Applied Cryptography
Week 2: Randomness and Cryptographic Security

Bernardo Portela

M:ERSI, M:SI - 23

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For Asymmetric Crypto

- Key generation algorithm $\rightarrow$ key pair
- Private key holder generates both keys; publishes public key
- Asymmetric keys are typically much larger
  - RSA keys take roughly 4096-bits for 128-bit security
  - Elliptic-curve keys take roughly 400-bits for 128-bit security

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Key wrapping

- Long-term keys are often *wrapped* before storage
- To encrypt with another key
- Password-based encryption (low security)
- Wrap with HW-protected master key (standard security)
- Master key stored in trusted hardware (high security)

To Be Random

Q1: Which of these numbers are random?

1. 00000000
2. 10101010
3. 00100100
4. 10011101

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Q2: Which of these numbers will more likely appear in a fair randomness generator?

Randomness Distributions

Randomized processes described using *randomness distributions*.

We start with the **uniform distribution** over a finite field S.

A process U samples from the uniform distribution if

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$$\tfrac{2}{2^8} \approx 0.0078$$

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It is, and is called *rejection sampling*. **Q3: what is the downside?**

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- It is maximized by the uniform distribution, with entropy λ

$$2^8 \cdot (-\frac{1}{2^8} \cdot \log_2(\frac{1}{2^8})) = 8$$

- Entropy here quantifies the number of uncertainty bits
  - In this example, we are uncertain of exactly 8 bits
- If a sampling is biased, it has less uncertainty, i.e. entropy

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- It starts with a physical process
  - A source of entropy, e.g., some natural process that is believed to sample l-bits from a high-entropy distribution
  - Typically $l >> \lambda$ where λ is the assumed entropy
  - Randomness extractors (often a hash function) compress such bit strings down to λ bits
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- The combined process is called a Random Number Generator
- High-security RNGs currently exploit quantum effects

Pseudorandom Generators - Part 1

Good randomness is hard to generate, so RNGs are usually slow

Pseudorandom Generators are crypto's response to this problem:

- PRG takes a small, uniform seed of length λ
- Generates long, random-looking bit strings $l >> \lambda$
- PRGs are deterministic algorithms!

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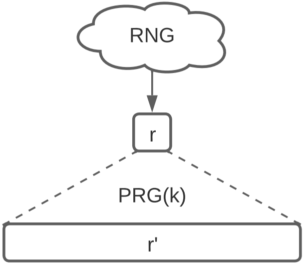
A Pseudorandom generator is a function $G : \{0,1\}^{\lambda} \rightarrow \{0,1\}^{l}$

Security: (without delving deep in probability) an attacker must be unable of distinguishing PRG outputs from a truly random string

Pseudorandom Generators - Part 2

$$PRG : \{0,1\}^{\lambda} \rightarrow \{0,1\}^{l}$$



Reasoning

- Use a strong RNG to generate seed r of (small) size λ
- Use the PRG on seed r to generate (much larger) r' of size l

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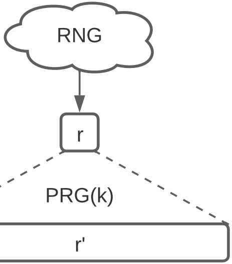

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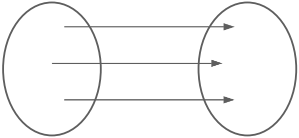
Q: Can we have secure PRGs (indistinguishable from uniform distribution), considering adversaries with unbound power?

Security of Pseudorandom Generators

$$U : \{0,1\}^l \rightarrow \{0,1\}^l$$



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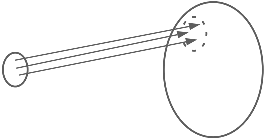

- An adversary can simply test all 2^λ cases
- Security refers to a computationally limited adversary
- One that cannot (realistically) test all possible PRG inputs

Security in Practice

Redefine "impossible to break"

- With *reasonable resources* (time, memory, HW power)
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Practical schemes are *computationally impossible* to break

Take an encryption scheme and an attacker that does not know k

- Attacker chooses non-repeating inputs X_i and gets
  - Y_i chosen uniformly at random if $b = 1$
  - $Y_i = E(k, X_i)$ if $b = 0$
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We define the adversary's advantage ϵ as

$$\epsilon = |\Pr[b' = 1|b = 1] - \Pr[b' = 1|b = 0]|$$

Best attack for $\epsilon = 2^{-40}$ takes 2^{80} steps

Concrete Numbers - Part 1

Some numbers for scale

- Not easy to perceive very very large numbers
- The estimated age of the universe in nanosecs is around 2^{88}
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- A common size for keys is 128 bits
- Consider the following events
  - Winning a lottery with 9 million participants (all of Portugal)
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Q2: By how much?

Concrete Numbers - Part 2

Security is defined as (t, ϵ)-security

- For some well-defined attack model
- Any attacker must run in at most t steps
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The more tries you get, the greater ϵ becomes: $(t, t/2^{128})$ security

Quantifying Security

Lower bound on the work required for a successful attack

Number of steps of the best attack

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  - **Q2: When?**
  - Best attack is more efficient than brute-force
  - Common in asymmetric cryptography
  - Keys must follow specific structures, not random bit strings
- Quantifying using *n*-bit security permits comparing schemes

Good Security Values for Real-world Crypto

The 2^{128} rule of thumb

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For how long do we need security to hold?

- Moore's law: computational power doubles every 2 years
- $n + 1$ bit security every 2 years
- This no longer seems to be true, but...
- Maybe we will have quantum computers soon

Long-term security: $\approx$ 256-bit keys

Short-term security: $\approx$ 80-bit keys may be OK

Stateful PRGs in Operating Systems

Randomness generation is statful

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Extract and expand randomness

- $st \leftarrow \text{init}()$: SO initializes state
- $st \leftarrow \text{refresh}(R, st)$: SO adds entropy (reseeds)
- $(C, st) \leftarrow \text{next}(N, st)$: SO returns N random bits

Dealing With a Compromised State

Backtracking $\Leftarrow$ resistance

- Suppose an adversary corrupts the PRG state
- Past randomness should not be compromised
  - We might have used it to generate cryptographic material
- A.k.a. *forward secrecy* (for past secret keys)

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Prediction $\Rightarrow$ resistance

- Suppose the adversary corrupts the PRG state
- SO adds extra (hidden) entropy to PRG state
- Future output should look random once more
- Hence refresh must be called regularly

Linux systems

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Link to code from **LibreSSL**

In some variants, there is a blocking */dev/random* based on an entropy simulator

- Check if there is “sufficient entropy”
- Blocks otherwise
- Current consensus indicates that, for most applications, this is not useful (see **this link** for more information)

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Irrelevant for Security

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- Totally insecure for cryptographic purposes

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Cryptographic PRGs come with a *proof of security*

- Goal: Given n bits of input, can an adversary guess bit $n + 1$?
- Secure PRGs used directly, or as building blocks to other PRGs

Security Assurance

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Provable Security

- Mathematical proof
- Breaking a scheme implies solving a hard problem
- A mathematical problem, or breaking another scheme!

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Assumption: mathematical problem P cannot be efficiently solved

Goal: Breaking scheme C cannot be efficiently done

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Methodology: building a reduction

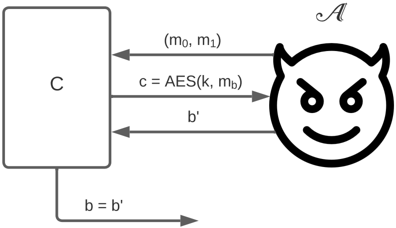
- Take any (hypothetical) attacker $\mathcal{A}$ that breaks C
- Construct (concrete) reduction $\mathcal{B}^{\mathcal{A}}$
- I.e. $\mathcal{B}$ uses $\mathcal{A}$ as a subroutine
- Show that $\mathcal{B}$ solves P when $\mathcal{A}$ succeeds

We never state that C is secure by itself

We state that C is as secure as the hardness of P

An Example of Provable Security - Part 1

Assume that AES is a semantic secure scheme, i.e.



An adversary with non-negligible victory probability (over $\frac{1}{2}$), i.e a successful $\mathcal{A}$ must not exist!

An Example of Provable Security - Part 2

Consider an encryption scheme that just repeats AES 2 times.

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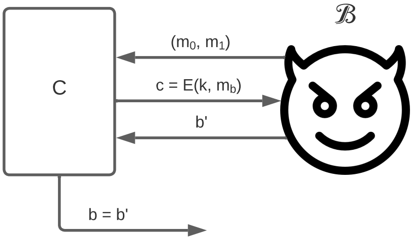

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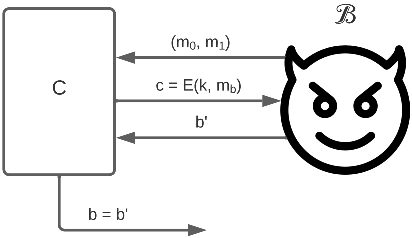
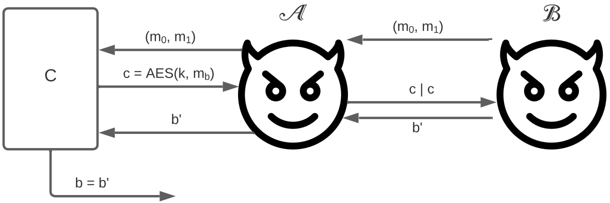


Q: given that AES is secure, is this secure?

- It should be...
- We are just repeating the encryption
- Can we demonstrate this?

An Example of Provable Security - Part 3



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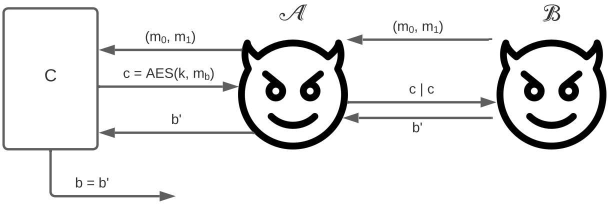

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Corollary

- No $\mathcal{B}^{\mathcal{A}}$ can exist (AES is secure)
- As such, no $\mathcal{A}$ can exist
- So, scheme E must be secure!

Caveats of Provable Security

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Proof assurance $\leq$ assumption assurance

- Proofs of security are relative to assumptions
- Security only holds if assumptions are true

Most of the assumptions are validated via **heuristic security**

Heuristic Security

Validating hardness assumptions is crucial for modern cryptography

Methodology for heuristic security has been progressing

- Standards take years to define
- Competitions where proposals are scrutinized
  - It is how AES was established as the *de facto* encryption standard for the overwhelming majority of applications
  - And is how PQ encryption schemes are being selected
- "My construction wins if I break your construction"
  - Yet again we see the value of the Kerckhoffs's principle!

Applied Cryptography
Week 2: Randomness and Cryptographic Security

Bernardo Portela

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