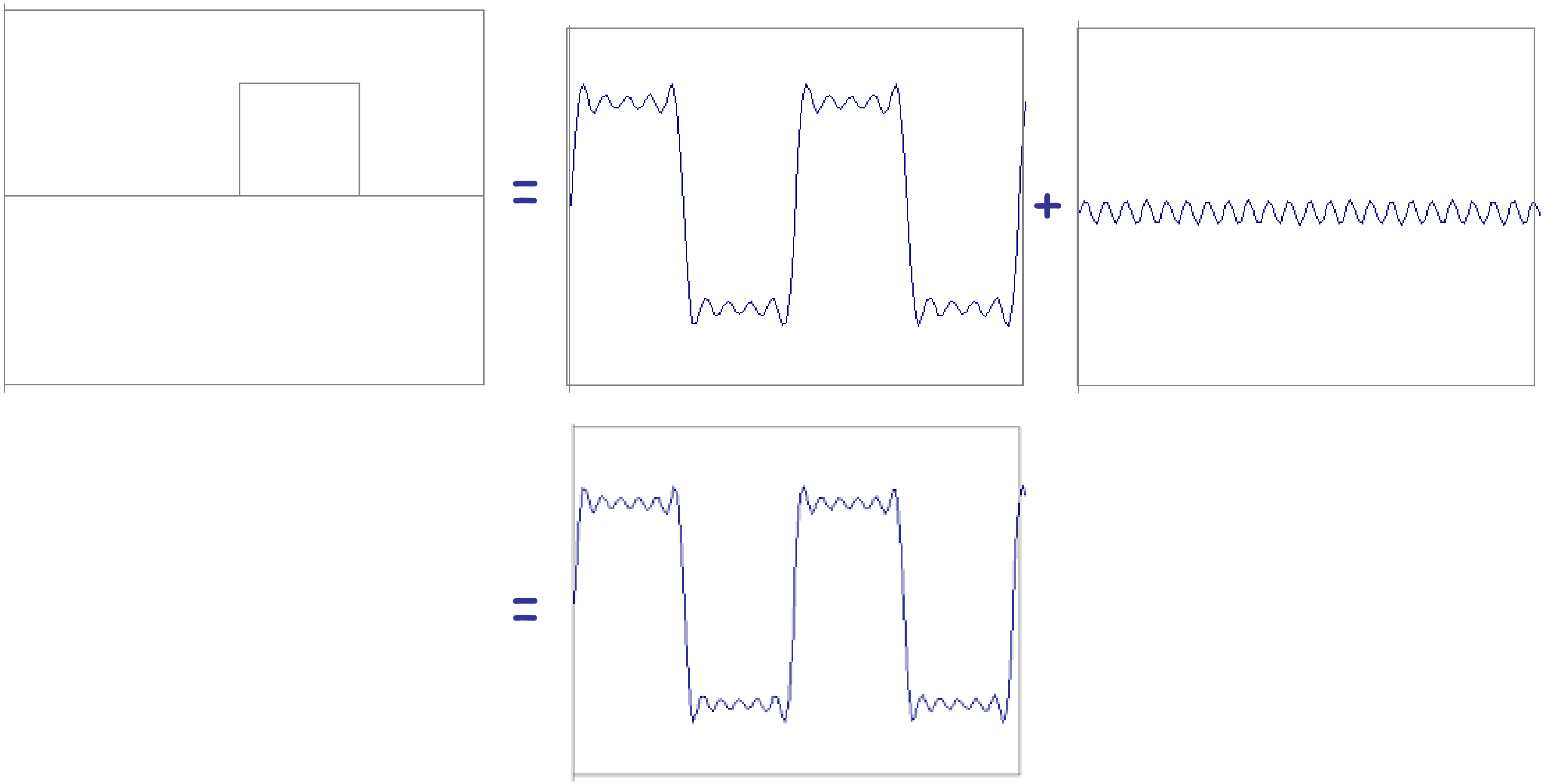
Frequency Spectra



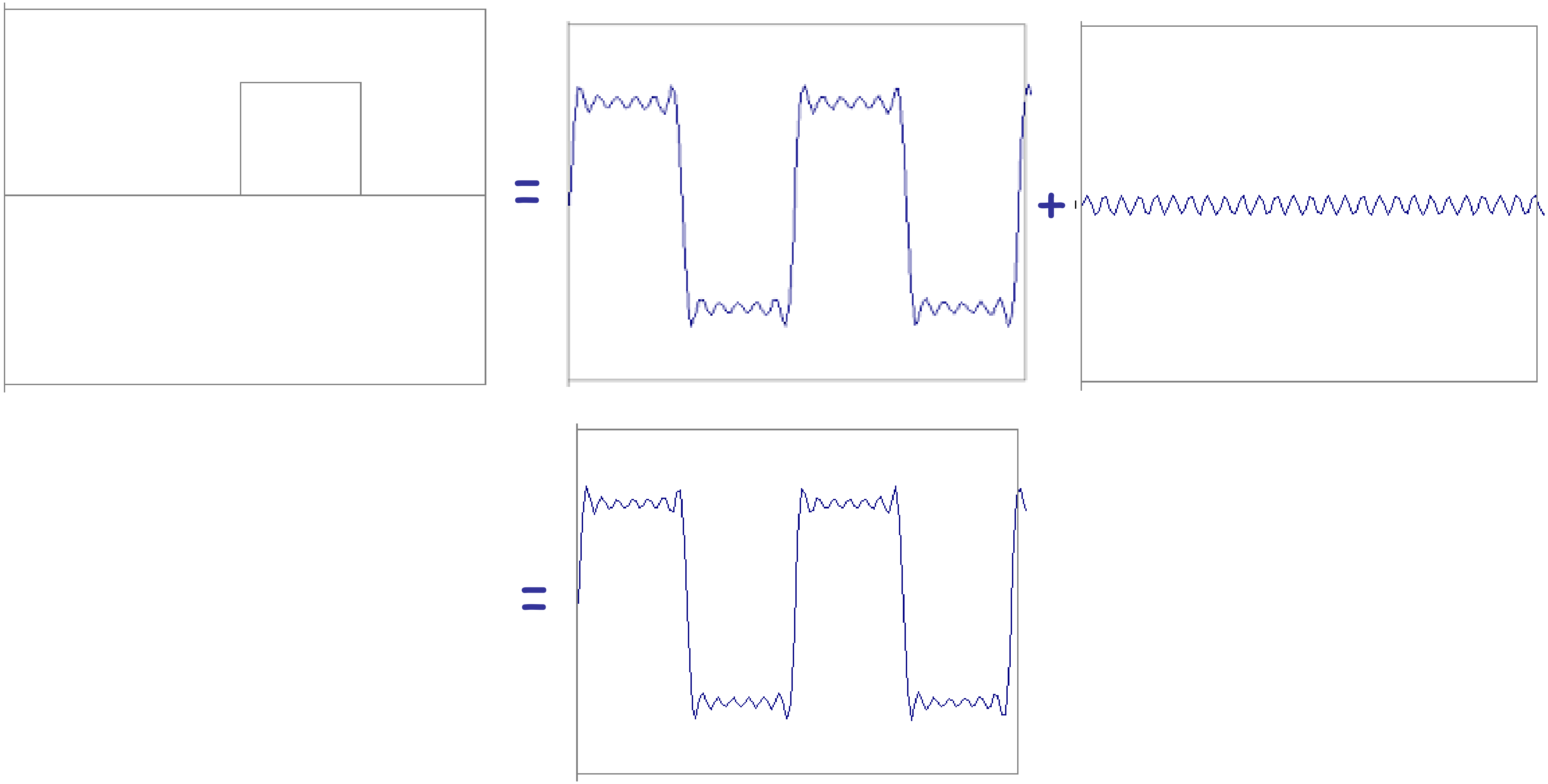
Frequency Spectra



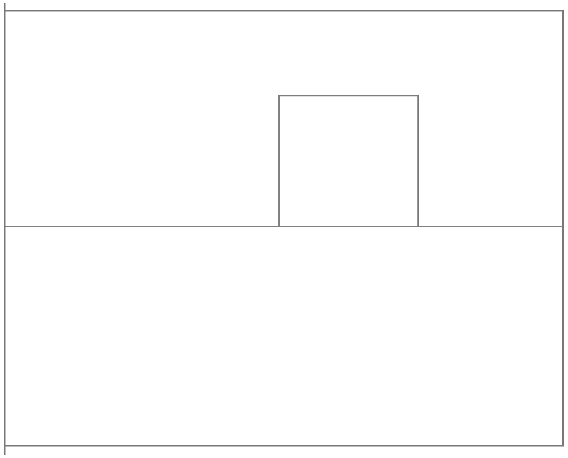
Frequency Spectra



Frequency Spectra

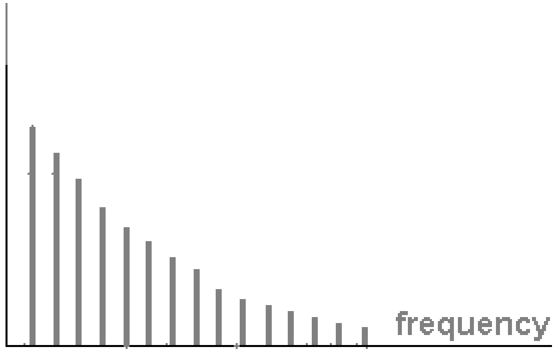


Frequency Spectra



=

$$\boxed{A\sum_{k=1}^{\infty}\frac{1}{k}\sin(2\pi kt)}$$



Fourier Transform – more formally

Represent the signal as an infinite weighted sum of an infinite number of sinusoids

$$F(u) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi ux} dx$$

Note: $e^{ik} = \cos k + i \sin k \qquad i = \sqrt{-1}$

Arbitrary function $\longrightarrow$ Single Analytic Expression

Spatial Domain (*x*) $\longrightarrow$ Frequency Domain (*u*)

(Frequency Spectrum *F(u)*)

Inverse Fourier Transform (IFT) $$f(x) = \int_{-\infty}^{\infty} F(u) e^{i2\pi ux} dx$$

Fourier Transform

- Also, defined as:

$$F(u) = \int_{-\infty}^{\infty} f(x) e^{-iux} dx$$

Note: $e^{ik} = \cos k + i \sin k \qquad i = \sqrt{-1}$

- Inverse Fourier Transform (IFT)

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(u) e^{iux} dx$$

Properties of Fourier Transform

	Spatial Domain	Frequency Domain
Linearity	$c_1 f(x) + c_2 g(x)$	$c_1 F(u) + c_2 G(u)$
Scaling	$f(ax)$	$\frac{1}{\lvert a \rvert} F\left(\frac{u}{a}\right)$
Shifting	$f(x - x_0)$	$e^{-i2\pi u x_0} F(u)$
Symmetry	$F(x)$	$f(-u)$
Conjugation	$f^*(x)$	$F^*(-u)$
Convolution	$f(x) * g(x)$	$F(u)G(u)$
Differentiation	$\frac{d^n f(x)}{dx^n}$	$(i2\pi u)^n F(u)$

Topic: Frequency Space

- Fourier Transform
- Frequency Space
- Spatial Convolution

How does this apply to images?

- We have defined the Fourier Transform as

$$F(u)=\int_{-\infty}^{\infty} f(x)e^{-iux}dx$$

- But images are:
  - Discrete.
  - Two-dimensional.

0	3	2	5	4	7	6	9	8
3	0	1	2	3	4	5	6	7
2	1	0	3	2	5	4	7	6
5	2	3	0	1	2	3	4	5
4	3	2	1	0	3	2	5	4
7	4	5	2	3	0	1	2	3
6	5	4	3	2	1	0	3	2
9	6	7	4	5	2	3	0	1
8	7	6	5	4	3	2	1	0

What a computer sees

2D Discrete FT

- In a 2-variable case, the discrete FT pair is:

$$F(u,v) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) \exp[-j2\pi(ux/M + vy/N)]$$

For u=0,1,2,...,M-1 and v=0,1,2,...,N-1

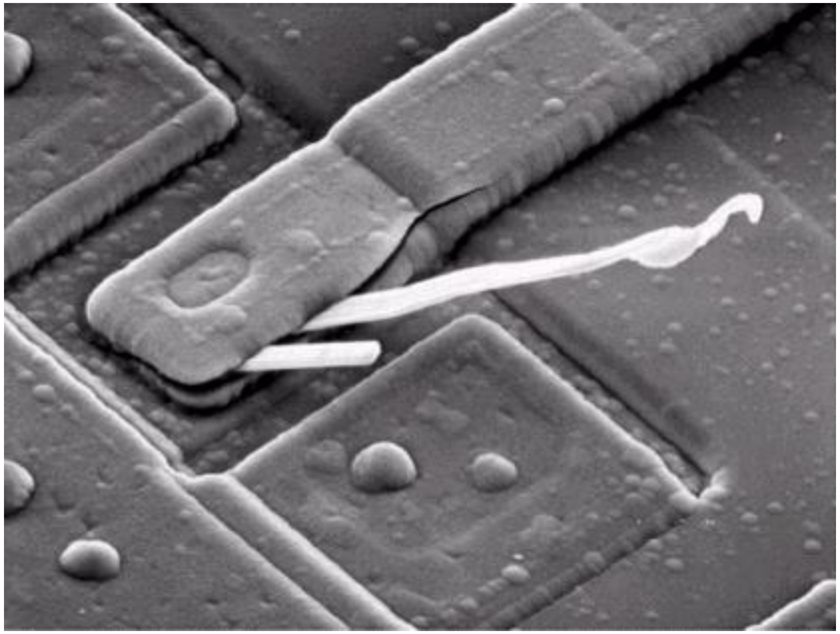
New matrix with the same size!

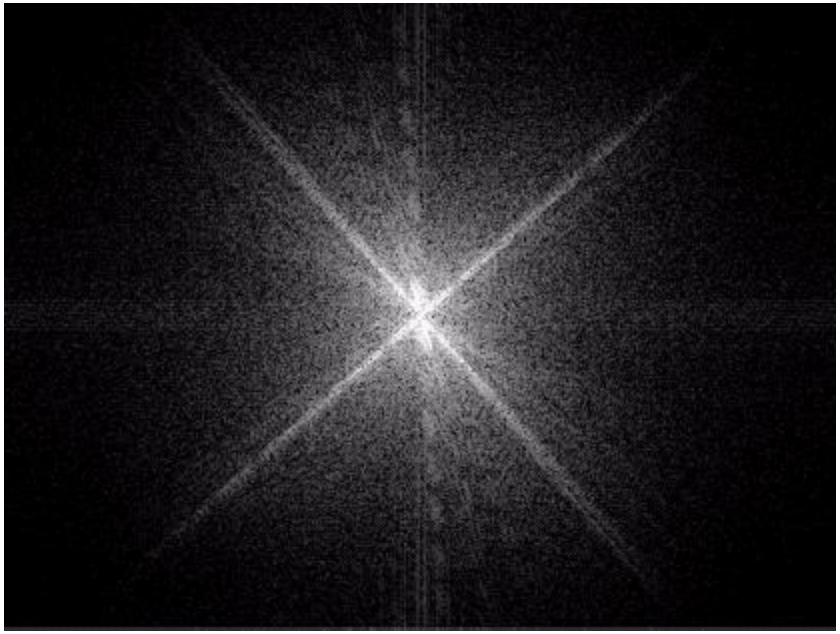
AND: $$f(x,y) = \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u,v) \exp[j2\pi(ux/M + vy/N)]$$

For x=0,1,2,...,M-1 and y=0,1,2,...,N-1

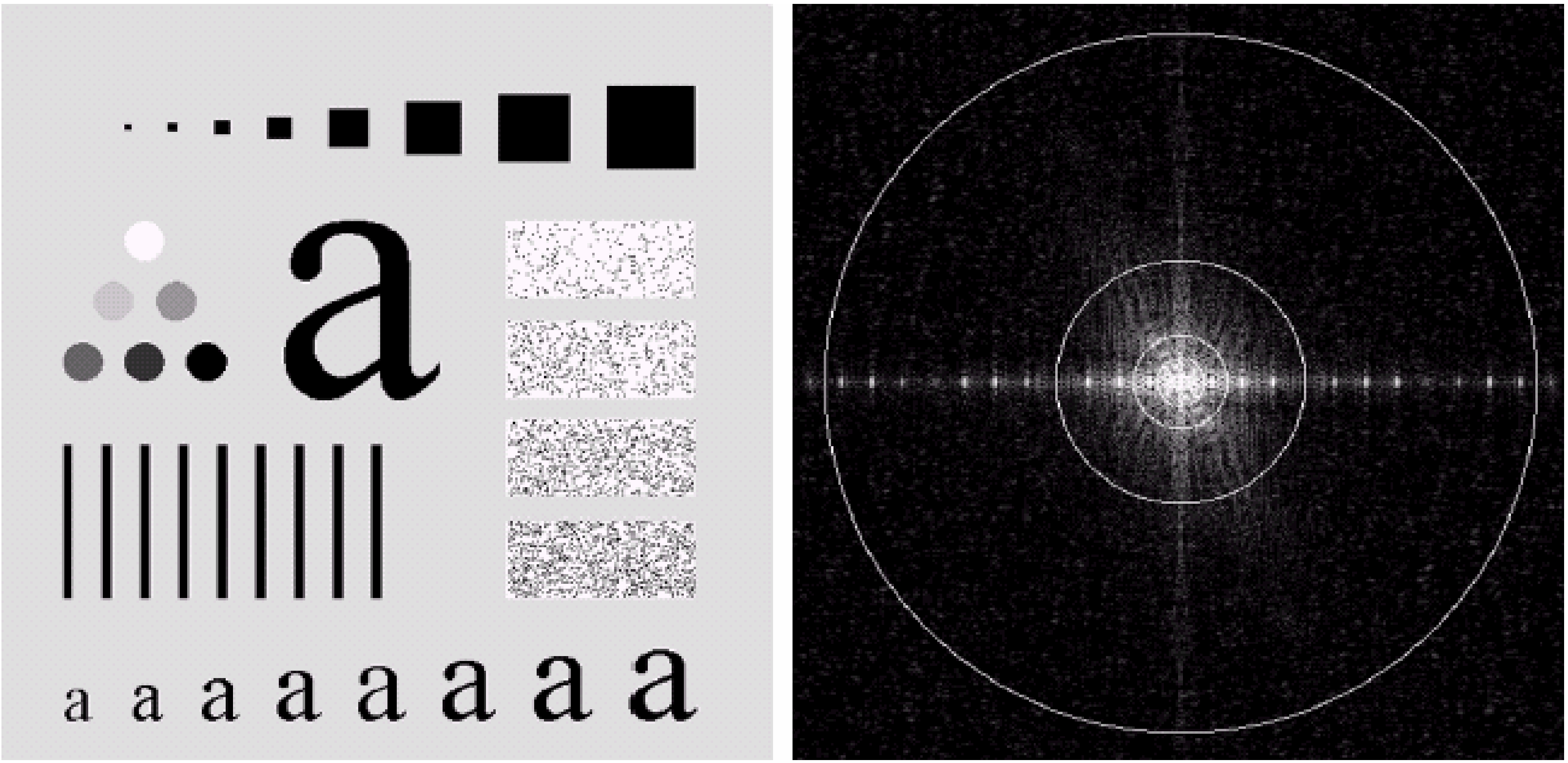
Frequency Space

- Image Space
  - f(x,y)
  - Intuitive
- Frequency Space
  - F(u,v)
  - What does this mean?



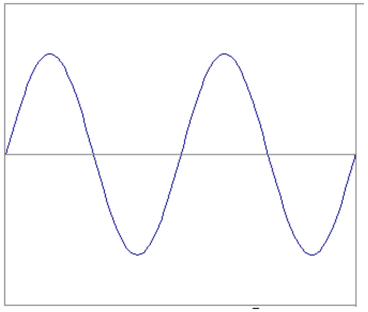


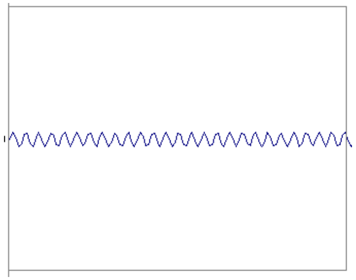
Power distribution



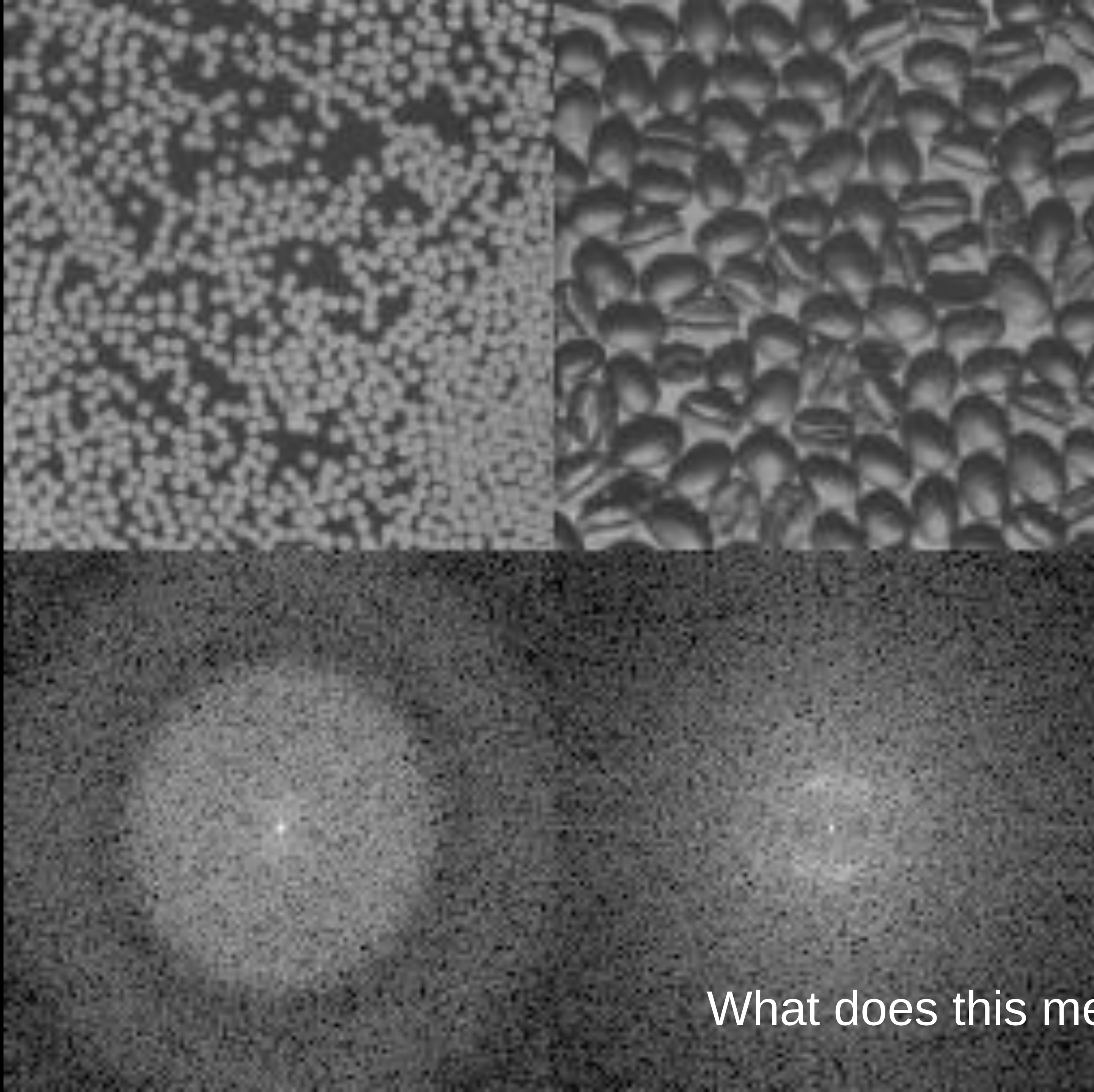
An image (500x500 pixels) and its Fourier spectrum. The super-imposed circles have radii values of 5, 15, 30, 80, and 230, which respectively enclose 92.0, 94.6, 96.4, 98.0, and 99.5% of the image power.

Power distribution

- Most power is in low frequencies.
- Means we are using more of this: 

  And less of this: 

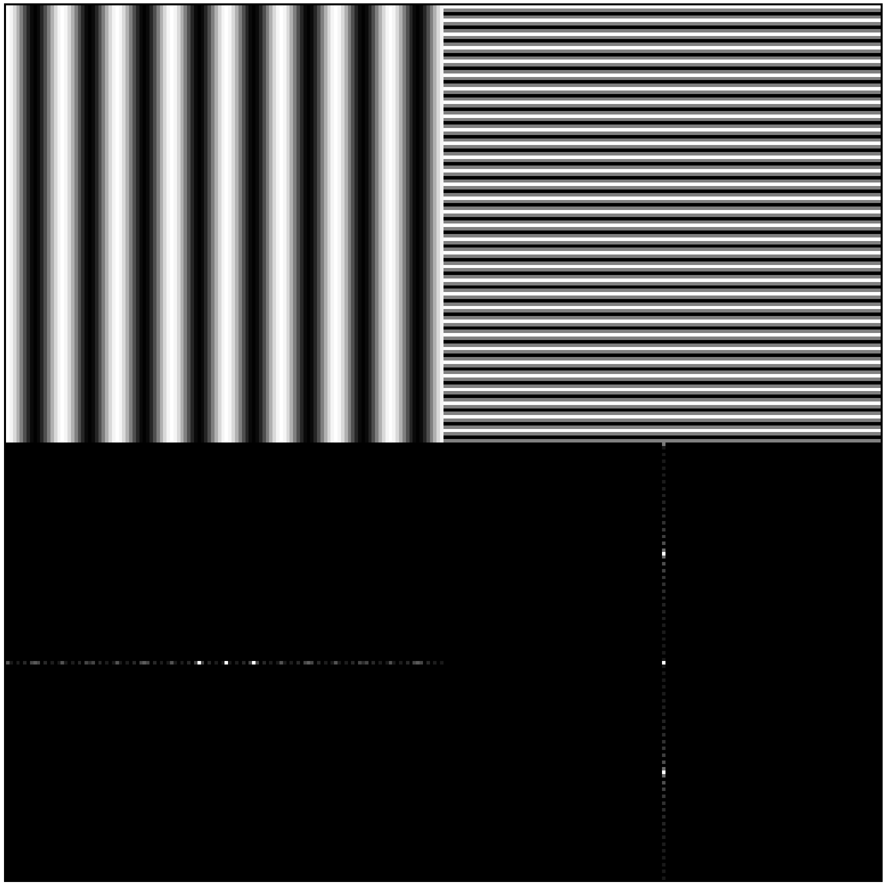
  To represent our signal.
- Why?

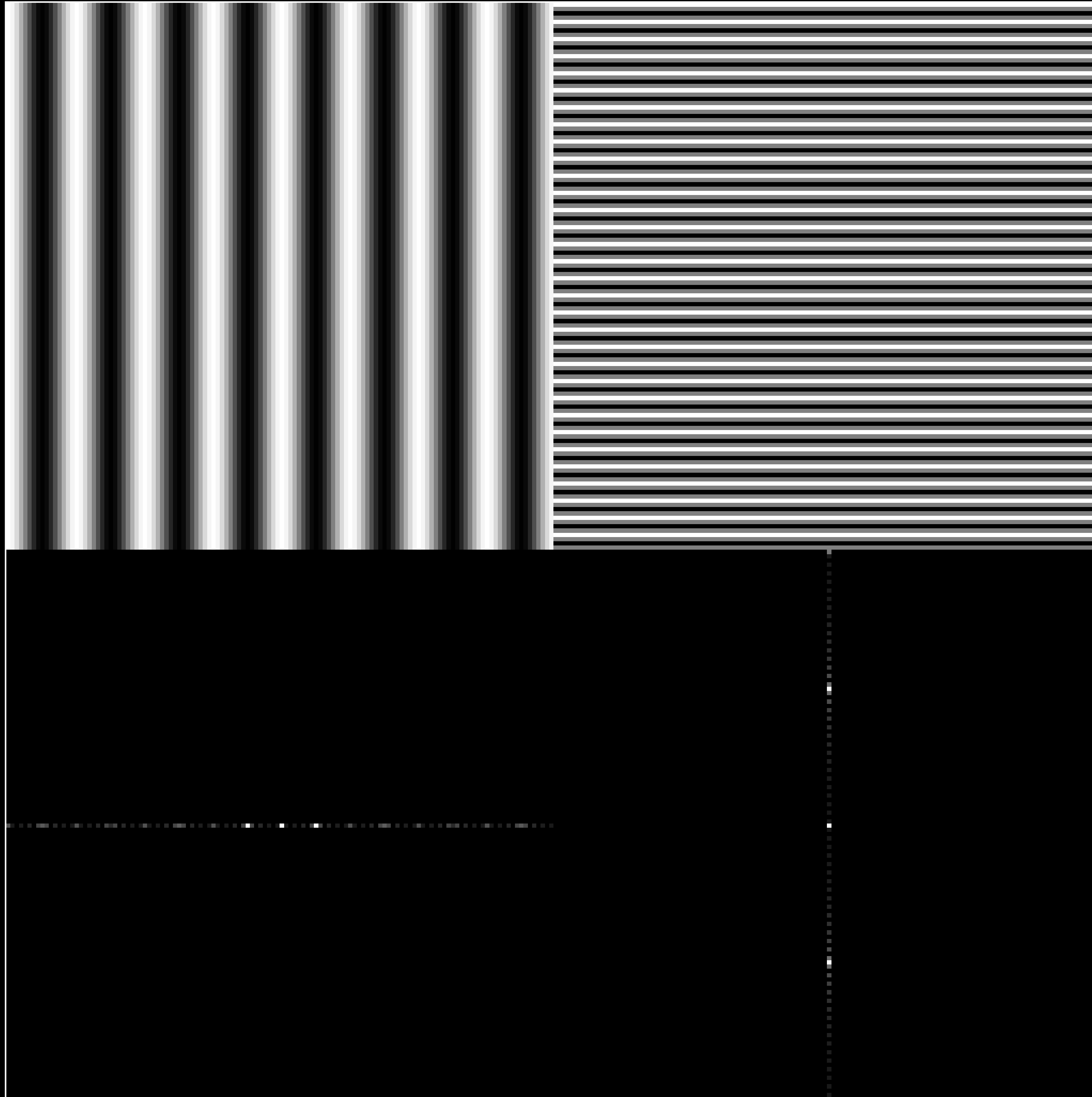


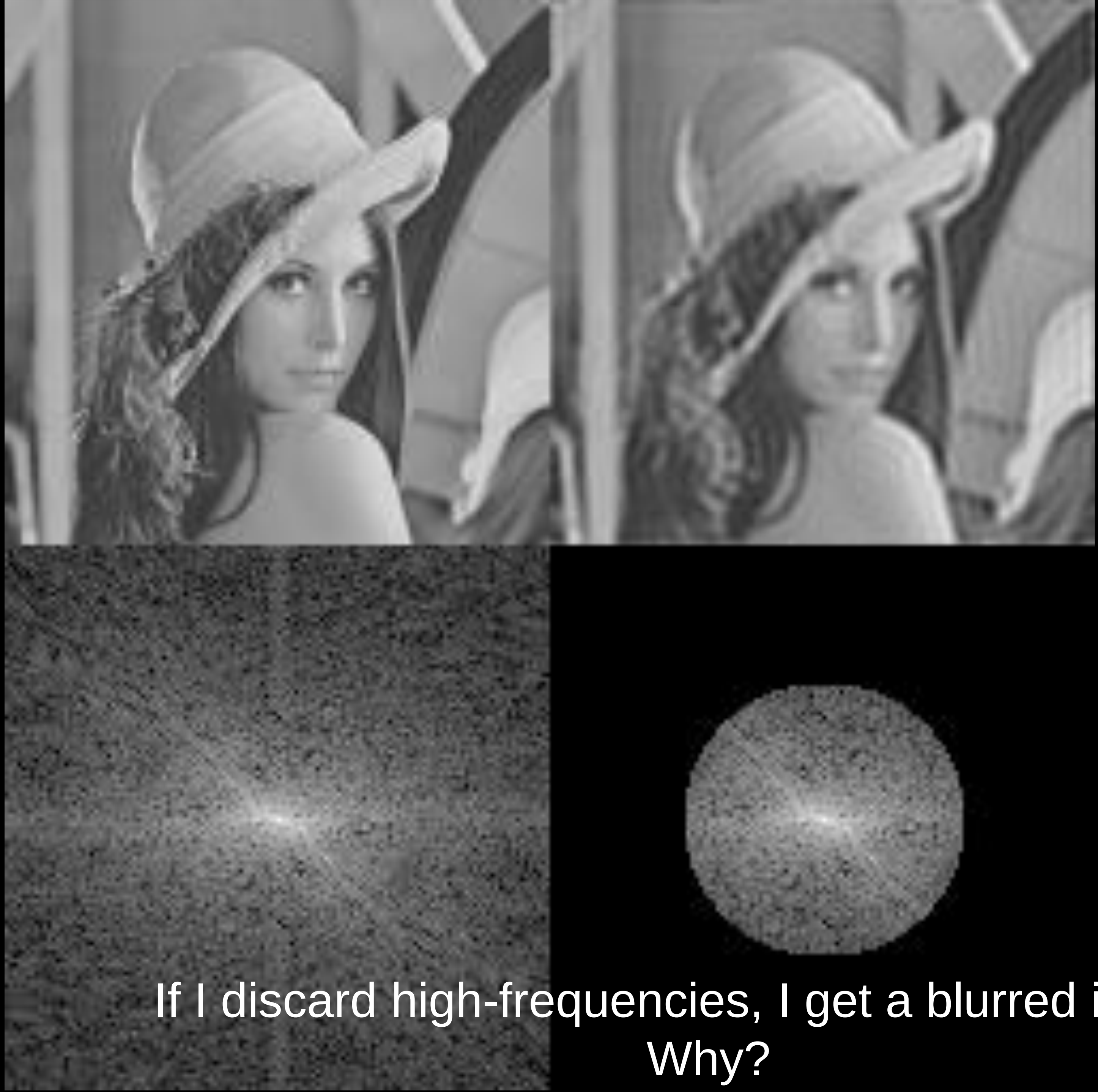
What does this mean??

Horizontal and Vertical Frequency

- Frequencies:
  - Horizontal frequencies correspond to horizontal gradients.
  - Vertical frequencies correspond to vertical gradients.
- What about diagonal lines?







If I discard high-frequencies, I get a blurred image...
Why?

Why bother with FT?

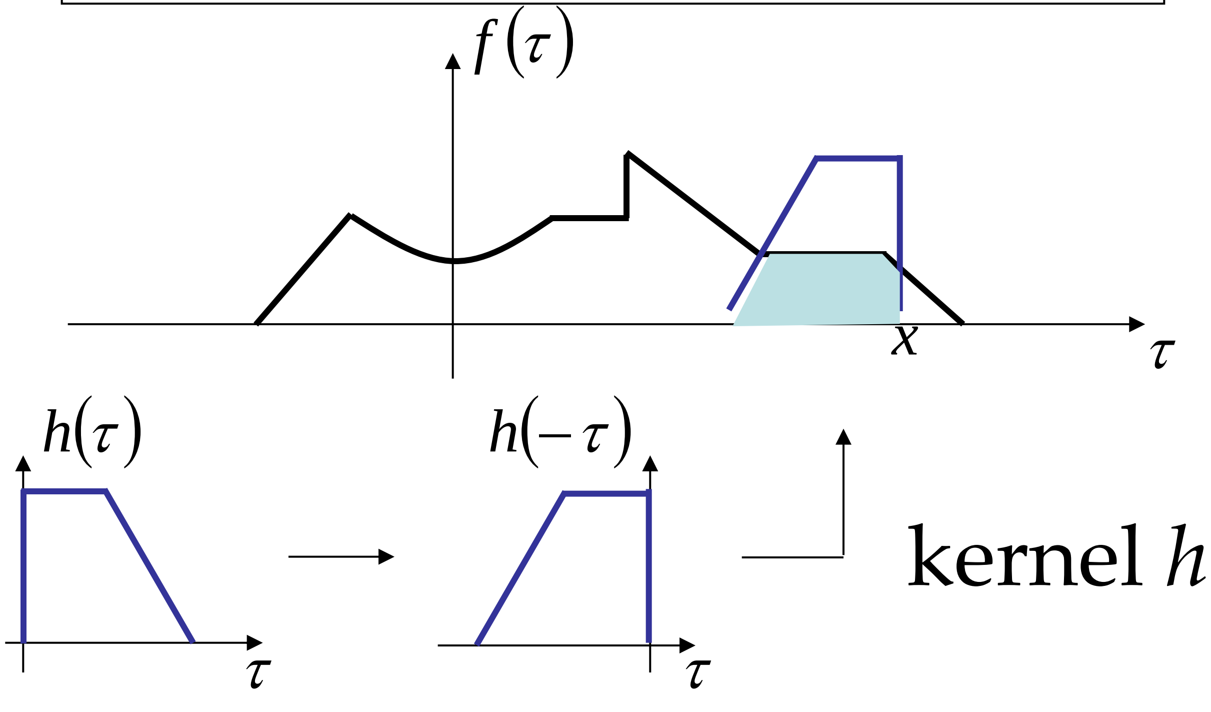
- Great for filtering.
- Great for compression.
- In some situations: Much faster than operating in the spatial domain.
- Convolutions are simple multiplications in Frequency space!
- ...

Topic: Spatial Convolution

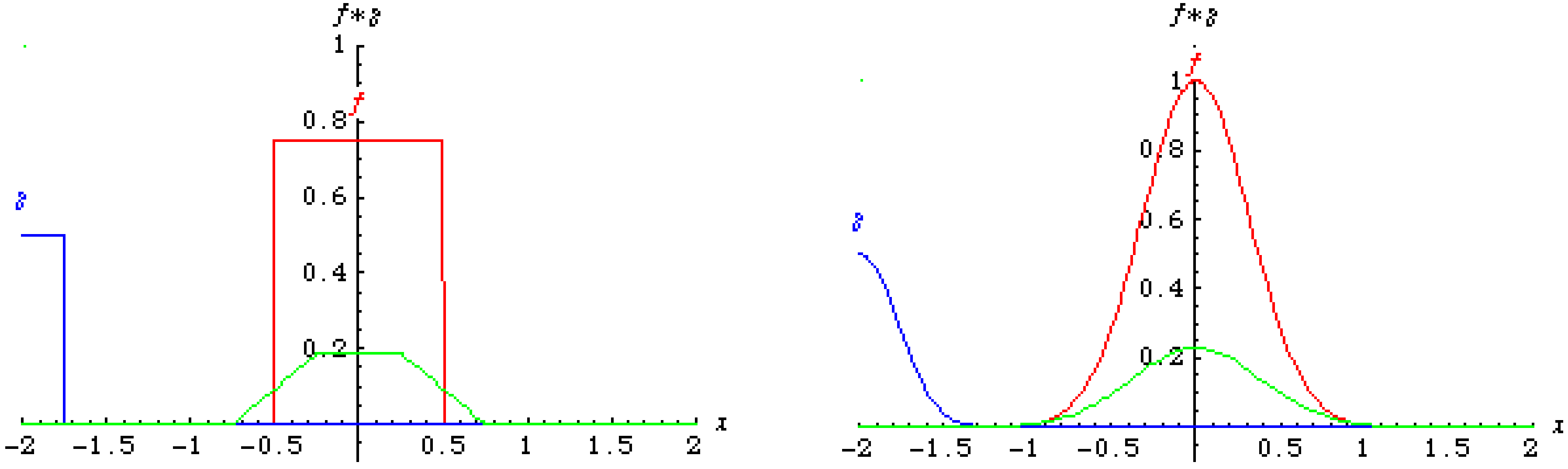
- Fourier Transform
- Frequency Space
- Spatial Convolution

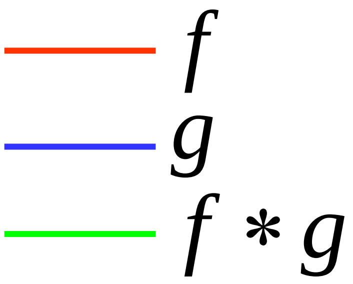
Convolution

$$g(x) = \int_{-\infty}^{\infty} f(\tau) h(x-\tau) d\tau \qquad g = f * h$$



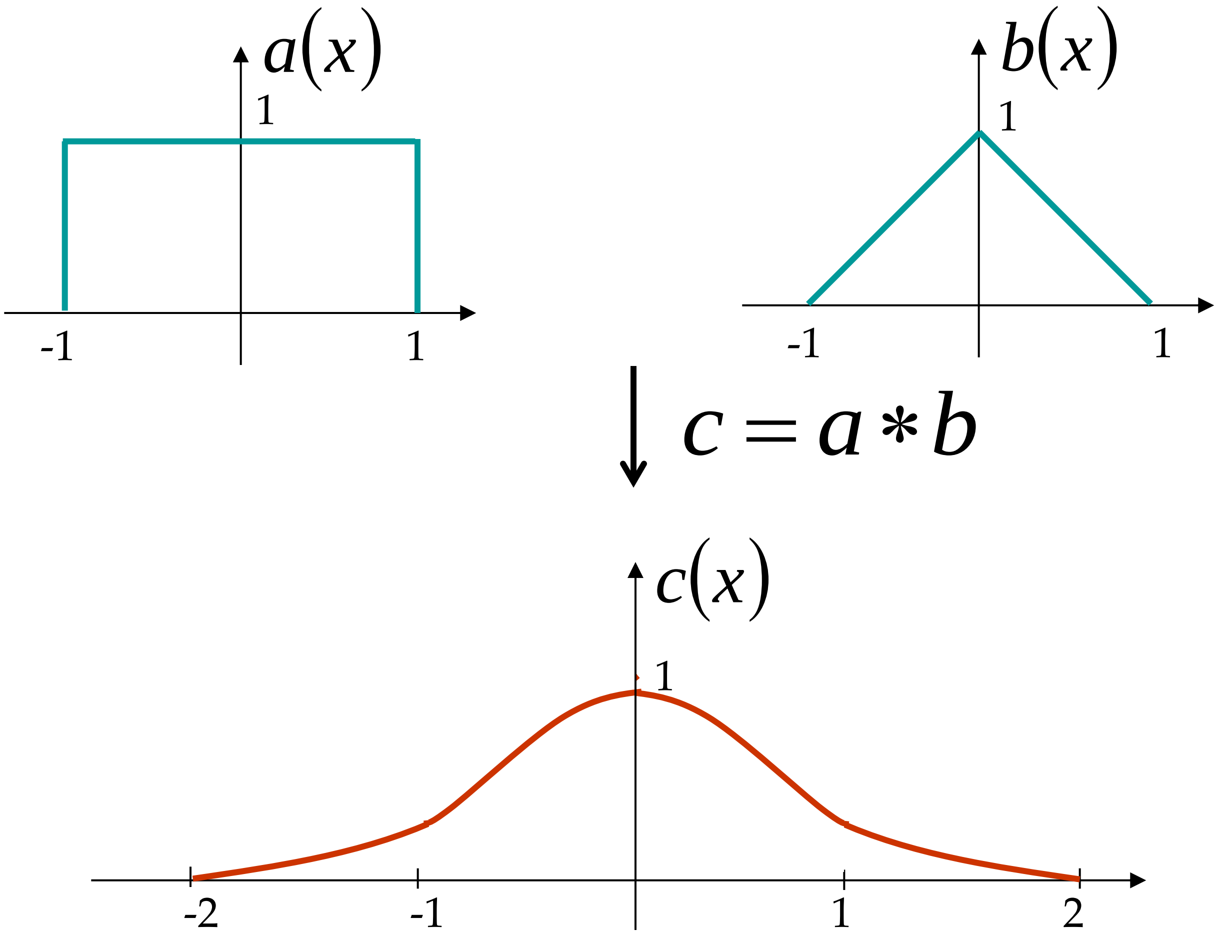
Convolution - Example



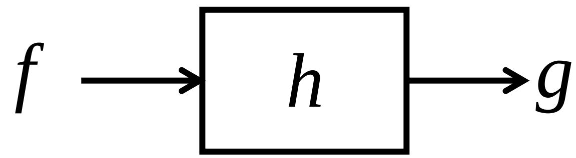


Eric Weinstein’s Math World

Convolution - Example



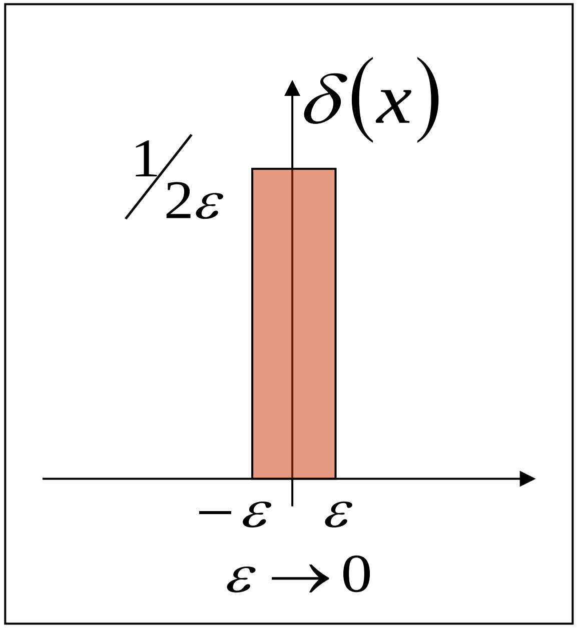
Convolution Kernel – Impulse Response



$$g = f * h$$

- What *h* will give us *g* = *f* ?

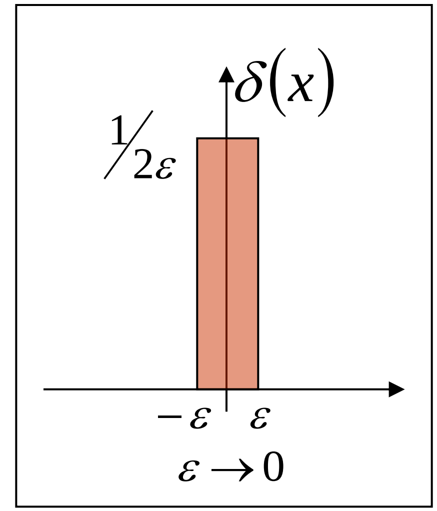
  Dirac Delta Function (Unit Impulse)



Point Spread Function

- Ideally, the optical system should be a Dirac delta function.

scene → Optical System → image

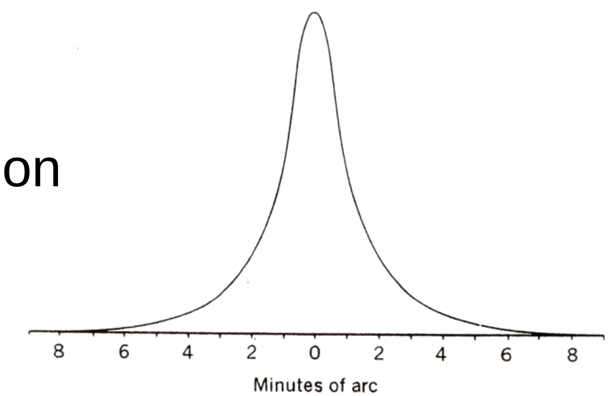


- However, optical systems are never ideal.

$\delta(x)$ → Optical System → $PSF(x)$

point source → point spread function

- Point spread function of Human Eyes.



Point Spread Function

normal vision

myopia

hyperopia

Images by Richmond Eye Associates

Properties of Convolution

- Commutative

$$a * b = b * a$$

- Associative

$$(a * b) * c = a * (b * c)$$

- Cascade system

$$f \longrightarrow \boxed{h_1} \longrightarrow \boxed{h_2} \longrightarrow g$$

$$= f \longrightarrow \boxed{h_1 * h_2} \longrightarrow g$$

$$= f \longrightarrow \boxed{h_2 * h_1} \longrightarrow g$$

Fourier Transform and Convolution

Let $g = f * h$ Then $G(u) = \int_{-\infty}^{\infty} g(x) e^{-i2\pi ux} dx$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\tau) h(x-\tau) e^{-i2\pi ux} d\tau dx$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[f(\tau) e^{-i2\pi u\tau} d\tau \right] \left[h(x-\tau) e^{-i2\pi u(x-\tau)} dx \right]$$

$$= \int_{-\infty}^{\infty} \left[f(\tau) e^{-i2\pi u\tau} d\tau \right] \int_{-\infty}^{\infty} \left[h(x') e^{-i2\pi ux'} dx' \right] \qquad = F(u) H(u)$$

Convolution in spatial domain

$\Leftrightarrow$ Multiplication in frequency domain

Fourier Transform and Convolution

Spatial Domain *(x)*		Frequency Domain *(u)*
$g = f * h$	$\longleftrightarrow$	$G = FH$
$g = fh$	$\longleftrightarrow$	$G = F * H$

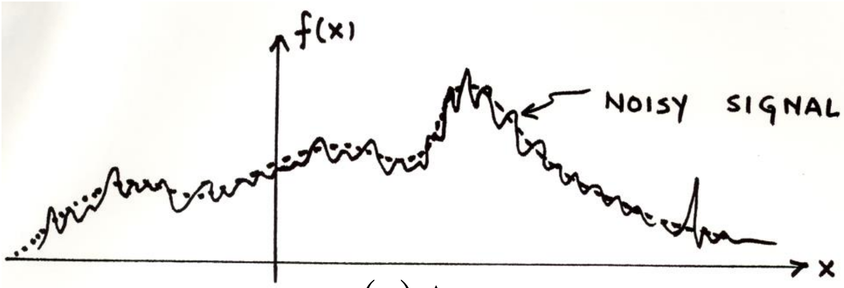
So, we can find *g(x)* by Fourier transform

$$
\begin{array}{ccccc}
g & = & f & * & h \\
\uparrow & & \downarrow & & \downarrow \\
\boxed{\textbf{IFT}} & & \boxed{\textbf{FT}} & & \boxed{\textbf{FT}} \\
| & & \downarrow & & \downarrow \\
G & = & F & \times & H
\end{array}
$$

Example use: Smoothing/Blurring

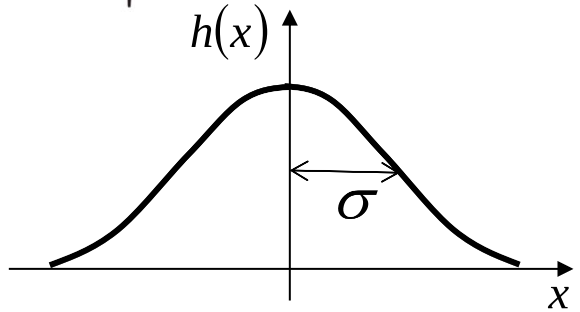
- We want a smoothed function of *f(x)*

$$g(x) = f(x) * h(x)$$



- Let us use a Gaussian kernel

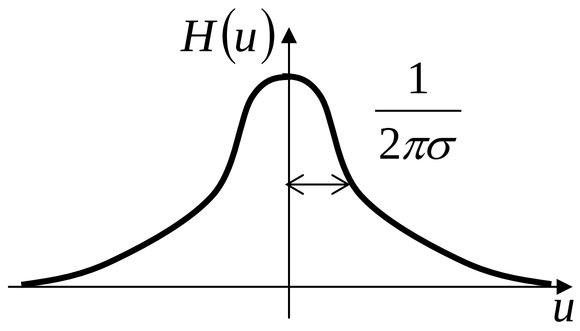
$$h(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2} \frac{x^2}{\sigma^2} \right]$$



- Then

$$H(u) = \exp\left[-\frac{1}{2} (2\pi u)^2 \sigma^2 \right]$$

$$G(u) = F(u) H(u)$$



Resources

- Russ – Chapter 6
- Gonzalez & Woods – Chapter 4