VC 19/20 – TP6
Frequency Space

Mestrado em Ciência de Computadores

Mestrado Integrado em Engenharia de Redes e Sistemas Informáticos

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Outline

- Fourier Transform
- Frequency Space
- Spatial Convolution

Topic: Fourier Transform

- Fourier Transform
- Frequency Space
- Spatial Convolution

How to Represent Signals?

- Option 1: Taylor series represents any function using polynomials.

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \ldots.$$

- Polynomials are not the best - unstable and not very physically meaningful.

- Easier to talk about “signals” in terms of its “frequencies” (how fast/often signals change, etc).

Jean Baptiste Joseph Fourier (1768-1830)

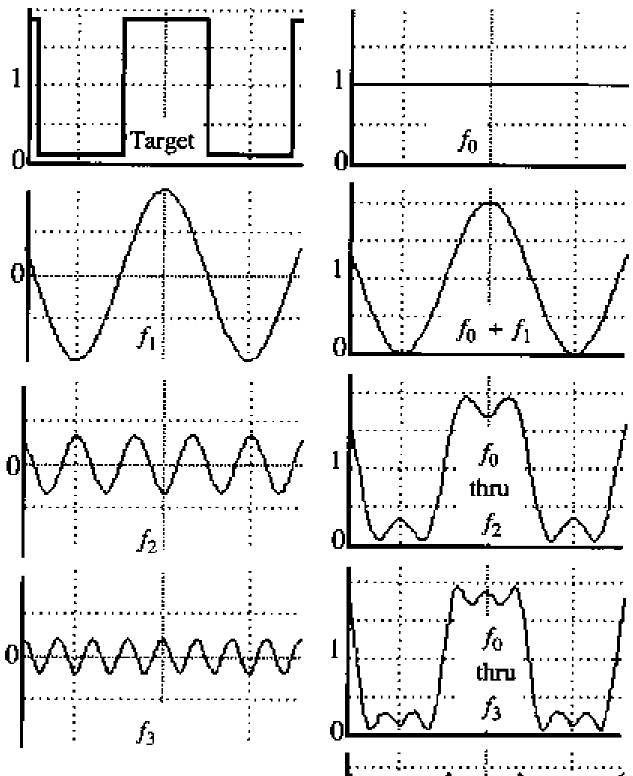
- Had a crazy idea (1807):
- **Any** periodic function can be rewritten as a weighted sum of Sines and Cosines of different frequencies.
- Don't believe it?
  - Neither did Lagrange, Laplace, Poisson and other big wigs
  - Not translated into English until 1878!
- But it's true!
  - called Fourier Series
  - Possibly the greatest tool used in Engineering

A Sum of Sinusoids

- Our building block:

$$A\sin(\omega x + \phi)$$

- Add enough of them to get any signal *f(x)* you want!
- How many degrees of freedom?
- What does each control?
- Which one encodes the coarse vs. fine structure of the signal?



f(target)=
$f_1 + f_2 + f_3 ... + f_n + ...$

Fourier Transform

- We want to understand the frequency ω of our signal. So, let's reparametrize the signal by ω instead of *x*:

$$f(x) \longrightarrow \boxed{\text{Fourier Transform}} \longrightarrow F(\omega)$$

- For every ω from 0 to inf, ***F(ω)*** holds the amplitude *A* and phase ϕ of the corresponding sine
  - How can *F* hold both? Complex number trick!

$$A\sin(\omega x + \phi)$$

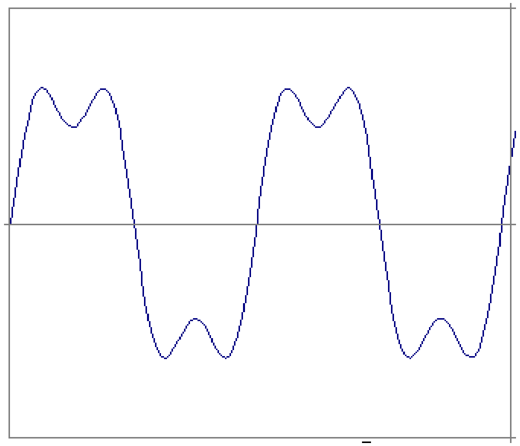
$$F(\omega) = R(\omega) + iI(\omega)$$

$$\phi = \tan^{-1}\frac{I(\omega)}{R(\omega)}$$

$$A = \pm\sqrt{R(\omega)^2 + I(\omega)^2}$$

Time and Frequency

- example : $g(t) = \sin(2pf\, t) + (1/3)\sin(2p(3f)\, t)$



Time and Frequency

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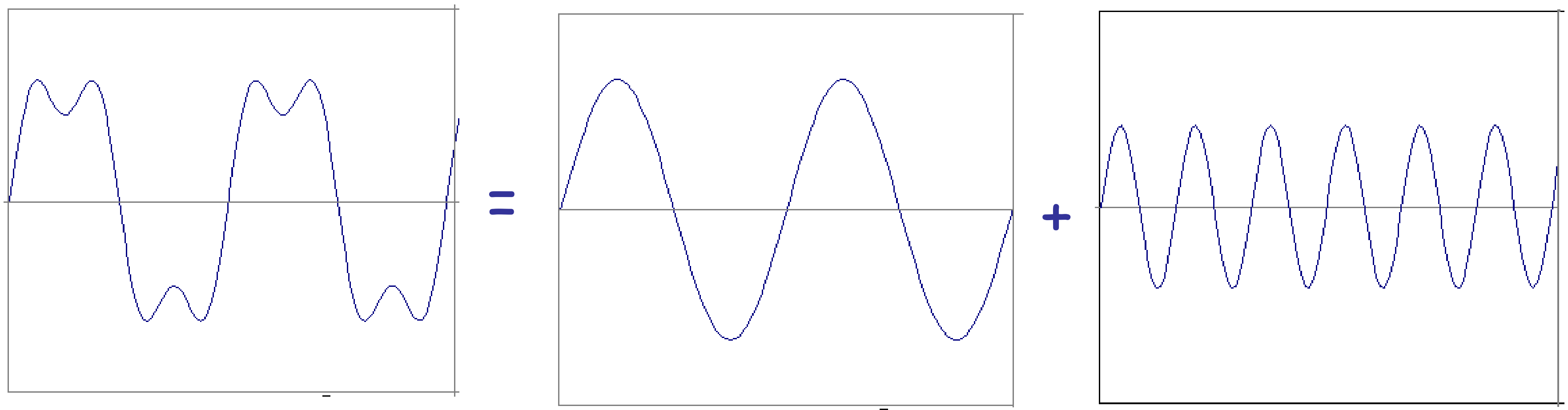
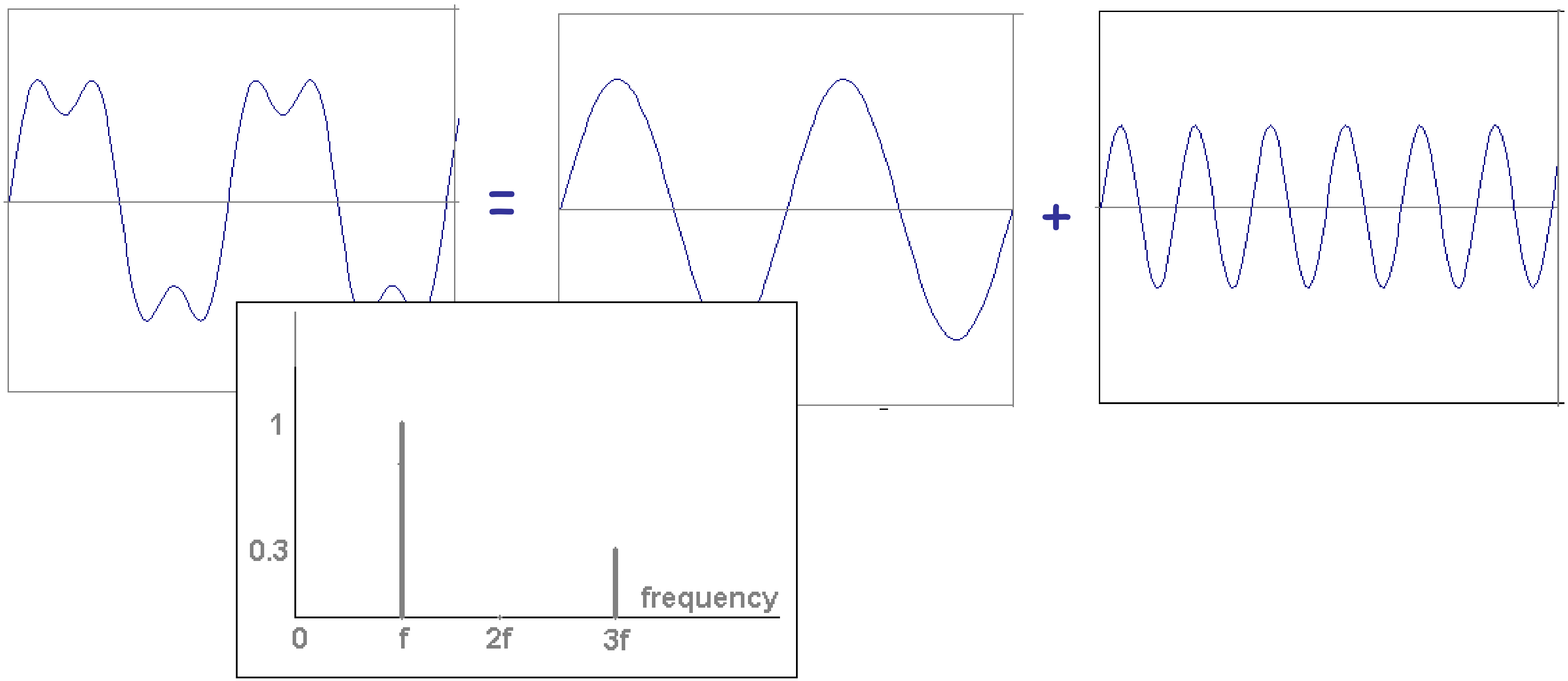


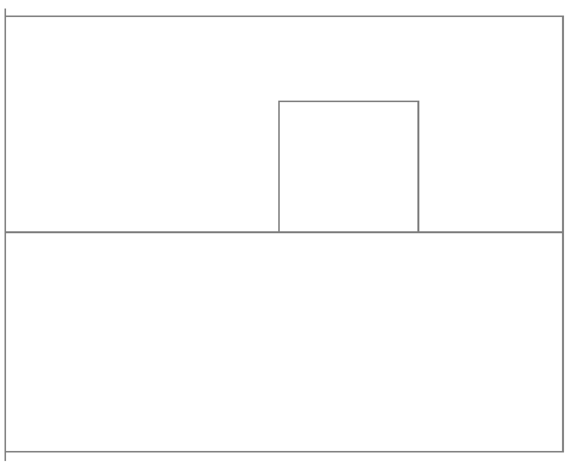
Frequency Spectra

- example : $g(t) = \sin(2pf\,t) + (1/3)\sin(2p(3f)\,t)$

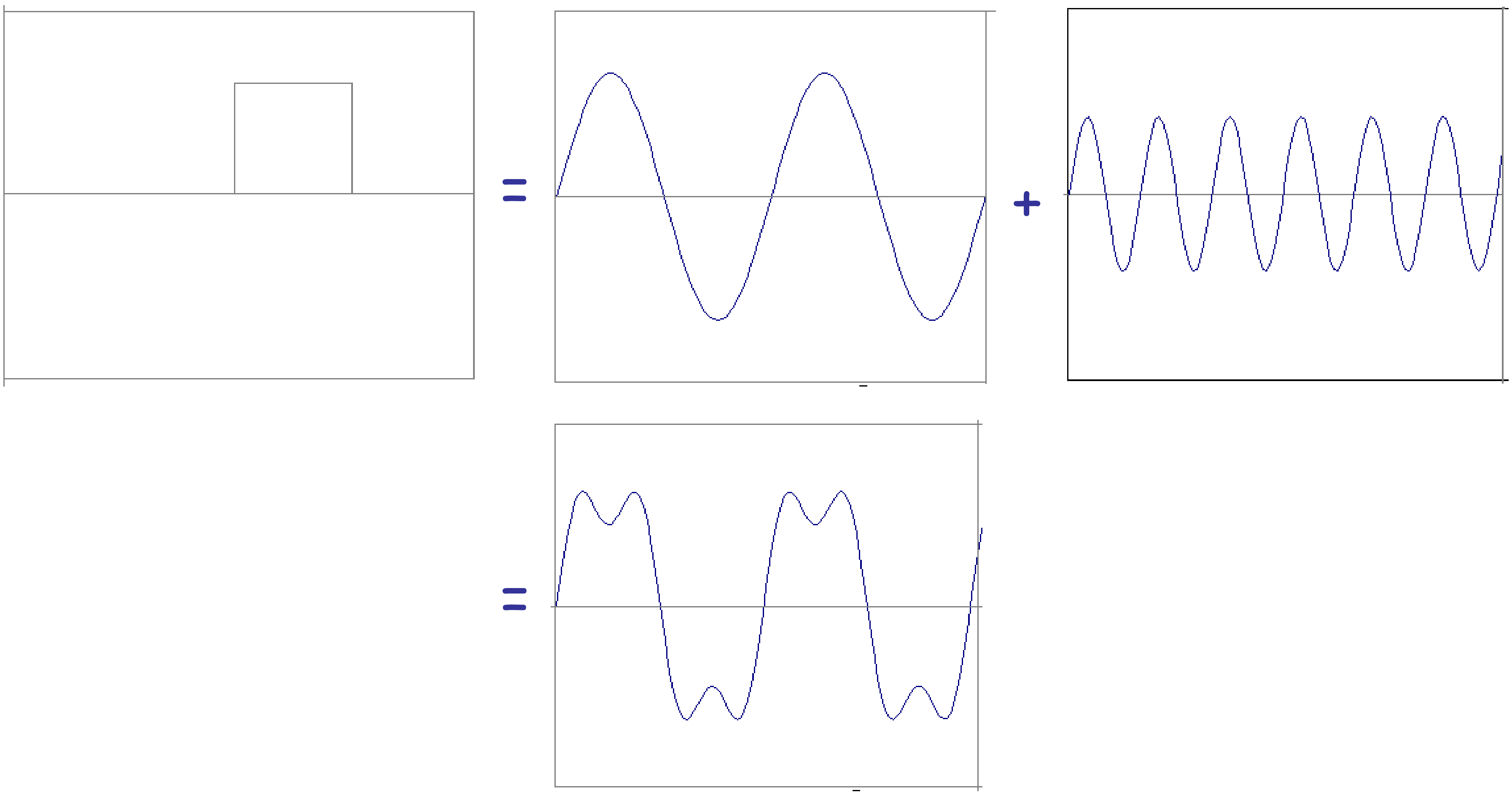


Frequency Spectra

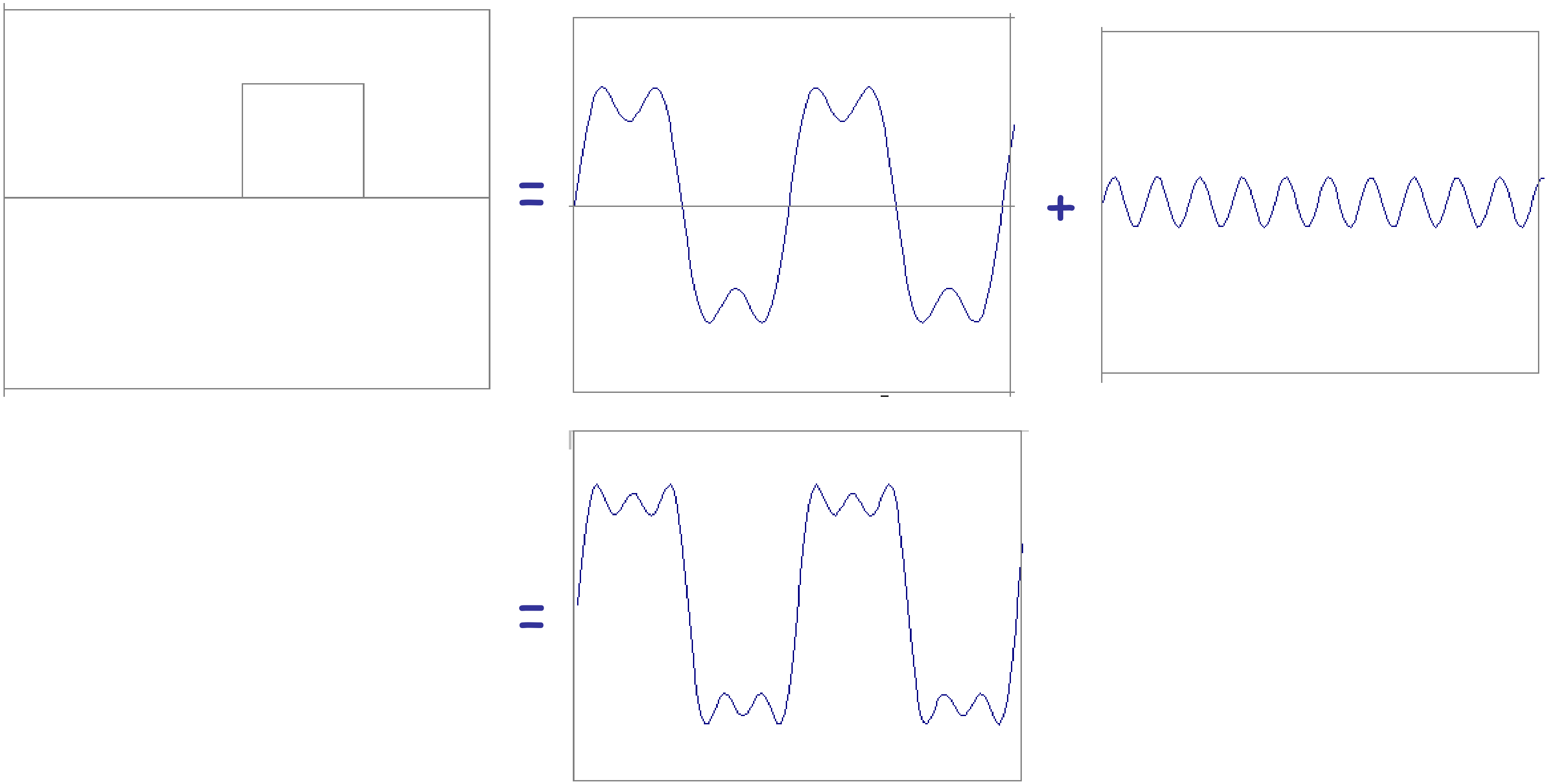
- Usually, frequency is more interesting than the phase



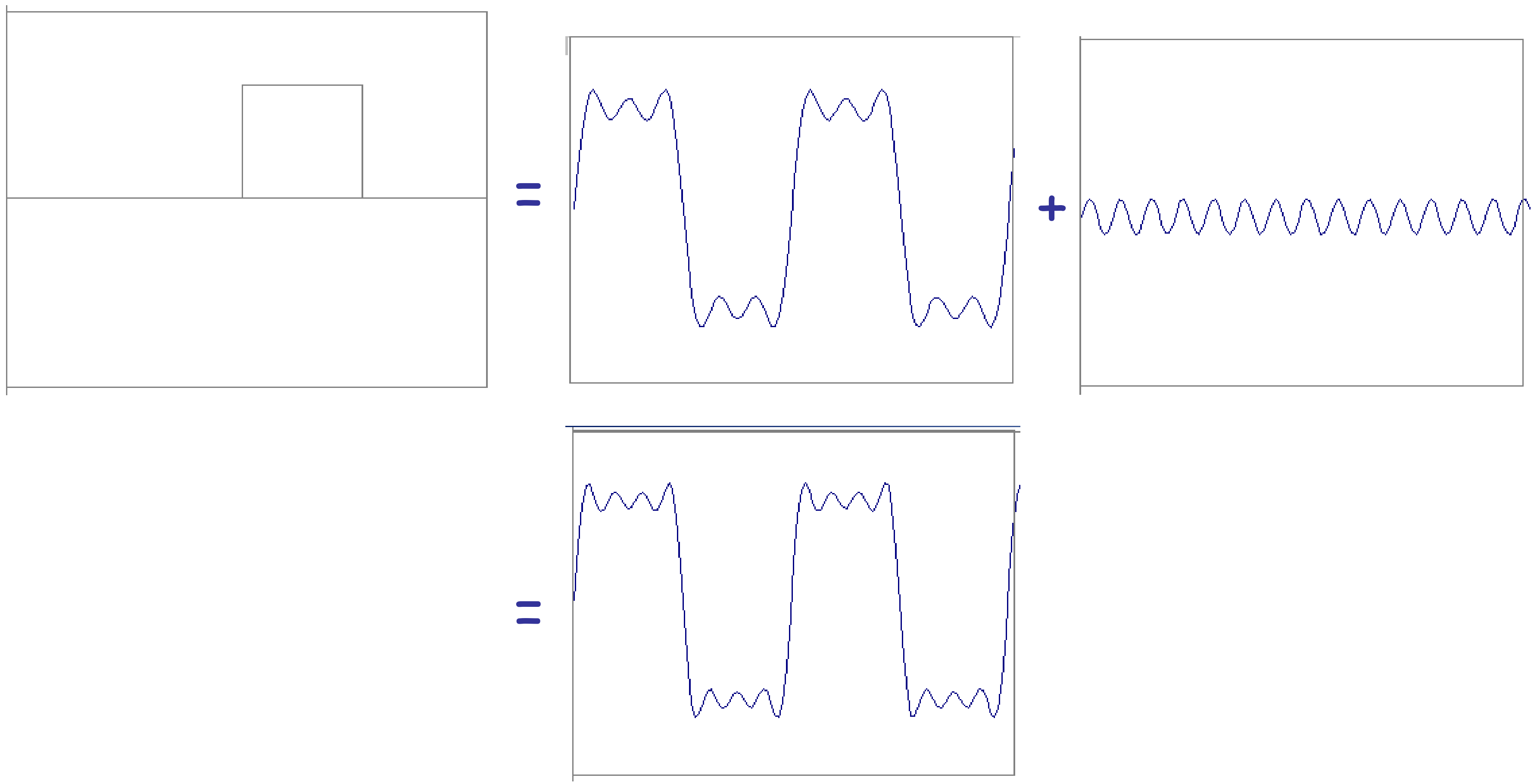
Frequency Spectra



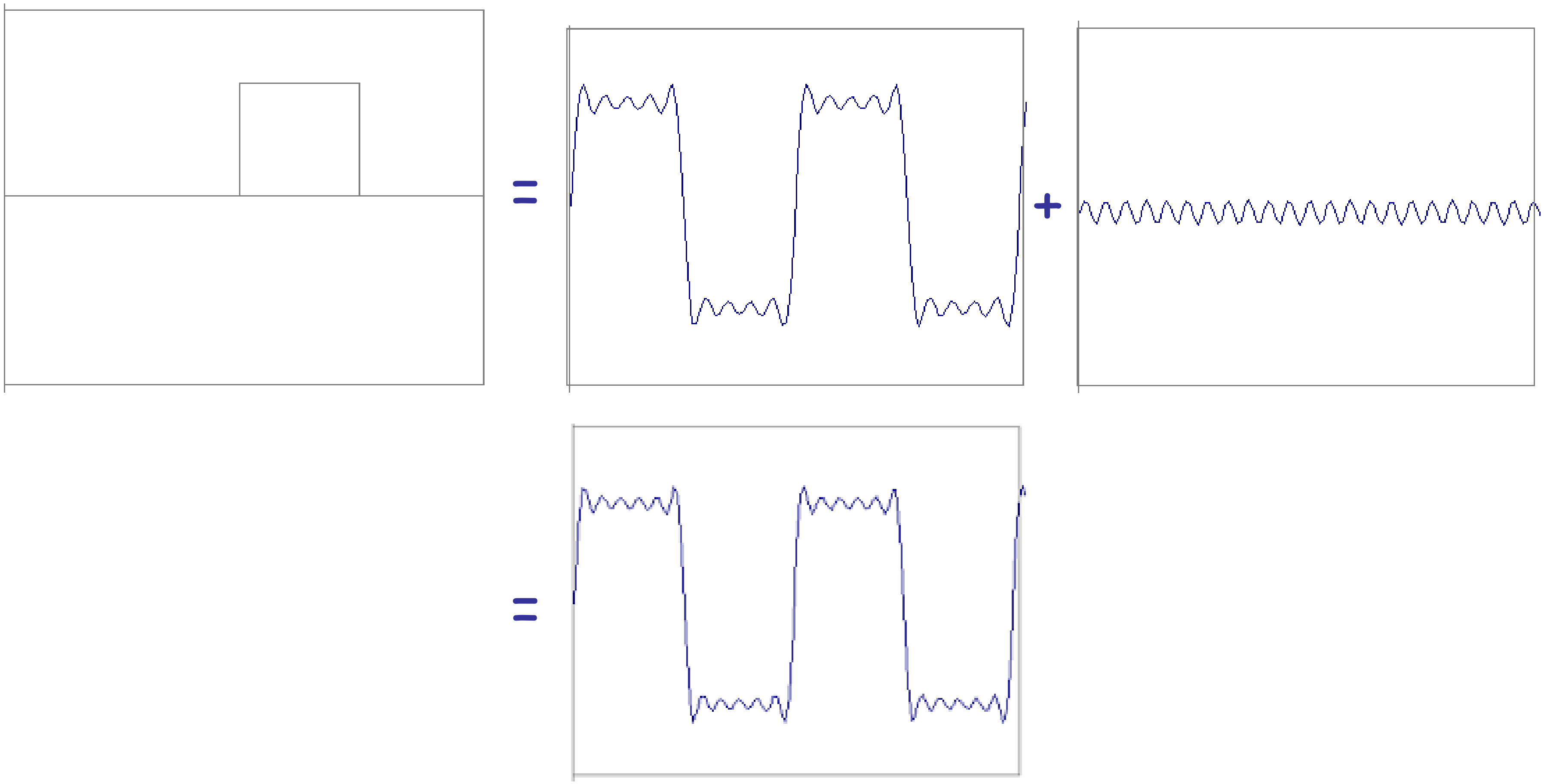
Frequency Spectra



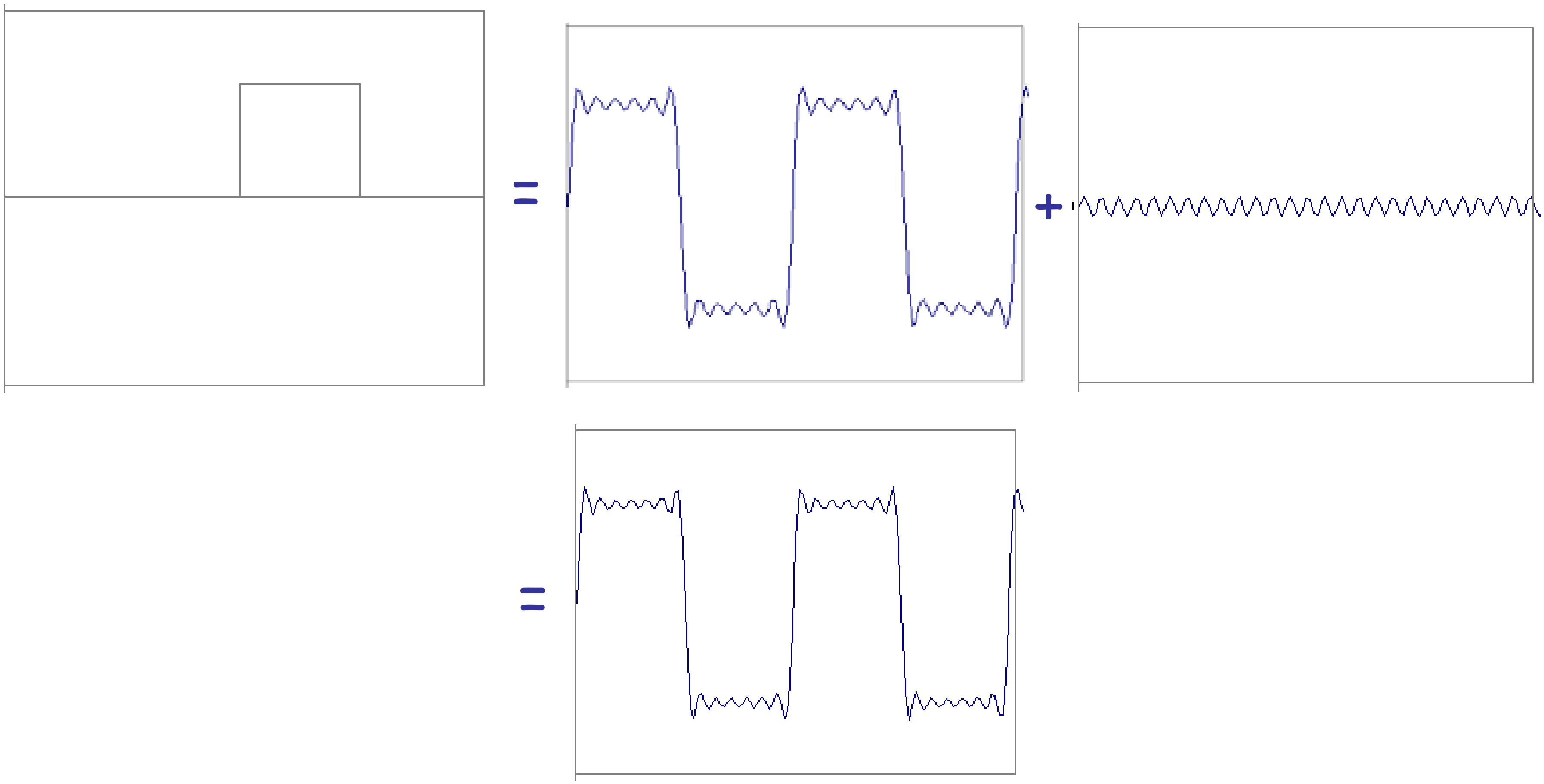
Frequency Spectra



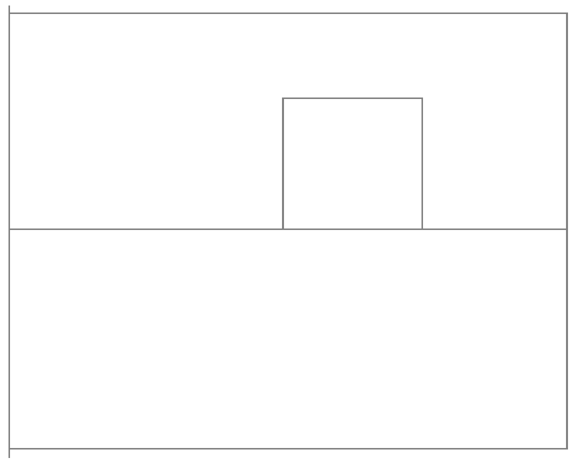
Frequency Spectra



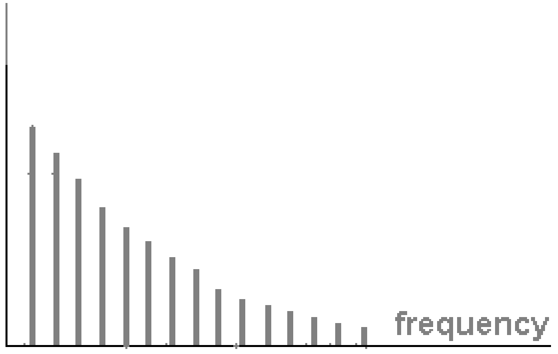
Frequency Spectra



Frequency Spectra



$$= \boxed{A\sum_{k=1}^{\infty}\frac{1}{k}\sin(2\pi kt)}$$



Fourier Transform – more formally

Represent the signal as an infinite weighted sum of an infinite number of sinusoids

$$F(u) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi ux} dx$$

Note: $e^{ik} = \cos k + i \sin k \qquad i = \sqrt{-1}$

Arbitrary function $\longrightarrow$ Single Analytic Expression

Spatial Domain (*x*) $\longrightarrow$ Frequency Domain (*u*)

(Frequency Spectrum *F(u)*)

Inverse Fourier Transform (IFT) $$f(x) = \int_{-\infty}^{\infty} F(u) e^{i2\pi ux} dx$$

Fourier Transform

- Also, defined as:

$$F(u) = \int_{-\infty}^{\infty} f(x) e^{-iux} dx$$

Note: $e^{ik} = \cos k + i \sin k \qquad i = \sqrt{-1}$

- Inverse Fourier Transform (IFT)

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(u) e^{iux} dx$$

Properties of Fourier Transform

	Spatial Domain	Frequency Domain
Linearity	$c_1 f(x) + c_2 g(x)$	$c_1 F(u) + c_2 G(u)$
Scaling	$f(ax)$	$\frac{1}{\lvert a \rvert} F\left(\frac{u}{a}\right)$
Shifting	$f(x - x_0)$	$e^{-i2\pi u x_0} F(u)$
Symmetry	$F(x)$	$f(-u)$
Conjugation	$f^*(x)$	$F^*(-u)$
Convolution	$f(x) * g(x)$	$F(u)G(u)$
Differentiation	$\frac{d^n f(x)}{dx^n}$	$(i2\pi u)^n F(u)$

Topic: Frequency Space

- Fourier Transform
- Frequency Space
- Spatial Convolution

How does this apply to images?

- We have defined the Fourier Transform as

$$F(u)=\int_{-\infty}^{\infty} f(x)e^{-iux}dx$$

- But images are:
  - Discrete.
  - Two-dimensional.

0	3	2	5	4	7	6	9	8
3	0	1	2	3	4	5	6	7
2	1	0	3	2	5	4	7	6
5	2	3	0	1	2	3	4	5
4	3	2	1	0	3	2	5	4
7	4	5	2	3	0	1	2	3
6	5	4	3	2	1	0	3	2
9	6	7	4	5	2	3	0	1
8	7	6	5	4	3	2	1	0

What a computer sees

2D Discrete FT

- In a 2-variable case, the discrete FT pair is:

$$F(u,v)=\frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)\exp[-j2\pi(ux/M+vy/N)]$$

For u=0,1,2,…,M-1 and v=0,1,2,…,N-1

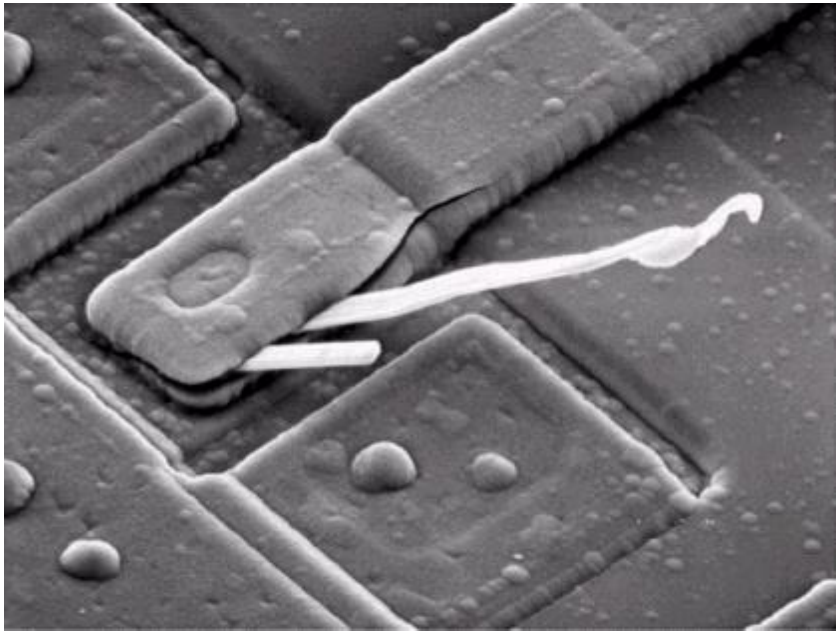
New matrix with the same size!

AND: $$f(x,y)=\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)\exp[j2\pi(ux/M+vy/N)]$$

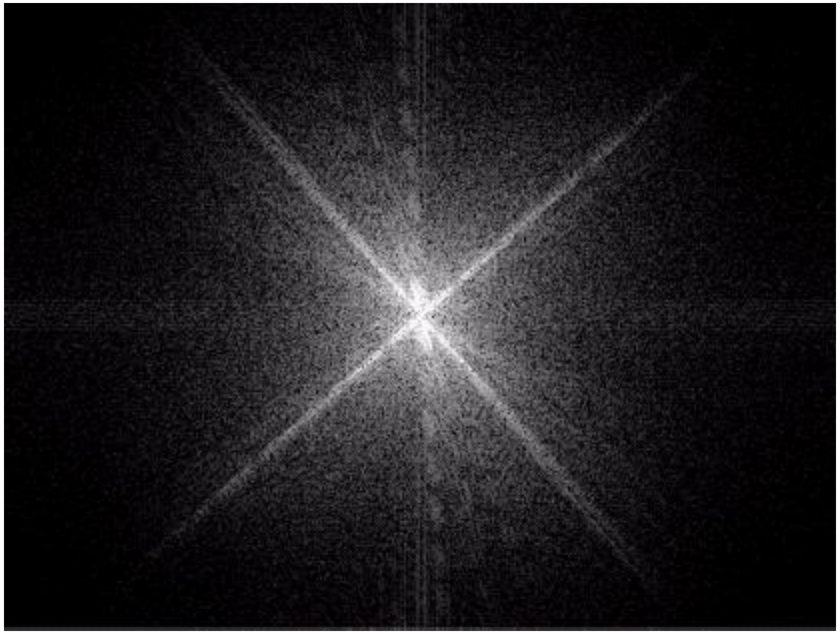
For x=0,1,2,…,M-1 and y=0,1,2,…,N-1

Frequency Space

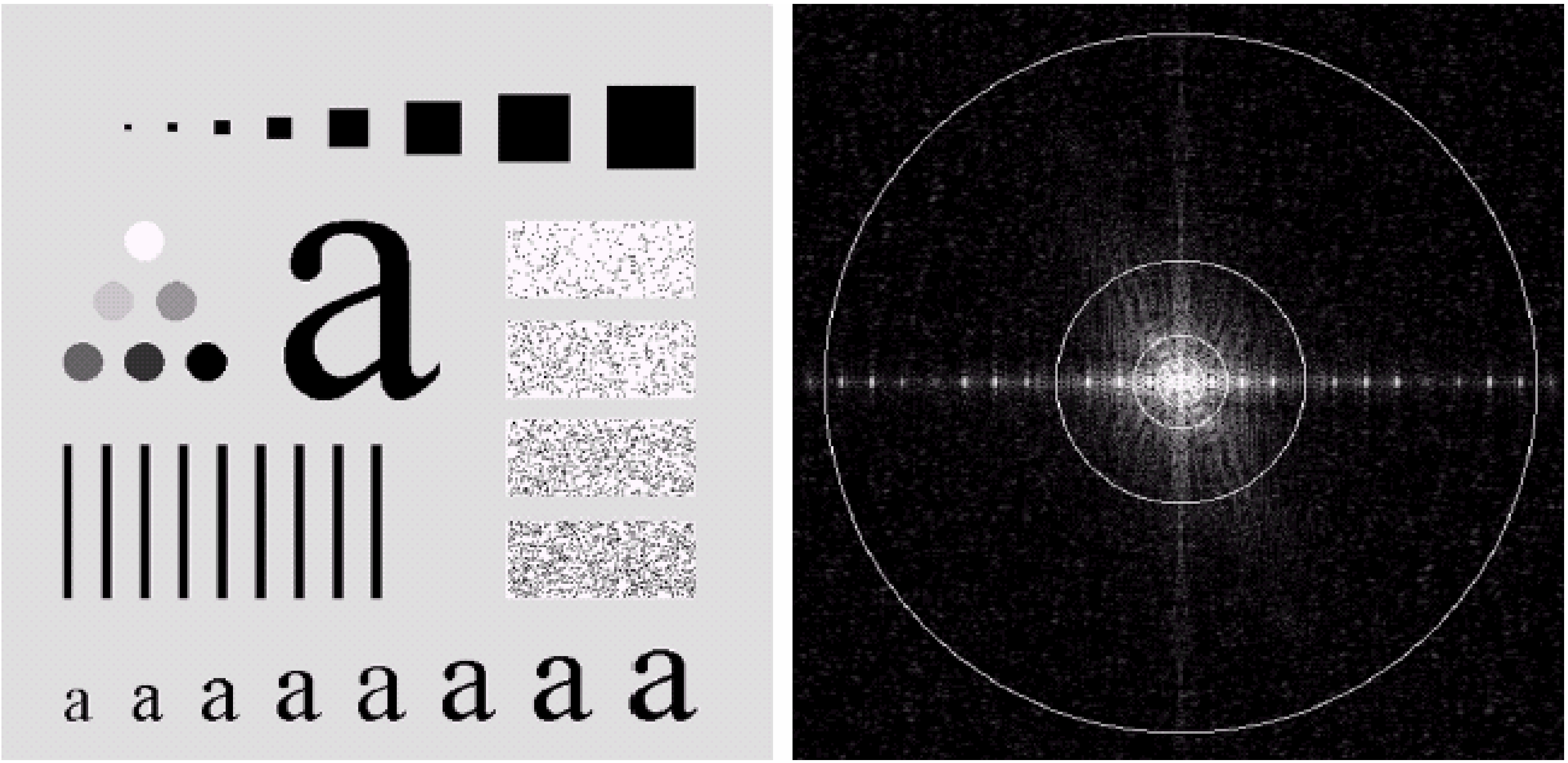
- Image Space
  - f(x,y)
  - Intuitive



- Frequency Space
  - F(u,v)
  - What does this mean?



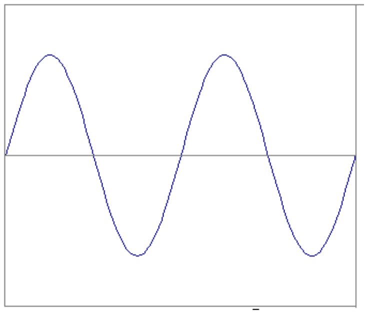
Power distribution



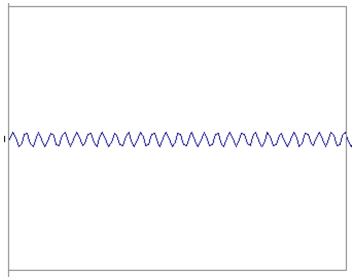
An image (500x500 pixels) and its Fourier spectrum. The super-imposed circles have radii values of 5, 15, 30, 80, and 230, which respectively enclose 92.0, 94.6, 96.4, 98.0, and 99.5% of the image power.

Power distribution

- Most power is in low frequencies.
- Means we are using more of this:

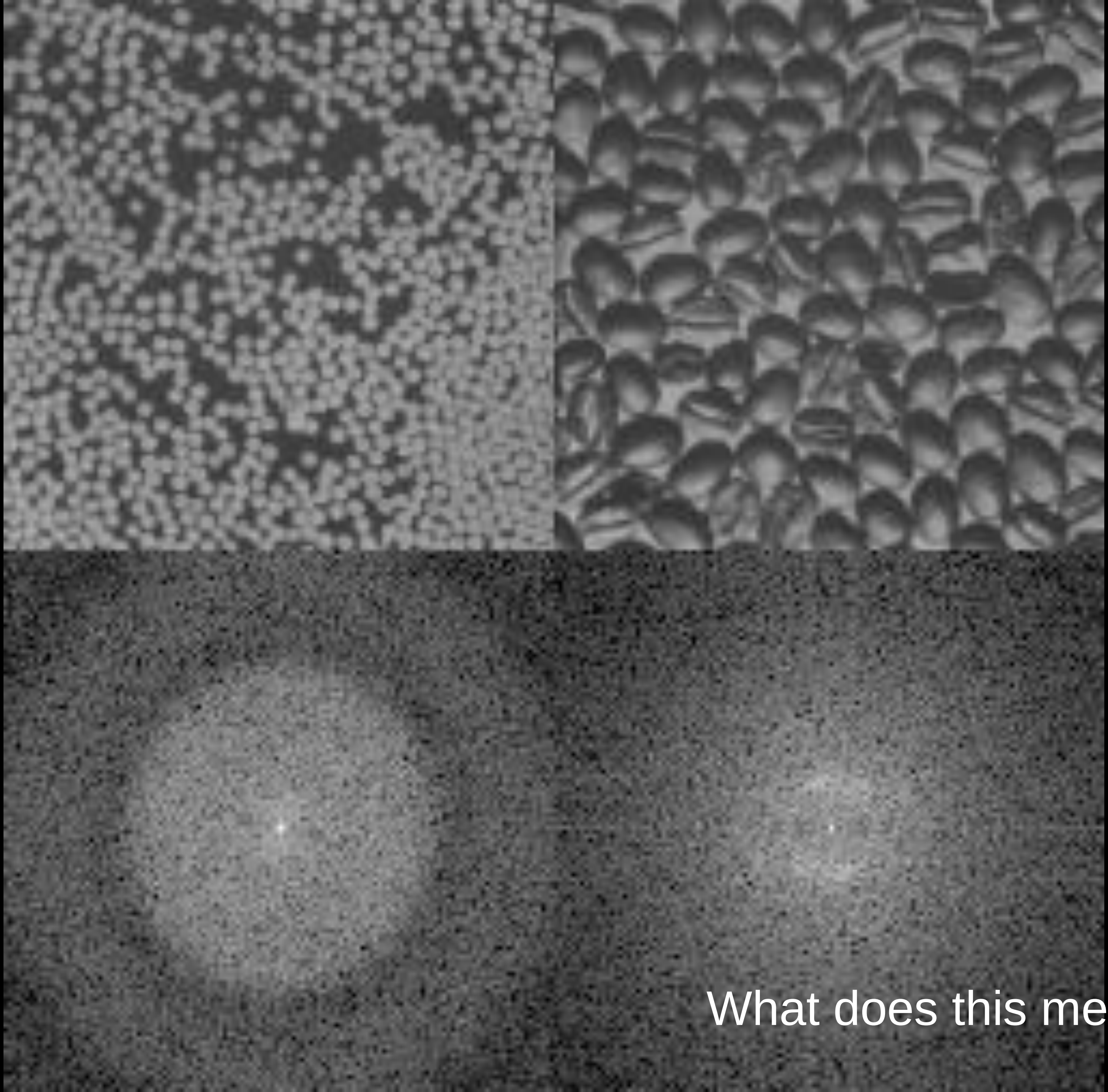


And less of this:



To represent our signal.

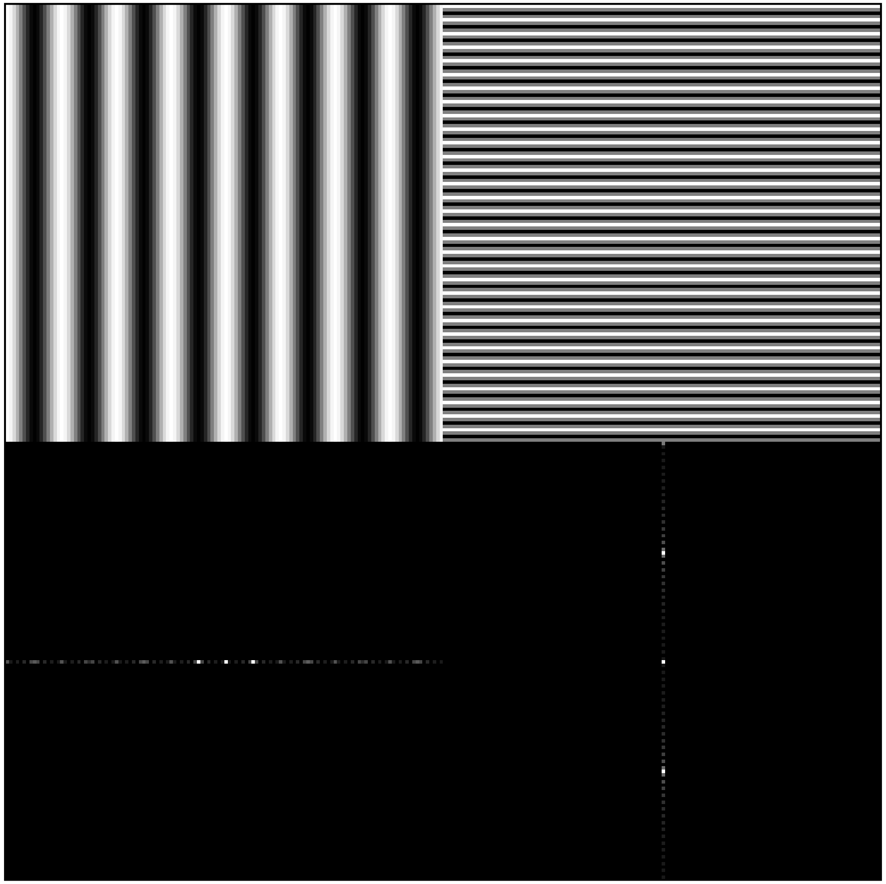
- Why?

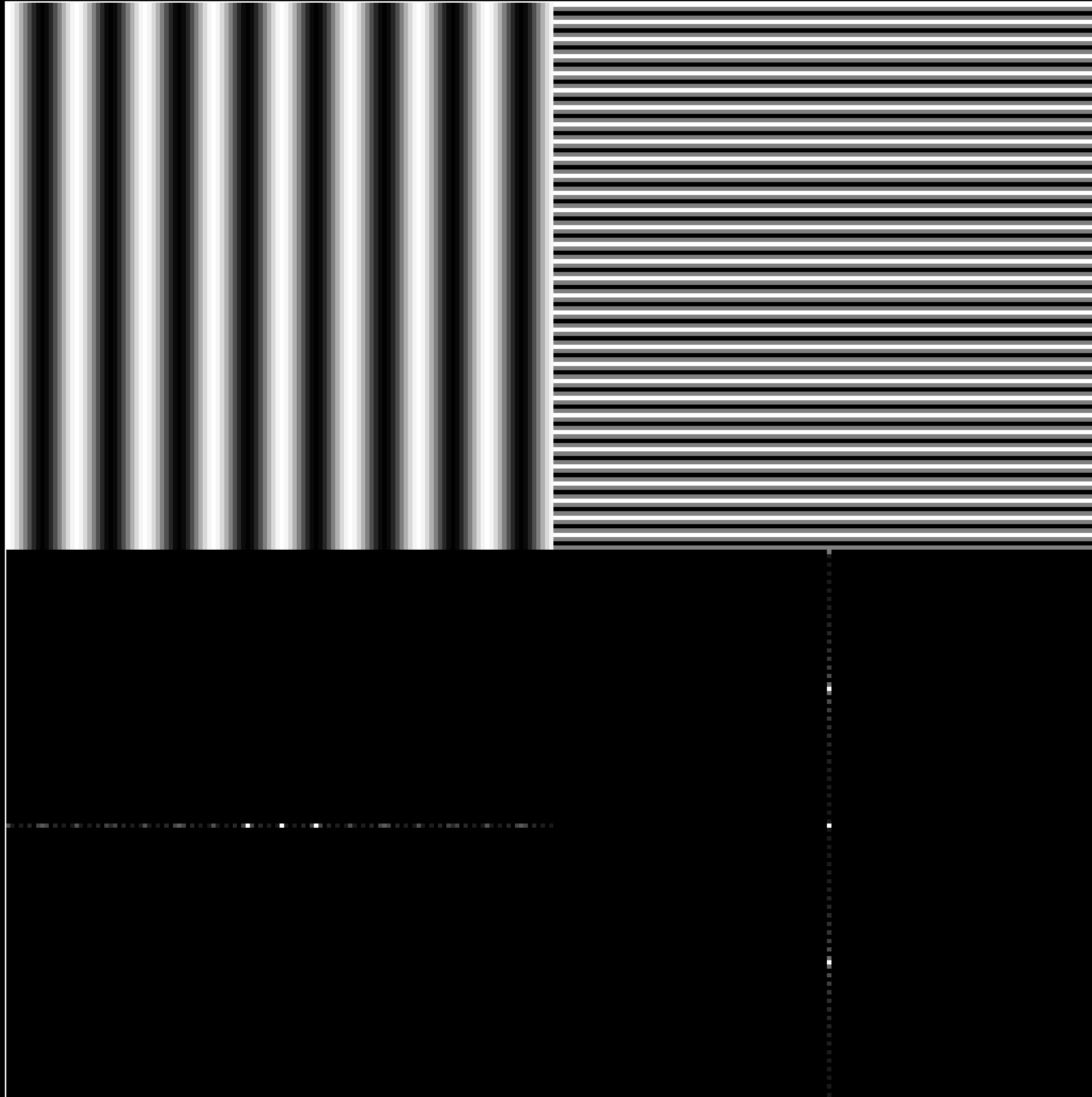


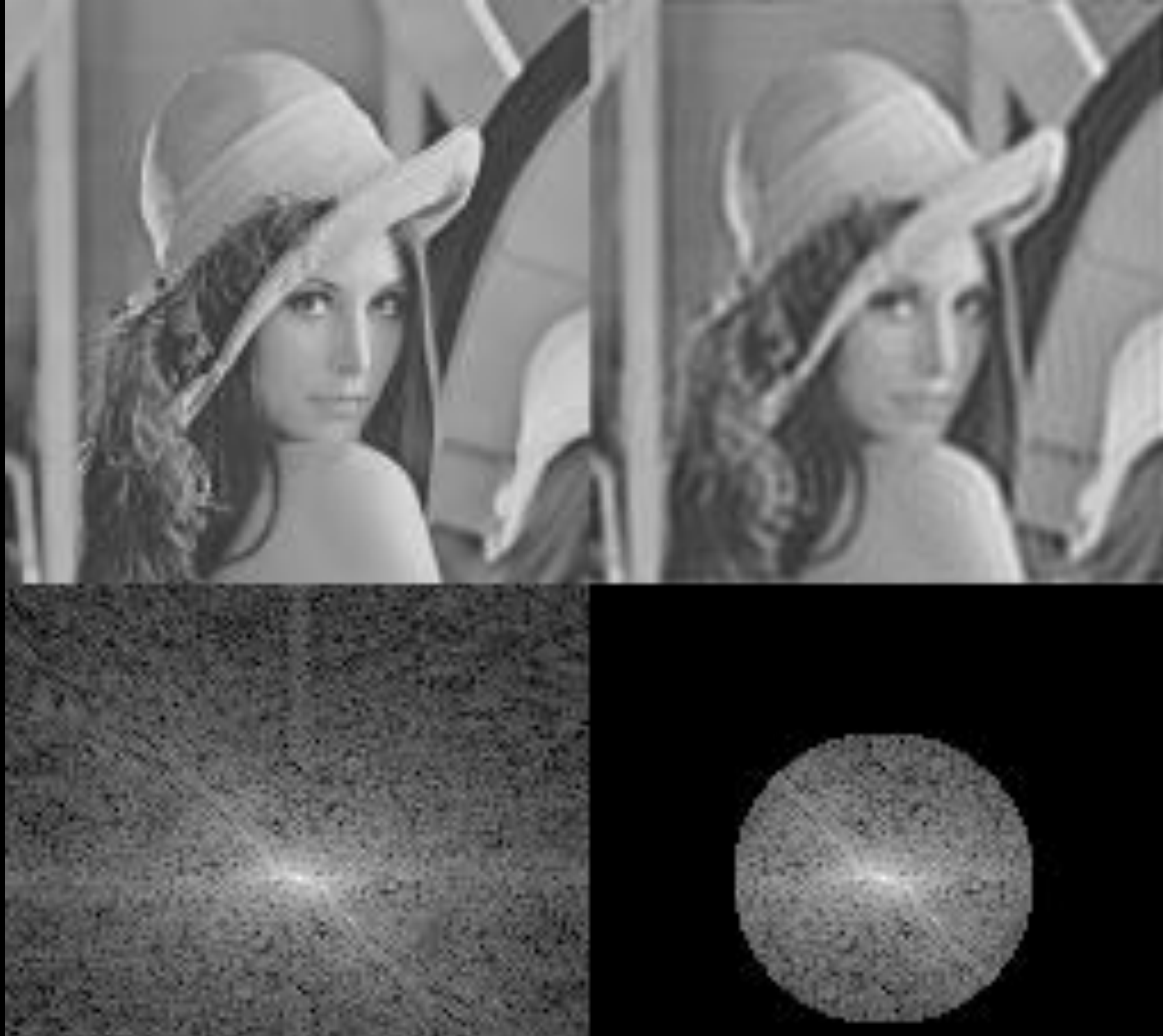
What does this mean??

Horizontal and Vertical Frequency

- Frequencies:
  - Horizontal frequencies correspond to horizontal gradients.
  - Vertical frequencies correspond to vertical gradients.
- What about diagonal lines?







If I discard high-frequencies, I get a blurred image...
Why?

Why bother with FT?

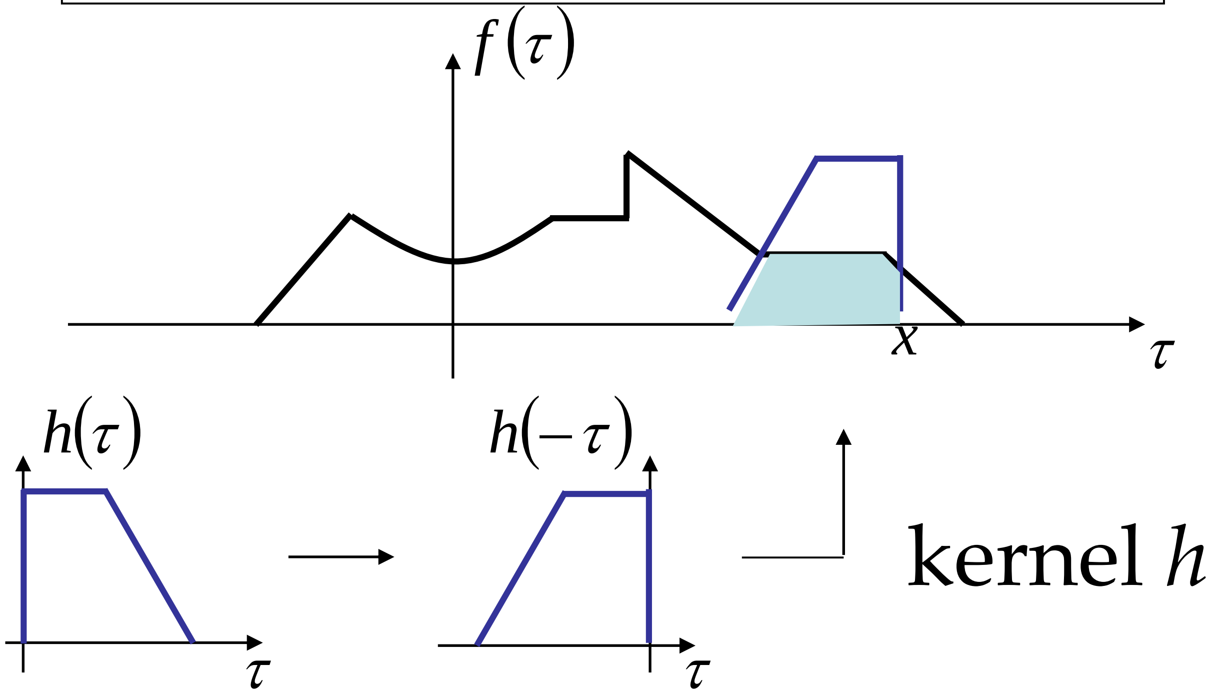
- Great for filtering.
- Great for compression.
- In some situations: Much faster than operating in the spatial domain.
- Convolutions are simple multiplications in Frequency space!
- ...

Topic: Spatial Convolution

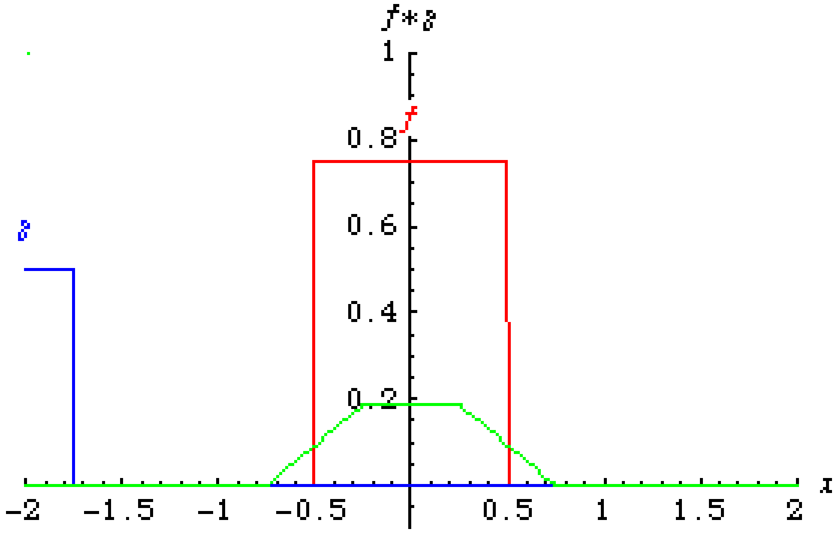
- Fourier Transform
- Frequency Space
- Spatial Convolution

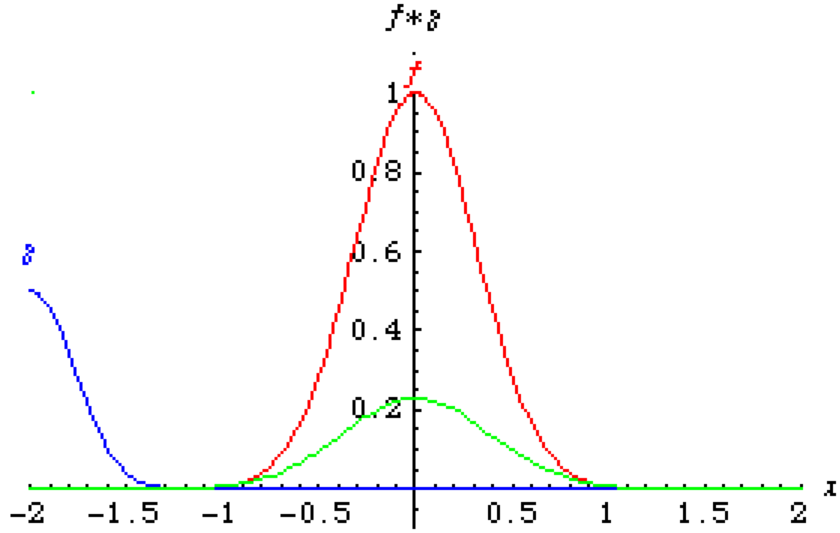
Convolution

$$g(x) = \int_{-\infty}^{\infty} f(\tau)h(x-\tau)d\tau \qquad g = f * h$$



Convolution - Example





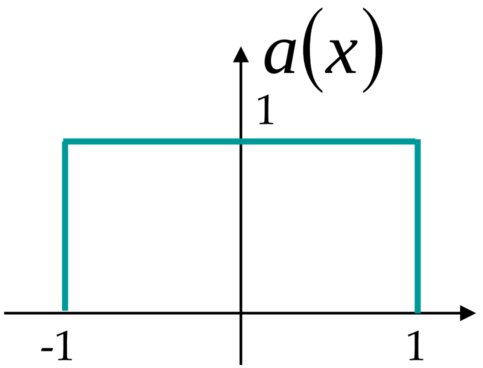
f

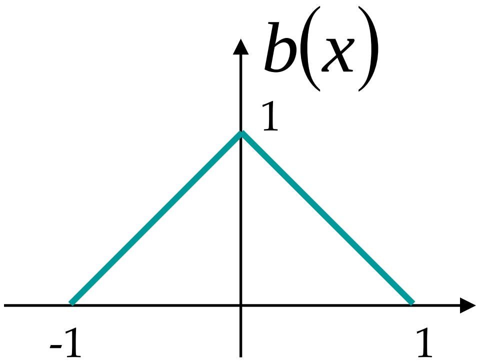
g

$f * g$

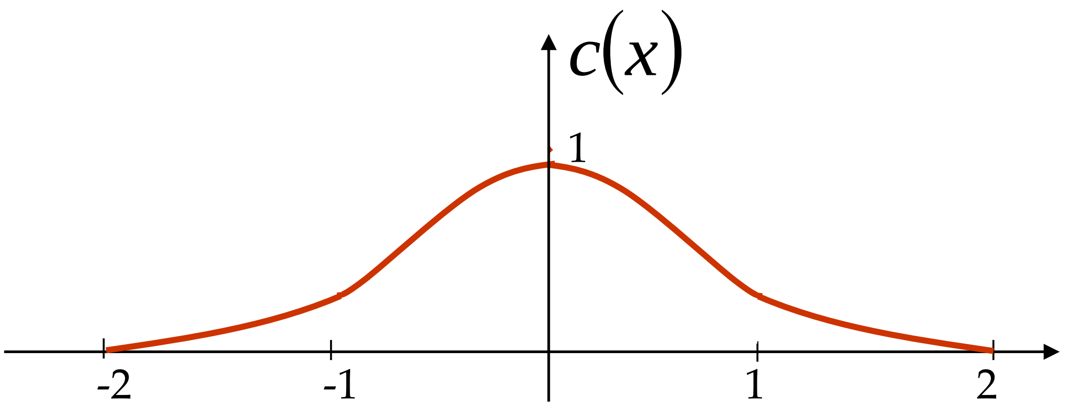
Eric Weinstein's Math World

Convolution - Example





$$c = a * b$$

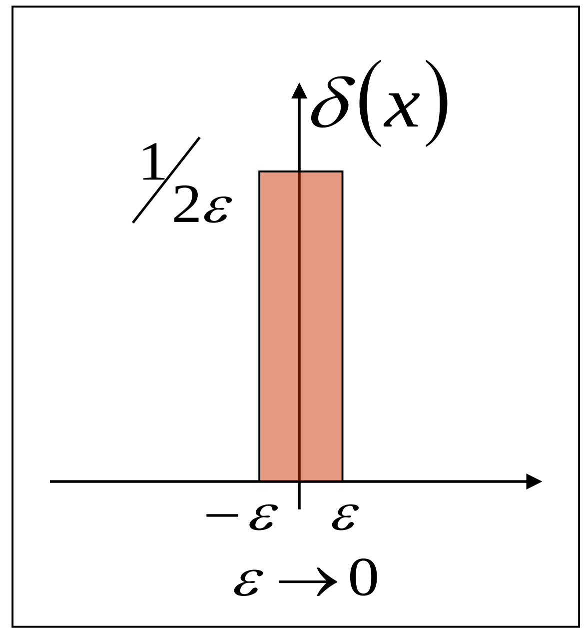


Convolution Kernel – Impulse Response

$$f \longrightarrow \boxed{h} \longrightarrow g \qquad\qquad g = f * h$$

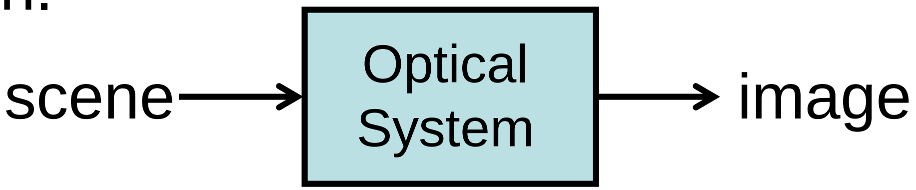
- What h will give us $g = f$?

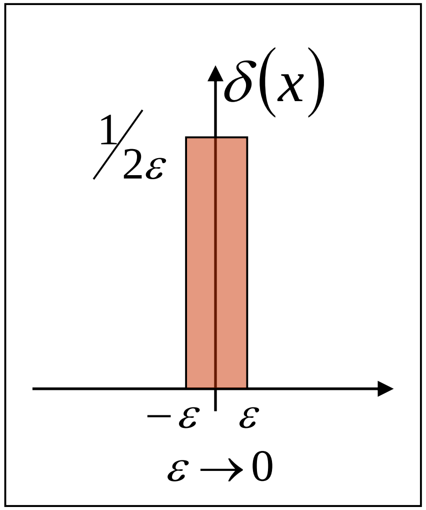
  Dirac Delta Function (Unit Impulse)



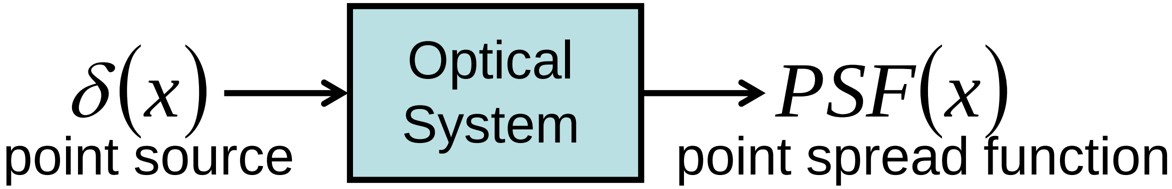
Point Spread Function

- Ideally, the optical system should be a Dirac delta function.

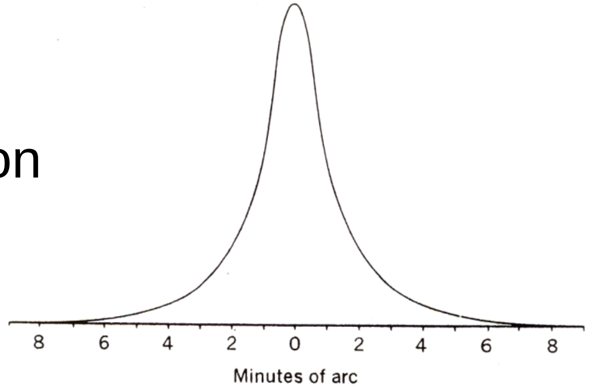




- However, optical systems are never ideal.



- Point spread function of Human Eyes.



Point Spread Function

normal vision

myopia

hyperopia

Images by Richmond Eye Associates

Properties of Convolution

- Commutative

$$a * b = b * a$$

- Associative

$$(a * b) * c = a * (b * c)$$

- Cascade system

$$f \longrightarrow \boxed{h_1} \longrightarrow \boxed{h_2} \longrightarrow g$$

$$= f \longrightarrow \boxed{h_1 * h_2} \longrightarrow g$$

$$= f \longrightarrow \boxed{h_2 * h_1} \longrightarrow g$$

Fourier Transform and Convolution

Let $g = f * h$ Then $G(u) = \int_{-\infty}^{\infty} g(x) e^{-i2\pi ux} dx$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\tau) h(x-\tau) e^{-i2\pi ux} d\tau dx$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[f(\tau) e^{-i2\pi u\tau} d\tau \right] \left[h(x-\tau) e^{-i2\pi u(x-\tau)} dx \right]$$

$$= \int_{-\infty}^{\infty} \left[f(\tau) e^{-i2\pi u\tau} d\tau \right] \int_{-\infty}^{\infty} \left[h(x') e^{-i2\pi ux'} dx' \right] \qquad = F(u) H(u)$$

Convolution in spatial domain

$\Leftrightarrow$ Multiplication in frequency domain

Fourier Transform and Convolution

Spatial Domain *(x)*		Frequency Domain *(u)*
$g = f * h$	$\longleftrightarrow$	$G = FH$
$g = fh$	$\longleftrightarrow$	$G = F * H$

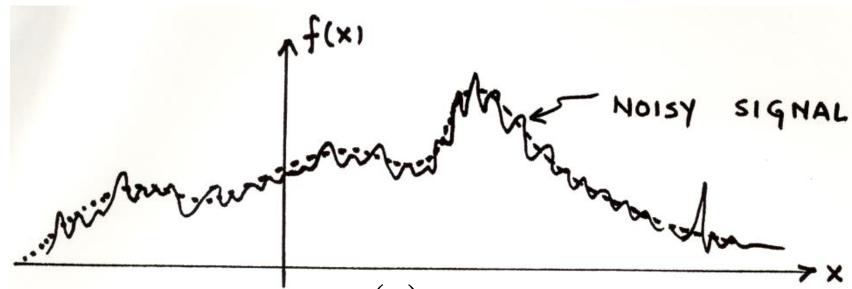
So, we can find *g(x)* by Fourier transform

$$\begin{tikzcd}
g & = & f \arrow[d, "\mathbf{FT}"] & * & h \arrow[d, "\mathbf{FT}"] \\
G \arrow[u, "\mathbf{IFT}"] & = & F & \times & H
\end{tikzcd}$$

Example use: Smoothing/Blurring

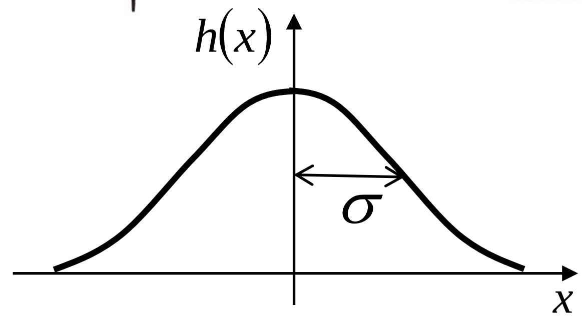
- We want a smoothed function of *f(x)*

$$g(x) = f(x) * h(x)$$



- Let us use a Gaussian kernel

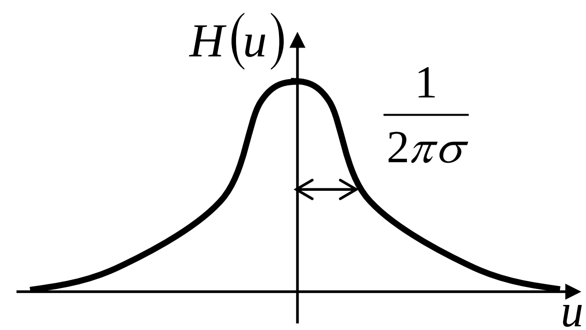
$$h(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2} \frac{x^2}{\sigma^2} \right]$$



- Then

$$H(u) = \exp\left[-\frac{1}{2} (2\pi u)^2 \sigma^2 \right]$$

$$G(u) = F(u) H(u)$$



Resources

- Russ – Chapter 6
- Gonzalez & Woods – Chapter 4