Glushkov and Equation Automata for KAT Expressions

Sabine Broda, António Machiavelo, Nelma Moreira, Rogério Reis
CMUP & DCC, Faculdade de Ciências da Universidade do Porto
Rua do Campo Alegre, 4169-007 Porto, Portugal

Version 1.0 April 2013
Glushkov and Equation Automata for KAT Expressions

Sabine Broda  António Machiavelo  Nelma Moreira  Rogério Reis
CMUP & DCC, Faculdade de Ciências da Universidade do Porto
Rua do Campo Alegre, 4169-007 Porto, Portugal

February 14, 2014

Abstract

Kleene algebra with tests (KAT) is an equational system that extends Kleene algebra, the algebra of regular expressions, and that is specially suited to capture and verify properties of simple imperative programs. In this paper we study two constructions of automata from KAT expressions: the Glushkov automaton ($A_{pos}$), and a new construction based on the notion of prebase (equation automata, $A_{eq}$). Contrary to other automata constructions from KAT expressions, these two constructions enjoy the same descriptional complexity behaviour as their counterparts for regular expressions, both in the worst-case as well as in the average-case. In particular, our main result is to show that, asymptotically and on average the number of transitions of the $A_{pos}$ is linear in the size of the KAT expression.

1 Introduction

Kleene algebra with tests (KAT) [Koz97] is an equational system for propositional program verification that combines Boolean algebra (BA) with Kleene algebra (KA), the algebra of the regular expressions. The equational theory of KAT is PSPACE-complete and can be reduced to the equational theory of KA, with an exponential cost [CKS96, KS96]. Several automata constructions from KAT expressions have been proposed in order to obtain feasible decision procedures for KAT expressions equivalence [Koz03, Wor08, Koz08, Sil10, ABM12]. Regular sets of guarded strings [Kap69] are the standard models for KAT (as regular languages are for KA) [Koz03]. A coalgebraic approach based on the notion of (partial) derivatives and automata on guarded strings were developed by Kozen [Koz08], and implemented, with slightly modifications, by Almeida et al. [ABM12]. Silva [Sil10] presented yet another automata construction, extending for KAT the Glushkov construction, well known for the conversion of regular expressions to nondeterministic finite automata [Glu61]. All the constructions of automata on guarded strings, with the exception of Silva’s, induce an exponential blow-up on the number of states/transitions of the automata. This is due to the use of all valuations of the boolean expressions that occur in a KAT expression, and also induces an extra exponential factor when testing the equivalence of two KAT expressions. In this paper, we present a new construction to obtain an automaton from a KAT expression, adapting the Mirkin construction of an equation automaton [Mir66]. For regular expressions, this construction coincides with Antimirov’s partial derivative automaton [CZ01], that it is known to be a quotient of the Glushkov automaton [CZ02]. The number of states of the Glushkov automaton equals to the number of occurrences of alphabetic symbols in the regular expression, and its number of transitions is, in the worst-case, quadratic in that number. Herein, we also observe that, in the worst-case, the number of transitions of the the Glushkov automaton is quadratic in the size of the KAT expression. Nicaud [Nic09] and Broda et al. [BMMR11, BMMR12b] studied the average size of these two automata for regular languages, using the framework of analytic combinatorics. Asymptotically, the average size of the Glushkov automaton is linear in the size of the regular expression, and the size of the equation (partial derivative) automaton is half that size. We show that similar results hold for their analogue
constructions for KAT expressions. The main outcome is the asymptotical linearity of the average number of transitions of the Glushkov automaton, i.e. for KAT expressions of size n it is $\Theta(n)$. This also provides an upper bound for the number of transitions of the equation automaton. These results come as a surprise, due to the bad behaviour of other automata constructions from KAT expressions, and can lead to more efficient decision procedures for KAT expressions equivalence. Note that the equation automaton can be more suitable to use in decision procedures based on coalgebraic methods, due to the fact that states correspond to combinations of subexpressions of the initial expressions.

2 KAT expressions, Automata, and Guarded Strings

Let $P = \{p_1, \ldots, p_k\}$ be a non-empty set of program symbols and $T = \{t_1, \ldots, t_l\}$ be a non-empty set of test symbols. The set of boolean expressions over $T$ together with negation, disjunction and conjunction, is denoted by $BExp$, and the set of KAT expressions with disjunction, concatenation, and Kleene star, by $Exp$. The abstract syntax of KAT expressions, over an alphabet $P \cup T$, is given by the following grammar, where $p \in P$ and $t \in T$,

$$BExp : \quad b \rightarrow \ 0 \ | \ 1 \ | \ \neg b \ | \ b + b \ | \ b \cdot b$$

(1)

$$Exp : \quad e \rightarrow \ p \ | \ b \ | \ e + e \ | \ e \cdot e \ | \ e^*.$$  

(2)

As usual, we omit the operator $\cdot$ whenever it does not give rise to any ambiguity. The size of a KAT expression $e$, denoted by $|e|$, is the number of symbols in the syntactic tree of $e$. The set $At$, of atoms over $T$, is the set of all boolean assignments to all elements of $T$, $At = \{x_1 \cdot \cdots \cdot x_l \ | \ x_i \in \{t_i, t\}, \ t_i \in T\}$. Now, the set of guarded strings over $P$ and $T$ is $GS = (At \cdot P)^* \cdot At$. Regular sets of guarded strings form the standard language-theoretic model for KAT [Koz03]. For $x = \alpha_1 p_1 \cdots p_m - \alpha_m$, $y = \beta_1 q_1 \cdots q_n - \beta_n$, $x \circ y = \{\alpha_1 p_1 \cdots p_m - \alpha_m q_1 \cdots q_n - \beta_n\}$, where $m, n \geq 1$, $\alpha_i, \beta_j \in At$ and $p_i, q_j \in P$, we define the fusion product

$$x \circ y = \begin{cases} 
\alpha_1 p_1 \cdots p_m - \alpha_m q_1 \cdots q_n - \beta_n, & \text{if } \alpha_m = \beta_1 \\
\text{undefined,} & \text{otherwise.}
\end{cases}$$

For sets $X, Y \subseteq GS$, $X \circ Y$ is the set of all $x \circ y$ such that $x \in X$ and $y \in Y$. Let $X^0 = At$ and $X^{n+1} = X \circ X^n$, for $n \geq 0$.

Given a KAT expression $e$ we define $GS(e) \subseteq GS$ inductively as follows:

$$GS(p) = \{\alpha p \beta \ | \ \alpha, \beta \in At\}$$

$$GS(b) = \{\alpha \ | \ \alpha \in At \wedge \alpha \leq b\}$$

$$GS(e_1 + e_2) = GS(e_1) \cup GS(e_2)$$

$$GS(e_1 \cdot e_2) = GS(e_1) \circ GS(e_2)$$

$$GS(e_1^*) = \cup_{n \geq 0} GS(e_1)^n, \quad \text{where } \alpha \leq b \text{ if } \alpha \rightarrow b \text{ is a propositional tautology.}$$

Example 1. Consider $e = t_1 + (\neg t_1)(t_2p)^*$, where $P = \{p\}$ and $T = \{t_1, t_2\}$. Then, $At = \{t_1t_2, t_1\overline{t_2}, \overline{t_1t_2}, \overline{t_1}\overline{t_2}\}$ and

$$GS(e) = GS(t_1) \cup GS(\neg t_1) \circ GS((t_2p)^*)$$

$$= \{t_1t_2, \overline{t_1t_2} \} \cup \{t_1t_2, \overline{t_1t_2} \} \cup \{ \overline{t_1}(t_2p)^n \alpha \ | \ n \geq 1, \alpha \in At \}$$

Given two KAT expressions $e_1$ and $e_2$, we say that they are equivalent, and write $e_1 = e_2$, if $GS(e_1) = GS(e_2)$.

A (non-deterministic) automaton over the alphabets $P$ and $T$ is a tuple $A = \langle S, s_0, o, \delta \rangle$, where $S$ is a finite set of states, $s_0 \in S$ is the initial state, $o : S \rightarrow BExp$ is the output function, and $\delta \subseteq 2^{S \times (BExp \times P) \times S}$ is the transition relation. A guarded string $\alpha_1 p_1 \cdots p_{n-1} \alpha_n$, with $n \geq 1$, is accepted by the automaton $A$ if and only if there is a sequence of states $s_0, s_1, \ldots, s_{n-1} \in S$, where $s_0 = s$, and, for $i = 1, \ldots, n-1$, one has $\alpha_i \leq b_i$ for some $(s_{i-1}, (b_i, p_i), s_i) \in \delta$, and $\alpha_n \leq o(s_{n-1})$.

The set of all guarded strings accepted by $A$ is denoted by $GS(A)$. Formally, we define $L : S \rightarrow GS \rightarrow \{0, 1\}$, by structural induction on $x \in GS$ as follows.

3
We recursively define the functions always remain unchanged.

The definition of the Glushkov automaton for a given expression \( e \) occurring in \( p, b, q \) triplets (i.e., last \( \alpha \) follow \( \alpha \)) such that \( \text{Exp} = \{ 1 \} \) = \( \text{Out}(1) \cup \text{Out}(2) \) otherwise.

Given \( s \in S \), let \( \text{GS}(s) = \{ x \in \text{GS} \mid L(s)(x) = 1 \} \). Then, \( \text{GS}(A) = \text{GS}(s_0) \). We say that a KAT expression \( e \in \text{Exp} \) is equivalent to an automaton \( A \), and write \( e = A \), if \( \text{GS}(A) = \text{GS}(e) \).

In the next two sections we present two different constructions of automata that are equivalent to a given KAT expression.

3 The Glushkov Automaton

The definition of the Glushkov automaton for KAT expressions follows closely the one given by Silva [Sil10]. Let \( \tilde{e} \) denote the KAT expression obtained by considering the number of elements of \( P \) occurring in \( e \), marking each one with its appearance number, that is called its position. The same notation is used to denote the removal of the markings, i.e., \( \tilde{e} = e \). The set of positions in an expression \( e \) is denoted by \( \text{pos}(e) \). Note that this marking does not apply to test symbols, which always remain unchanged.

Example 2. Consider \( e = t_1p(pq^*t_2 + t_3q)^* \), where \( P = \{ p, q \} \) and \( T = \{ t_1, t_2, t_3 \} \). Then, \( \tilde{e} = t_1p_1(pq^*_t_2 + t_3q^*_t_3)^* \).

Definition 3. We recursively define the functions first, follow, last, and out according to grammar (2)

\[
L(s)(\alpha) = \begin{cases} 1 & \text{if } \alpha \leq \alpha(s), \\ 0 & \text{otherwise.} \end{cases} \quad L(s)(\alpha px) = \begin{cases} 1 & \text{if } (s, (b, p), s') \in \delta \text{ for } s' \in S, \\ b \text{ s.t. } \alpha \leq b, \text{ and } L(s')(x) = 1, \\ 0 & \text{otherwise.} \end{cases}
\]

where, for \( X \subseteq \text{BExp} \times \text{Exp}, Y \subseteq \text{Exp} \times \text{BExp} \text{ and } b \in \text{BExp}, \) we have \( b \cdot X = \{ (bb', p) \mid (b', p) \in X \} \), \( Y \cdot b = \{ (bb', p) \mid (p, b') \in Y \} \) and \( Y \otimes X = \{ (p, p, b') \mid (p, b) \in Y, (b', p) \in X \} \), with the caveat that \( 0 \cdot 0 = 0 = 0 + 1 + b = b + 1 = 1 \).

Informally, given \( e \in \text{Exp} \), the elements of first(e) are pairs \( (b, p) \) such that \( \alpha px \in \text{GS}(e) \) and \( \alpha \leq b \); the elements of last(e) are pairs \( (p, b) \) such that \( xpxo \in \text{GS}(e) \) and \( \alpha \leq b \); the elements of follow(e) are triplets \( (p, b, q) \) such that \( xpxoqy \in \text{GS}(e) \) and \( \alpha \leq b \); and out(e) \in \text{BExp} corresponds to the values \( \alpha \in \text{At} \) such that \( \alpha \leq e \).
Example 4. Consider the expression $\tilde{e}$ of Example 2. One has,

$$\begin{align*}
\text{first}(\tilde{e}) &= \{(t_1, p_1)\} \\
\text{last}(\tilde{e}) &= \{(p_1, 1), (p_2, t_2), (q_3, t_2), (q_4, 1)\} \\
\text{follow}(\tilde{e}) &= \{(p_1, 1, p_2), (p_1, t_3, q_4), (p_2, 1, q_3), (p_2, t_2, p_2), (p_2, t_2 t_3, q_4) \}
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ quad \\
\text{out}(\tilde{e}) &= \text{out}(t_1) \cdot \text{out}(p_1) \cdot \text{out}(p_2) = t_1 \cdot 0 \cdot 1 = 0
\end{align*}$$

Definition 5 (Glushkov Automaton). For $e \in \text{Exp}$, we define the Glushkov automaton $A_{\text{pos}}(e) = (\text{pos}(e) \cup \{0\}, 0, o, \delta_{\text{pos}})$, where $o(0) = \text{out}(\tilde{e})$, $o(i) = b$ if $i > 0$ and $(p_i, b) \in \text{last}(\tilde{e})$, and $o(i) = 0$, otherwise; and

$$\delta_{\text{pos}} = \{ (0, (b, p), j) \mid (b, p_j) \in \text{first}(\tilde{e}), p = p_j \} \cup \{ (i, (b, p), j) \mid (p_i, b, p_j) \in \text{follow}(\tilde{e}), p = p_j \}.$$

Analogously to what happens for regular expressions, given an expression $e$ the Glushkov automaton $A_{\text{pos}}(e)$ has exactly $|e|p + 1$ states, where $|e|p$ denotes the number of occurrences of program symbols (elements of P) in the expression $e$. This means that the boolean parts of an expression do not affect the number of states in its corresponding Glushkov automaton, contrary to what happens in other constructions, cf. [Koz03, Koz08]. Furthermore, the number of transitions of $A_{\text{pos}}(e)$ is in the worst-case $\mathcal{O}(|e|^{3/2})$. This results from the fact that, for every marked expression $\tilde{e}$ and for every marked program symbols $p_i$ and $p_j$, there is at most one pair $(p_i, b)$ in first($\tilde{e}$) and at most one pair $(p_j, b')$ in last($\tilde{e}$), for $b, b' \in \text{BEExp}$; and there is at most one tuple $(p_i, b, p_j)$ in follow($\tilde{e}$), for $b \in \text{BEExp}$.

Example 6. Consider again the expression $e$ of Example 2 and the functions computed in Example 4. In this case, one has $\text{pos}(e) = \{1, 2, 3, 4\}$ and the $A_{\text{pos}}(e)$, is the following:

![Graph](image_url)

Proposition 7. [Sil10, Th. 3.2.7] For every KAT expression, $e \in \text{Exp}$, one has $\text{GS}(A_{\text{pos}}(e)) = \text{GS}(e)$.

4 The Equation Automaton

In this section, we give a definition of the equation automaton for a KAT expression, extending the notion of prebase of a regular expression due to Mirkin [Mir66]. Here we do not consider the equivalence of this construction to the partial derivative automata [Koz08, ABM12], since KAT derivatives are considered with respect to $\alpha p$, ($\alpha \in \text{At}$) and we want to avoid the possible exponential blow-up associated to the set of atoms.

Definition 8. Given $e \in \text{Exp}$, a set of non-null expressions $E = \{e_1, \ldots, e_n\} \subseteq \text{Exp}$ is called a support of $e$, if the following system of equations holds:

$$\begin{align*}
e & \equiv e_0 = P_{01} e_1 + \cdots + P_{0n} e_n + \text{out}(e_0) \\
e_1 & = P_{11} e_1 + \cdots + P_{1n} e_n + \text{out}(e_1) \\
& \vdots \\
e_n & = P_{n1} e_1 + \cdots + P_{nn} e_n + \text{out}(e_n),
\end{align*}$$

where $P_{ij} = \sum_{k=1}^{n} b_{ij, p_k}$, for $0 \leq i, j \leq n$. For the components $bp$ of a sum $P_{ij}$, we write $bp < P_{ij}$.

If $E$ is a support of $e$, then the set $E_0 = E \cup \{e\}$ is called a prebase of $e$. 

5
Note that, if $E$ is a support of $e$, we may have $e \in E$. The system of equations (3) can be written in matrix form $E_0 = PE + O(E_0)$, where $P$ is the $(n + 1) \times n$ matrix with entries $P_{ij}$, and $E, E_0$ and $O(E_0)$ are, respectively, the column matrices

\[
E = \begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix}, \quad E_0 = \begin{bmatrix} e_0 \\ \vdots \\ e_n \end{bmatrix} \quad \text{and} \quad O(E_0) = \begin{bmatrix} \text{out}(e_0) \\ \vdots \\ \text{out}(e_n) \end{bmatrix}.
\]

In the following, we will arbitrarily interchange the matrix and the set notation for the above values. Considering the notion of KAT expression equivalence, we have the following lemma.

**Lemma 9.** Given $e \in \text{Exp}$ and $E_0 = \{e = e_0, e_1, \ldots, e_n\}$ a prebase of $e$, then

a) $\alpha \in \text{GS}(e_i)$ iff $\alpha \leq \text{out}(e_i)$;

b) $\alpha px \in \text{GS}(e_i)$ iff there are $j \in \{0, \ldots, n\}$ and $b \in \text{BExp},$ such that $bp \prec P_{ij}, \alpha \leq b,$ and $x \in \text{GS}(e_j)$.

Given a prebase $E_0 = \{e = e_0, e_1, \ldots, e_n\}$, an NFA can be defined by

\[
\mathcal{A}_{E_0}(e) = \langle \{e = e_0, e_1, \ldots, e_n\}, e, \text{out}, \delta \rangle,
\]

where

\[
\delta = \{ (e_i, (b_{ij}, p_{e}), e_j) \mid 0 \leq i, j \leq n, \ 1 \leq r \leq k \}.
\]

Using Lemma 9, it is easy to see that $\text{GS}(\mathcal{A}_{E_0}(e)) = \text{GS}(e)$.

**Definition 10 (Equation Automaton).** For $e \in \text{Exp}$, we define the equation automaton $\mathcal{A}_{\text{eq}}(e) = (\{e\} \cup \pi(e), e, \text{out}, \delta_{\text{eq}})$ with $\delta_{\text{eq}} = \{ (e_i, (b, p), e_j) \mid bp \prec P_{ij} \in \text{P}(e) \}$ and where $\pi : \text{Exp} \to 2^{\text{Exp}}$ is defined by induction on the structure of $e$ as follows:

\[
\begin{align*}
\pi(p) &= \{1\} \\
\pi(b) &= \emptyset \\
\pi(e + f) &= \pi(e) \cup \pi(f) \\
\pi(e \cdot f) &= \pi(e)f \cup \pi(f) \\
\pi(e^*) &= \pi(e)^*.
\end{align*}
\]

and $\text{P}(e)$ is a $(n + 1) \times n$ matrix with entries $P_{ij} \in \text{P},$ $n = |\pi(e)|,$ and which is inductively defined on the structure of $e$ by

\[
\begin{align*}
P(p) &= \begin{bmatrix} 1p \\ 0 \end{bmatrix} \\
P(b) &= 0 \\
P(e + f) &= \begin{bmatrix} P(e) & P(f)|_0 \\ 0 & P(f)|_{1 \ldots m} \end{bmatrix} \\
P(e \cdot f) &= \begin{bmatrix} \frac{P(e)}{0} & O(\pi(e)) \odot P(f)|_0 \\ 0 & P(f)|_{1 \ldots m} \end{bmatrix} \\
P(e^*) &= P(e) + \begin{bmatrix} 0 & O(\pi(e)) \odot P(e)|_0 \\ 0 & P(e)|_{1 \ldots m} \end{bmatrix}.
\end{align*}
\]

where $P(f)$ is an $(m + 1) \times m$ matrix, for some $m > 0$; $P(f)|_0$ denotes the first row of matrix $P(f)$; $P(f)|_{1 \ldots m}$ denotes the matrix $P(f)$ without the first row; and the $\odot$ operator is defined as follows

\[
\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \odot \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} x_1y_1 & \ldots & x_1y_m \\ \vdots & \ddots & \vdots \\ x_ny_1 & \ldots & x_ny_m \end{bmatrix}.
\]

Note, that the above definition of $\pi$ follows closely the one for regular expressions by Mirkin. Analogously, it can be easily shown that for a KAT expression $e$, one has $|\pi(e)| \leq |e|_p$. Consequently, the number of states in the equation automaton of an expression $e$ is $|\pi(e)| \leq |e|_p + 1$, thus smaller or equal than the number of states of the Glushkov automaton.
Example 11. For the expression $e = t_1 p e_1^*$ of Example 2, where $e_1 = pq^* t_2 + t_3 q$. One has $\pi(e) = \{q^* t_2 e_1^*, e_1^*\}$ and the $A_{eq}(e)$ is the following:

One has $|e|_p + 1 = 5$, while the number of states of $A_{eq}(e)$ is $|\pi(e) \cup \{e\}| = 3$.

In order to show that the equation automaton $A_{eq}(e)$ is equivalent to $e$ it is enough to show that the function $\pi$ is a support of $e$.

Proposition 12. Given $e \in \text{Exp}$, one has $\text{GS}(A_{eq}(e)) = \text{GS}(e)$.

5 Average Size of Glushkov and Equation Automata

In this section, we estimate the asymptotic average size of the Glushkov and equation automata. This is done by the use of the standard methods of analytic combinatorics as expounded by Flajolet and Sedgewick [FS08]. These apply to generating functions $A(z) = \sum a_n z^n$ for a combinatorial class $A$ with $a_n$ objects of size $n$, denoted by $[z^n] A(z)$, and also bivariate functions $C(u, z) = \sum b(\alpha) z^{\alpha}$, where $c(\alpha)$ is some measure of the object $\alpha \in A$.

In order to apply this method, it is necessary to have an unambiguous description of the objects of the combinatorial class. One can see that the grammar (2) for KAT expressions is ambiguous. But, we can use the following non-ambiguous grammar for KAT expressions, where expressions $\text{BExp}$ correspond to KAT expressions with at least one program symbol $p \in P$. For simplicity, we exclude expressions that contain subexpressions of the form $b^*$, as their semantics correspond to $A t$ and thus are equivalent to 1.

$$\text{BExp} : \quad b \rightarrow 0 \mid 1 \mid t \mid \neg b \mid b + b \mid b \cdot b$$

$$\text{AExp} : \quad a \rightarrow p \mid a + a \mid a + b \mid b + a \mid a \cdot a \mid a \cdot b \mid b \cdot a \mid a^*$$

$$\text{Exp} : \quad e \rightarrow b \mid a.$$  

From the definitions above, one can compute the generating functions $B_t(z)$, $A_{k,l}(z)$, and $E_{k,l}(z)$, for the number of boolean expressions $B\text{Exp}$, expressions $A\text{Exp}$, and KAT expressions $\text{Exp}$, respectively. However, it is easy to see that $B_t(z)$ and $E_{k,l}(z)$ coincide with the generating function of standard regular expressions. Considering the following grammar for regular expressions with an alphabet $\Sigma$ of size $m$,

$$r \rightarrow 0 \mid 1 \mid \sigma \in \Sigma \mid r + r \mid r \cdot r \mid r^*,$$  

its generating function is given by

$$R_m(z) = \frac{1 - z - \sqrt{\Delta_m(z)}}{4z}, \quad \text{where} \quad \Delta_m(z) = 1 - 2z - (15 + 8m)z^2.$$  

We have $B_t(z) = R_t(z)$ and $E_{k,l}(z) = R_{k+l}(z)$. The first equality is due to the similarity of the grammars (8) and (11), where one has the negation operator and the other, the star operator. This fact and the exclusion of expressions of the form $b^*$ leads to the second equality. As a consequence, we can easily adapt most of the results obtained for regular expressions [Nic09, BMMR11, BMMR12b] to KAT expressions.

Using the technique presented in Section 5 of Broda et al. [BMMR12a] applied to (12), the asymptotic estimates for the number of regular expressions of size $n$ is

$$[z^n] R_m(z) \sim \frac{\sqrt{\rho_m \sqrt{2m + 4}}}{4\sqrt{\pi}} \rho_m^{-\frac{n+1}{2}} (n+1)^{-\frac{3}{2}}.$$  

where $\rho_m = \frac{1+2\sqrt{m+1}}{2}$ is the radius of convergence of $R_m(z)$. This estimate differs from the one presented by Nicaud, and exhibits a slightly faster convergence in experimental tests.
The generating function $L_m(u, z)$ for the number of alphabetic symbols in regular expressions satisfies the following:

$$L_m(u, z) = (2 + mu)z + zL_m(u, z) + 2zL_m(u, z)^2,$$

which yields

$$L_m(u, z) = \frac{1 - z - \sqrt{1 - 2z - (15 + 8mu)z^2}}{4z}.$$  

In order to obtain the generating function that gives the total number of alphabetic symbols occurring in all regular expressions of a given size, it is enough to evaluate

$$L_m(z) = \frac{\partial}{\partial u} \bigg|_{u=1} L_m(u, z) = \frac{mz}{\Delta_m(z)}.$$  

(14)

Analogously,

$$[z^n]L_m(z) \sim \frac{m}{2\sqrt{\rho_m\pi}\sqrt{2m + 4}n^{-(n-1)/2}}.$$  

(15)

In KAT expressions we can estimate both the number of test symbols in boolean expressions as well as the number of program symbols in KAT expressions. Let $T_l(z)$ and $P_{k,l}(z)$ be, respectively, their generating functions. We have, $T_l(z) = L_l(z)$ and $P_{k,l} = \frac{1}{k+l} L_{k+l}(z)$. Therefore, the probability, for a uniform distribution, that a symbol in a boolean expression of size $n$ is a test symbol is

$$\frac{[z^n]T_l(z)}{n[z^n]B_l(z)} \sim \frac{(4l + 8 - \sqrt{2l + 4}) l}{(15 + 8l)(l + 2)} \left(1 + \frac{1}{n}\right)^{3/2}$$  

(16)

and the probability that a symbol in a KAT expression of size $n$ is a program symbol is

$$\frac{[z^n]P_{k,l}(z)}{n[z^n]E_{k,l}(z)} \sim \frac{(4k + l + 8 - \sqrt{2(k + l) + 4}) k}{(15 + 8(k + l))(k + l + 2)} \left(1 + \frac{1}{n}\right)^{3/2} = \eta_{k,l,n}.$$  

(17)

The average number of test symbols in a boolean expression grows to about half of their size, as $l$ tends to $\infty$. The average number of program symbols for a growing value of $k + l$ tends to $\frac{2}{2(1+c)}$, where $c = \frac{3}{6}$. For instance, if $l = k$, $l = 2k$, and $l = \frac{2}{3}k$, this limit is, respectively, $\frac{1}{3}, \frac{1}{6}$, and $\frac{1}{4}$. Furthermore, for any ratio $c$, the asymptotic average number of states in Glushkov automata is less than half the size of the corresponding expressions.

**Average State Complexity of the Equation Automaton.** Now, we will estimate the average number of states of an equation automaton. Our approach follows the one for regular expressions presented by Broda et al. [BMMR11]. Considering Definition 6, one notes that whenever $1 \in \pi(e) \cap \pi(f)$, two states are merged into a single one in $\pi(e + f)$. Analogously, for $\pi(ef)$, when $1 \in \pi(e)$ and $f \in \pi(f)$. The following two grammars generate, respectively, KAT expressions $e_u$ such that $1 \in \pi(e_u)$, and $e_r$ such that $e_r \in \pi(e_r)$.

$$e_u := p \in P \mid e_u + e \mid e_u e \mid e_u u \mid e \cdot e_u \mid e_u \cdot 1$$

$$e_r := e_r e \mid e_r u.$$  

The generating function for $e_u$, $E_{u,k,l}$, satisfies

$$E_{u,k,l}(z) = k z + z E_{u,k,l}(z) E_{k,l}(z) + z (E_{k,l}(z) - E_{u,k,l}(z)) E_{u,k,l}(z) + z E_{k,l}(z) E_{u,k,l}(z) + z E_{u,k,l}(z),$$

from which one gets

$$E_{u,k,l}(z) = \frac{(z^2 + 3z E_{k,l}(z) - 1) + \sqrt{(z^2 + 3z E_{k,l}(z) - 1)^2 + 4k z^2}}{2z}.$$  

(18)
which is similar to the relation satisfied by the generating function for the set of regular expressions for which \( \varepsilon \in \pi(\alpha_x) \), considered in [BMMR11] (see equation (10) on p. 119).

On the other hand, the generating function for \( e_r, E_{r,k,l} \), satisfies

\[
E_{r,k,l}(z) = zE_{r,k,l}(z)E_{k,l}(z) + zE_{u,k,l}(z)
\]

from which one gets

\[
E_{r,k,l}(z) = \frac{zE_{u,k,l}(z)}{1 - zE_{k,l}(z)}.
\]

Putting

\[
i(p) = i(b) = 0
\]

\[
i(e_u + e_u) = i(e_u) + i(e_u) + 1
\]

\[
i(e_u + e_{\overline{r}}) = i(e_u) + i(e_{\overline{r}})
\]

\[
i(e_{\overline{r}} + e_r) = i(e_{\overline{r}}) + i(e_r) + 1
\]

\[
i(e_u + e_r) = i(e_u) + i(e_r)
\]

\[
i(e_{\overline{r}} + e) = i(e_{\overline{r}}) + i(e)
\]

\[
i(e^*) = i(e).
\]

The cumulative generating function of the mergings, \( I_k(z) \), can now be obtained from these equations...

\[
I_k(l) = \frac{zE_{u,k,l}(z)(E_{u,k,l}(z) + E_{r,k,l}(z))}{1 - 4zE_{k,l}(z) - z}
\]

Putting

\[
\Delta_{k,l}(z) = \Delta_l(z) - 8kz^2 = 1 - 2z - (15 + 8l + 8k)z^2,
\]

\[
\Lambda_{k,l}(z) = (z^2 + 3zE_{k,l}(z) - 1)^2 + 4kz^2,
\]

with the help of an algebraic manipulator, one gets

\[
I_k(l) = \frac{\left(\frac{1}{2(3z^2 + \sqrt{\Delta_{k,l}(z) - \Lambda_{k,l}(z)})}\right)^2}{\left(\frac{1}{\sqrt{\Delta_{k,l}(z) - \Lambda_{k,l}(z)}}\right)^2}
\]

### 5.1 Average Transition Complexity of the Glushkov Automaton

Now we compute an upper bound for the asymptotic average of the number of transitions in a Glushkov automaton with respect to the size of the corresponding KAT expression. As observed before, the number of transitions of \( A_{bas}(e) \) is, in the worst-case, quadratic in \( |e|p \). Below, we show that it is on average linear in \( |e| \). As the number of transitions must be at least equal to the number of states minus 1, on average, that number should be \( \Omega(n) \) for KAT expressions of size \( n \).

By Definition 5, the number of transitions is the sum of the sizes of the sets first and follow. In order to obtain a sufficiently accurate upper bound, we have to identify the KAT expressions \( e \) such that \( \text{out}(e) = 0 \). We begin to define the grammars that generate, respectively, “guaranteed” tautologies \( b_1 \), “guaranteed” falsities \( b_0 \), and, based on these, KAT expressions \( e_0 \) such that \( \text{out}(e_0) = 0 \). As usual, \( e_{\overline{r}} \) denotes an KAT expression that is not generated by this grammar, etc.

\[
\begin{align*}
b_1 & \rightarrow 1 \mid \neg b_0 \mid b_1 + b \mid b_{\overline{r}} + b_1 \mid b_1 \cdot b_1 \\
b_0 & \rightarrow 0 \mid \neg b_1 \mid b_0 + b_0 \mid b_0 \cdot b \mid b_{\overline{r}} \cdot b_0 \\
a_0 & \rightarrow p \mid a_0 + a_0 \mid a_0 + b_0 \mid b_0 + a_0 \mid a_0 \cdot a \mid a_{\overline{r}} \cdot a_0 \mid a_0 \cdot b \mid a_{\overline{r}} a_0 \mid b_0 \cdot a \mid b_{\overline{r}} \cdot a_0 \\
e_0 & \rightarrow b_0 \mid a_0
\end{align*}
\]
The corresponding generating functions $B_{1,l}(z)$, $B_{0,l}(z)$, $A_{0,k,l}(z)$, and $E_{0,k,l}(z)$ satisfy the following equations,

\[
\begin{align*}
B_{1,l}(z) &= z + zB_{0,l}(z) + 2zB_l(z)B_{1,l}(z) \\
B_{0,l}(z) &= z + zB_{1,l}(z) + 2zB_l(z)B_{0,l}(z) \\
A_{0,k,l}(z) &= kz + 2zA_{k,l}(z)B_{0,l}(z) + 2zA_{0,k,l}(z)B_{k,l}(z) + 2zA_{0,k,l}(z)A_{k,l}(z) \\
E_{0,k,l}(z) &= B_{0,l}(z) + A_{0,k,l}(z)
\end{align*}
\]

from which we obtain $B_{1,l}(z) = B_{0,l}(z) = \frac{B_l(z)}{(1+z)}$, $A_{0,k,l}(z) = \frac{kz + 2zA_{k,l}(z)A_{0,k,l}(z)}{1 - 2zE_{k,l}(z)}$. Finally the generating function for a lower bound of the number of expressions $e$ such that $\text{out}(e) = 0$ is

\[
E_{0,k,l}(z) = \frac{k(1+z)l + (1-2zB_l(z))B_l(z)}{(l+2)(1-2zE_{k,l}(z))}.
\]

Now, we can compute the generating functions of first($e$) and last($e$), $F_{k,l}(z)$ and $S_{k,l}(z)$, respectively, which coincide with the ones for regular expressions [Nic09], except that they depend on the function $E_{0,k,l}(z)$,

\[
F_{k,l}(z) = S_{k,l}(z) = \frac{kz}{1+z - 4zE_{k,l}(z) + zE_{0,k,l}(z)}.
\]

In Definition 3, follow($e$) is defined using non-disjoint unions for the case of $e^*$, and that does not allow an exact counting. Broda et al. [BMMR12b] presented a new recursive definition without non-disjoint unions which yielded an exact generating function for the number of transitions of the Glushkov automaton (for regular expressions). Since $E_{0,k,l}(z)$ corresponds to lower bounds for the number of expressions $e$ s.t. $\text{out}(e) = 0$, here we use a slightly simplified version and obtain an upper bound for the size of follow($e$). Our approximation $\text{Fol}(e)$ of the follow($e$) set is given by the recursive definition below, where there is no need to distinguish between $a$ and $e$ expressions. We have,

\[
\text{Fol}(p) = \text{Fol}(b) = \emptyset \\
\text{Fol}(e + f) = \text{Fol}(e) \cup \text{Fol}(f) \\
\text{Fol}(e \cdot f) = \text{Fol}(e) \cup \text{Fol}(f) \cup \text{last}(e) \otimes \text{first}(f) \\
\text{Fol}(e^*) = \text{Fol}^*(e) \\
\text{Fol}^*(b) = \emptyset \\
\text{Fol}^*(p) = \{(p, 1, p)\} \\
\text{Fol}^*(e + f) = \text{Fol}^*(e) \cup \text{Fol}^*(f) \cup \text{Cross}(e, f) \\
\text{Fol}^*(e \cdot f) = \text{Fol}^*(e) \cup \text{Fol}^*(f) \cup \text{Cross}(e, f) \\
\text{Fol}^*(e^*) = \text{Fol}^*(e),
\]

with $\text{Cross}(e, f) = \text{last}(e) \otimes \text{first}(f) \cup \text{last}(e) \otimes \text{first}(f)$. The corresponding generating functions satisfy the following equations,

\[
\begin{align*}
\text{Fol}_{k,l}(z) &= 4z\text{Fol}_{k,l}(z)E_{k,l}(z) + zF_{k,l}(z) + 2z\text{Fol}^*_{k,l}(z) \\
\text{Fol}^*_{k,l}(z) &= kz + 4z\text{Fol}^*_{k,l}(z)E_{k,l}(z) + 4zF_{k,l}(z) + 2z\text{Fol}^*_{k,l}(z).
\end{align*}
\]

Solving these, one gets

\[
\text{Fol}_{k,l}(z) = \frac{z \left( kz + F_{k,l}(z) \left( 1 + 3z - 4zE_{k,l}(z) \right) \right)}{1 - z - 8zE_{k,l}(z) + 4z^2E_{k,l}(z) + (4zE_{k,l}(z))^2}.
\]

By the definition of first($e$) it is straightforward to see that the size of this set is at most $|e|$.

Consequently, we can ignore the contribution of $F_{k,l}(z)$ in the computation of the upper bound for
the number of transitions. Concerning $F_{\text{ol}}(z)$ it is possible to see, with the help of an algebraic
symbolic manipulator, that this function has the form

$$F_{\text{ol}}(z) = \frac{U_{k,l}(z)}{V_{k,l}(z) \Delta_{k,l}(z)},$$

where $U_{k,l}(z), V_{k,l}(z)$ are defined in a neighbourhood of 0 with radius larger than $\rho_{k+l}$. This shows
that $z = \rho_{k+l}$ is the singularity of $F_{\text{ol}}(z)$ closest to the origin, and there is no other in the
circumference $|z| = \rho_{k+l}$. Using the same technique as exposed in [BMMR11, BMMR12a], one gets

$$[z^n]F_{\text{ol}}(z) \sim \frac{c_{k,l}}{\sqrt{\pi(2 - 2\rho_{k+l})}} \rho_{k+l}^{-n - 1/2},$$

(27)

where $c_{k,l}$ is a constant that depends on $k$ and $l$, through a rather complicated expression. It turns
out that $\lim_{k,l \to \infty} \frac{c_{k,l}}{\sqrt{k+l}} = \frac{17}{8} \sqrt{2} \simeq 3.182$.

Using now (13), one obtains

$$\frac{[z^n]F_{\text{ol}}(z)}{[z^n]E_{k,l}(z)} \sim \frac{4c_{k,l}}{\sqrt{2 - 2\rho_{k+l}}} \sqrt{2k + 2l + 4} \left(1 + \frac{1}{n}\right)^{3/2} n.$$  (28)

This means that the average number of transitions per automaton is approximately the size of
the original KAT expression.

References

[ABM12] Ricardo Almeida, Sabine Broda, and Nelma Moreira. Deciding KAT and Hoare logic
with derivatives. In Marco Faella and Aniello Murano, editors, Proc. 3rd GANDALF,

[BMMR11] Sabine Broda, António Machiavelo, Nelma Moreira, and Rogério Reis. On the average

[BMMR12a] Sabine Broda, António Machiavelo, Nelma Moreira, and Rogério Reis. An introduction
to descriptional complexity of regular languages through analytic combinatorics.

[BMMR12b] Sabine Broda, António Machiavelo, Nelma Moreira, and Rogério Reis. On the average

[CKS96] Ernie Cohen, Dexter Kozen, and Frederick Smith. The complexity of Kleene algebra
with tests. Technical Report TR96-1598, Computer Science Department, Cornell
University, 07 1996.

[CZ01] J. M. Champanraud and D. Ziadi. From Mirkin’s prebases to Antimirov’s word partial


1961.


