A Note on Sensor Location for Traffic Counting at Roundabouts: Solutions for a Particular Cost Function

Ana Paula Tomás

Technical Report Series: DCC-01-03

Departamento de Ciência de Computadores – Faculdade de Ciências

&

Laboratório de Inteligência Artificial e Ciência de Computadores

Universidade do Porto

Rua do Campo Alegre, 823 4150 Porto, Portugal

Tel: +351+2+6078830 – Fax: +351+2+6003654

http://www.ncc.up.pt/fcup/DCC/Pubs/treports.html

A Note on Sensor Location for Traffic Counting at Roundabouts — Solutions for a Particular Cost Function

Ana Paula Tomás*
DCC & LIACC, Univ. do Porto
R. do Campo Alegre, 823
4150-180 Porto, Portugal
apt@ncc.up.pt

March 2001 (rev. July 2001)

Abstract

We investigate the so-called *sensor location problem* to perform traffic counts at urban intersections so as to obtain OD data. A particular cost function is considered by which counting each of the directional flows $q_{i\,i+1}$ is given a relatively small cost. We show that the optimal solutions admit an exact characterization.

1 The Problem

A roundabout where n roads intersect may be identified by a string $R_1R_2\ldots R_n \in \{E,S,D\}^\star$ that we suppose to satisfy $R_1 \neq E$ and $R_n \neq S$, without loss of generality. All the strings obtained from $R_1R_2\ldots R_n$ by rotation denote exactly the same roundabout, but for different numerations of the roads. Each $R_i \in \{\mathrm{E},\mathrm{D},\mathrm{S}\}$ indicates whether road i is just an entry (E), just an exit (S) or both an entry and exit (D) road. The ordered sets $\mathcal{O}$ and $\mathcal{D}$, of origins and destinations, are defined as $\mathcal{O} = \{\imath_1,\ldots,\imath_e\}$ and $\mathcal{D} = \{\jmath_1,\ldots,\jmath_s\}$, respectively, with $\imath_e = n$ and $\jmath_1 = 1$.

The traffic flow from the entry i to the exit j is denoted by q_{ij}, for $i \in \mathcal{O}$ and $j \in \mathcal{D}$. In order to obtain origin-destination (OD) data, that is to determine all these flows q_{ij}, it is often necessary to carry out OD surveys. Manually recording of registration numbers or video surveys, may be thus required, in combination with some number plate tagging scheme, to obtain OD data.

The directional volumes are related to the total traffic volumes at entries, exits and passing through the cross-sections of the circulatory roadway in frontal alignment with the intersecting roads (respectively, O_i, D_j and F_k) by (1)–(3).

$$\sum_{j\in\mathcal{D}} q_{ij} = O_i, \text{ for } i \in \mathcal{O} \tag{1}$$

$$\sum_{i\in\mathcal{O}} q_{ij} = D_j, \text{ for } j \in \mathcal{D} \tag{2}$$

$$\sum_{i\in\mathcal{O}\setminus\{k\}} \; \sum_{j\in\mathcal{D},\; k\prec j\preceq i} q_{ij} = F_k, \text{ for } 1 \leq k \leq n \tag{3}$$

In (3), $j \in \mathcal{D},\ k \prec j \preceq i$ stands for the exits between road k and road i, being k excluded. In this report, we present the results from our investigation on the problem of finding the minimal number of exits and entries where OD surveys must be made, when some particular subset of traffic counts can be obtained at a relatively negligible cost. The major conclusion is the existence of an exact

*The work presented here has been partially supported by funds granted to LIACC through the *Programa de Financiamento Plurianual, Fundação para a Ciência e Tecnologia* and *Programa POSI*.

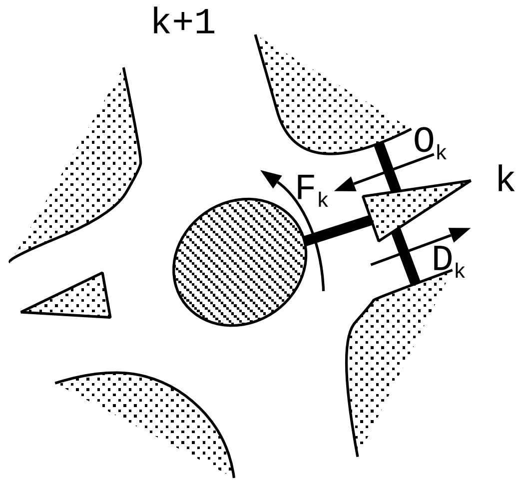


Figure 1: Measuring O_k, D_k and F_k

characterization of the minimal number for each type of hypothetical roundabout $R_1 R_2 \dots R_n$, when the following assumptions hold.

Assumptions.

1. Each of the required traffic counts can be obtained with accuracy during a certain period of time, meaning that the system that results from replacing the observed traffic volumes in (1)–(3) is consistent. This is a quite strong condition in real practice.

2. Counting the volumes O_i's, D_j's and F_k's is of negligible cost when compared to other measuring tasks, being supposed of null cost.

3. The directional flows q_{ij} can be fully observed by someone when the road j immediately follows road i, and thus measuring such traffic volumes is also assigned null cost.

Under this latter condition, it is possible in some situations to reduce the number of entries and exits where OD surveys must be performed, which is in general $(e-1)+(s-1)$. That is, the registration takes place at all the entries but one and at all the exits but one. Clearly, this reduction is achieved at the cost of having some of the $q_{i\,i+1}$ counted by other means (such as, an observer).

To simplify notation, we use $i+1$ to refer to the road that immediately follows road i, in the ordering naturally induced by the way traffic circulates in the roundabout, and $i-1$ to refer to the road that immediately precedes i. In particular, when $i=n$, the road $i+1$ is actually the road 1, whereas when $i=1$, the road $i-1$ is the road n.

2 The results – a catalog for hypothetical roundabouts

Tables 1 and 2 contain a summary of the results, which were obtained as explained in the following sections. It is tabulated the optimal cost for each type of roundabout $R_1 R_2 \dots R_n$, with e entries and s exits, and a given number of directional flows of type $q_{i\,i+1}$. Expressions of the form $(e-1)+(s-2)+2^\star$ and $(e-1)+(s-1)-1^\circ$ are used to represent costs. The intended meaning of $(e-1)+(s-2)+2^\star$ is that number plate recording takes place at $e-1$ entries and $s-2$ exits, and in addition one has to count two flows of type $q_{i\,i+1}$. Similarly, $(e-1)+(s-1)-1^\circ$ stands for number plate recording at $e-1$ entries and $s-1$ exits but, in this case, one does not need to measure one of the directional flows q_{ij}. In general, only particular combinations of entries, exits and flows $q_{i\,i+1}$ make an optimal solution. Details on the optimal solutions are given in Section 5.

e	s	$q_{i\,i+1}$	Pattern	Optimal Cost
≥ 5	≥ 5	1	$S^{s-1}(D+SE)E^{e-1}$	$(e-1)+(s-1)$
		≥ 2	—	$(e-1)+(s-1)-1^{\circ}$
4	≥ 5	1	$S^{s-1}(D+SE)EEE$	$(e-1)+(s-1)$
		2	$S^{k_1}(D+SE)S^{k_2}(D+SE)EE$ with $k_1+k_2\geq 3$	$(e-1)+(s-1)-1^{\circ}$
		2	$S^{k_1}(D+SE)ES^{k_2}(D+SE)E$ with $k_1+k_2\geq 3$	$(e-1)+(s-2)+2^{\star}$
		3	$S^{k_1}(D+SE)S^{k_2}(D+SE)S^{k_3}(D+SE)E$ with $k_1+k_2+k_3\geq 2$	$(e-1)+(s-2)+2^{\star}$
		4	$S^{k_1}(D+SE)S^{k_2}(D+SE)S^{k_3}(D+SE)S^{k_4}(SE+D)$ with $k_1+k_2+k_3+k_4\geq 1$	$(e-1)+(s-2)+2^{\star}$
3	≥ 5	1	$S^{s-1}(D+SE)EE$	$(e-1)+(s-1)$
		2	$S^{k_1}(D+SE)S^{k_2}(D+SE)E$ with $k_1+k_2\geq 3$	$(e-1)+(s-2)+1^{\star}$
		3	$S^{k_1}(D+SE)S^{k_2}(D+SE)S^{k_3}(D+SE)$ with $k_1+k_2+k_3\geq 2$	$(e-1)+(s-3)+3^{\star}$
2	≥ 5	1	$S^{s-1}(D+SE)E$	$(e-1)+(s-2)+1^{\star}$
		2	$S^{s-2}(D+SE)(D+SE)$	$(e-1)+(s-3)+1^{\star}$
		2	$SS^{k_1}(D+SE)SS^{k_2}(D+SE)$ with $k_1+k_2\geq 1$	$(e-1)+(s-4)+2^{\star}$

Table 1: Roundabouts with $s\geq 5$

e	s	$q_{i\,i+1}$	Pattern	Optimal Cost
4	4	4	$(D+SE)(D+SE)(D+SE)(D+SE)$	$(e-2)+(s-2)+4^\star$
4	4	3	$S^{k_1}(D+SE)S^{k_2}(D+SE)S^{k_3}(D+SE)E$ with $k_1+k_2+k_3=1$	$(e-1)+(s-2)+2^\star$ $(e-2)+(s-1)+2^\star$
4	4	2	$SS(D+SE)(D+SE)EE$	$(e-1)+(s-1)-1^\circ$
4	4	2	$S(D+SE)S(D+SE)EE$	$(e-2)+(s-1)+2^\star$
4	4	2	$(D+SE)SS(D+SE)EE$	$(e-1)+(s-1)-1^\circ$
4	4	2	$S(D+SE)ES(D+SE)E$	$(e-2)+(s-1)+2^\star$ $(e-1)+(s-2)+2^\star$
4	4	2	$SS(D+SE)E(D+SE)E$	$(e-1)+(s-2)+2^\star$
3	4	3	$S(D+SE)(D+SE)(D+SE)$	$(e-1)+(s-3)+3^\star$
3	4	2	$SS(D+SE)E^{k_1}(D+SE)E^{k_2}$ with $k_1+k_2=1$	$(e-1)+(s-2)+1^\star$
3	4	2	$S(D+SE)S(D+SE)E$	$(e-1)+(s-2)+1^\star$
3	3	3	$(D+SE)(D+SE)(D+SE)$	$3^\star$
3	3	2	$S(D+SE)(D+SE)E$	$(e-2)+(s-1)+1^\star$ $(e-1)+(s-2)+1^\star$
3	3	2	$(D+SE)S(D+SE)E$	$(e-2)+(s-2)+2^\star$
2	4	2	$S(D+SE)S(D+SE)$	$2^\star$
2	4	2	$SS(D+SE)(D+SE)$	$(e-1)+(s-3)+1^\star$
2	3	2	$S(D+SE)(D+SE)$	$1^\star$
2	2	2	$(D+SE)(D+SE)$	0

Table 2: Roundabouts with $e \leq s \leq 4$

The patterns of the hypothetical roundabouts are described by abbreviated regular expressions. We write $S^{s-1}(D+SE)E^{e-1}$ with $e \geq 5$ and $s \geq 5$, rather than $SSSSS^\star(D+SE)EEEEE^\star$, where $+$ denotes union (disjunction) and $\star$ is an abbreviation for zero ou more occurrences. Hence, we have, for example

$$S^{s-1}(D+SE)E^{e-1} \equiv S^{s-1}DE^{e-1} \ + \ S^{s-1}SEE^{e-1}$$

and

$$S^{k_1}(D+SE)S^{k_2}(D+SE)E \equiv S^{k_1}DS^{k_2}(D+SE)E \ + \ S^{k_1}SES^{k_2}(D+SE)E \equiv$$

$$\equiv S^{k_1}DS^{k_2}DE \ + \ S^{k_1}DS^{k_2}SEE \ + \ S^{k_1}SES^{k_2}DE \ + \ S^{k_1}SES^{k_2}SEE$$

being

$$S^{k_1}DS^{k_2}DE \equiv \underbrace{S\cdots S}_{k_1 \text{ times}} \ D \ \underbrace{S\cdots S}_{k_2 \text{ times}} \ DE$$

The number of patterns contained in Tables 1 and 2 is relatively small. This is in part due to the fact that all the strings obtained from $R_1R_2\ldots R_n$ by rotation denote the same roundabout. In this way, for instance, the expressions $SS(D+SE)E(D+SE)E$ and $(D+SE)ESS(D+SE)E$ denote exactly the same set of roundabouts.

$$SS(D+SE)E(D+SE)E \ \stackrel{\textbf{rot}}{\equiv} \ (D+SE)ESS(D+SE)E$$

In addition, there exists a kind of symmetry between certain pairs of patterns, which allows us to deduce the cost and optimal solutions for one case from those of the other.

This is, for example, the case of $(D+SE)EE^{k_2}(D+SE)EE^{k_1}$ and $SS^{k_1}(D+SE)SS^{k_2}(D+SE)$, with $k_1+k_2 \geq 1$. The optimal cost for all the roundabouts described by

$$SS^{k_1}(D+SE)SS^{k_2}(D+SE), \text{ with } k_1+k_2 \geq 1$$

is of the form $(e-1)+(s-4)+2^\star$, and from the results in Section 3, we may conclude that the optimal cost for

$$(D+SE)EE^{k_1}(D+SE)EE^{k_2}, \text{ with } k_1+k_2 \geq 1$$

is of the form $(e-4)+(s-1)+2^\star$. More accurately, it is show that if

$$(e_r - n_1) + (s_r - n_2) + n_3^\star$$

is the optimal costs for r then

$$(e_{\Psi(r)} - n_2) + (s_{\Psi(r)} - n_1) + n_3^\star$$

is the cost for $\Psi(r)$, being Ψ defined as follows.

The transformation.

$$\begin{array}{rcl}
\Psi(E) & = & S \\
\Psi(S) & = & E \\
\Psi(D) & = & D \\
\Psi(r^k) & = & (\Psi(r))^k, \ \text{ with } k \text{ a non-negative integer or } \star \\
\Psi(r+s) & = & \Psi(r)+\Psi(s), \ \text{ for all expressions } r \text{ and } s \\
\Psi(rs) & = & \Psi(s)\Psi(r), \ \text{ for all expressions } r \text{ and } s
\end{array}$$

Note that $\Psi(r)$ is the expression obtained by exchanging E with S in r and reading the result from right to left. Thus, $e_r = s_{\Psi(r)}$ and $s_r = e_{\Psi(r)}$, and it may be shown that both r and $\Psi(r)$ have the same number of flows $q_{i\,i+1}$.

For the example above we have the following.

$$\begin{aligned}\Psi(SS^{k_1}(D+SE)SS^{k_2}(D+SE)) &= (D+SE)E^{k_2}E(D+SE)E^{k_1}E \\ &\equiv (D+SE)EE^{k_2}(D+SE)EE^{k_1}\end{aligned}$$

The cases $e \geq 5$ and $s \leq e \leq 4$. The optimal costs for both the cases $e \geq 5$ and $s \leq e \leq 4$ can be obtained from those tabulated for $s \geq 5$ and $e \leq s \leq 4$, respectively. When the cost is given by

$$(e_r - 1) + (s_r - 1) - 1^\circ$$

then, for $\Psi(r)$, the cost is $(e_{\Psi(r)} - 1) + (s_{\Psi(r)} - 1) - 1^\circ$, which is in fact, $(s_r - 1) + (e_r - 1) - 1^\circ$.

It is important to remark that, besides the OD data that is collected by the number plate recording and some observation of directional flows $q_{i\,i+1}$, we need a given number of independent volumes O_i's, D_j's and F_k's. It can be seen that this number is $e+s-1$ if and only if the roundabout is described by $S^\star(D+SE)E^\star$, being $e+s$ in the remaining cases [5].

3 Solving techniques

Part of the work we developed to solve this problem was carried out in computer. It seems almost impossible to abstract the form of the solutions by analysing the output solutions. Because of that, we adopted a quite different approach, which follows straightforwardly from some mathematical properties of this problem. This results were found in a previous work, which is presented in [1, 5].

Since we want this report to be as much as possible self-contained, the main theoretical results used are briefly presented in this section. Some of them are shown in [5], whereas other simply state properties of the matrix of the Transportation Problems studied in Linear Programming (our reference book was [2]).

Property 1 *(shown in [5]) The matrix of the system defined by (1)–(3), in the variables q_{ij}, has rank $e+s$ if and only if none of equations in (3) is of the form $F_k = 0$, being $e+s-1$ otherwise. The matrix of the subsystem (1)–(2) has rank $e+s-1$. When (1)–(3) are consistent, the system is equivalent to the following subsystem, which is formed by equations (1)–(2) and any of the equations in (3).*

$$\begin{aligned}\sum_{j\in\mathcal{D}} q_{ij} &= O_i,\ \textit{for } i \in \mathcal{O} \\ \sum_{i\in\mathcal{O}} q_{ij} &= D_j,\ \textit{for } j \in \mathcal{D} \\ \sum_{i\in\mathcal{O}\setminus\{k\}} \ \sum_{j\in\mathcal{D},\ k\prec j\preceq i} q_{ij} &= F_k\end{aligned}$$

(Usually, we refer to the case $k=1$, where the last equation is the one defining F_1.) Any of the first $e+s$ equations is redundant. The last equation in the subsystem is redundant if and only if the system has rank $e+s-1$.

Under the hypothesis that the required traffic counts can be obtained with accuracy during a certain period of time, (1)–(3) admits a unique solution if $es-(e+s)$ of the q_{ij}'s are known and the columns of the system matrix associated to the remaining $e+s$ ones are linearly independent. Clearly, when the matrix has rank $e+s-1$, these numbers are $es-(e+s-1)$ and $e+s-1$, respectively.

Let $\mathbf{P}'$ be the matrix of the subsystem defined in Property 1, being $\mathbf{p}'_{ij}$ the column relating to the variable q_{ij}. Let $\mathbf{P}$ be the matrix of the subsystem consisting of equations (1)–(2) and $\mathbf{p}_{ij}$ be the column of q_{ij}. Clearly, $\mathbf{p}'_{ij}$ has just one more element than $\mathbf{p}_{ij}$, that is the coefficient of q_{ij} in the equation defining F_k. We denote such element by σ_{ij}.

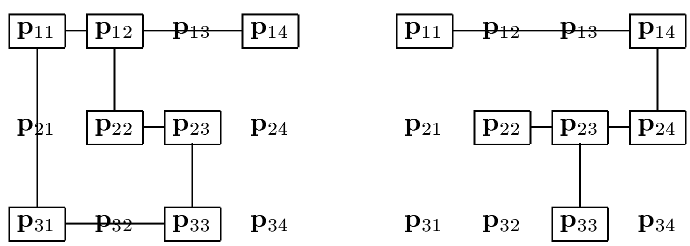
Property 2 below, which gives a simple graphical method to check the linear independence of a subset of columns of $\mathbf{P}$, refers to the following representation of the matrix $\mathbf{P}$ in tableau form.

$$\begin{array}{cccc}
\mathbf{p}_{\imath_1\jmath_1} & \mathbf{p}_{\imath_1\jmath_2} & \cdots & \mathbf{p}_{\imath_1\jmath_s} \\
\mathbf{p}_{\imath_2\jmath_1} & \mathbf{p}_{\imath_2\jmath_2} & \cdots & \mathbf{p}_{\imath_2\jmath_s} \\
\vdots & & & \vdots \\
\mathbf{p}_{\imath_e\jmath_1} & \mathbf{p}_{\imath_e\jmath_2} & \cdots & \mathbf{p}_{\imath_e\jmath_s}
\end{array}$$

Properties 2 and 3 state well known results of Transportation type problems in Linear Programming (as e.g.,[2]).

Property 2 *A subset $\mathcal{B}$ of the columns of $\mathbf{P}$ is free if and only if the graph G built in the following way is acyclic. The set of vertices correspond to the elements of $\mathcal{B}$, and the edges are obtained by linking each vertex in a given row (respectively, column) to the vertex in the same row (respectively, column) that is closer to that one in the tableau, if there is some. Moreover, if G is acyclic, connected and such that it contains at least a vertex in each column and in each row of the tableau, then $\mathcal{B}$ is a basis of the subspace spanned by the columns of $\mathbf{P}$.*

To illustrate this property, we analyse the following examples, where the columns in $\mathcal{B}$ are those in framed boxes. The conclusion is that the subset shown on the right is a basis of the subspace spanned by the columns of $\mathbf{P}$, whereas the one on the left is not even a free subset.



Property 3 *If $\mathcal{B}$ is a basis of the columns of $\mathbf{P}$, the linear combination of the columns in $\mathcal{B}$ that gives $\mathbf{p}_{ij} \notin \mathcal{B}$ is of the form*

$$\mathbf{p}_{ij} = \mathbf{p}_{ij_1} - \mathbf{p}_{i_1j_1} + \mathbf{p}_{i_1j_2} - \cdots - \mathbf{p}_{i_kj_k} + \mathbf{p}_{i_kj}$$

In terms of the tableau, $\mathbf{p}_{ij}$ and the columns in $\mathcal{B}$ that occur on the right-hand side form a cycle, consecutive edges being orthogonal.

In the example above on the right, $\mathbf{p}_{12} = \mathbf{p}_{14} - \mathbf{p}_{24} + \mathbf{p}_{22}$, whereas on the left, we have $\mathbf{p}_{11} = \mathbf{p}_{12} - \mathbf{p}_{22} + \mathbf{p}_{23} - \mathbf{p}_{33} + \mathbf{p}_{31}$. If we remove $\mathbf{p}_{11}$ from these selected columns, we also obtain a basis.

Property 4 gives an extension of the above results to $\mathbf{P}'$, being shown in [5].

Property 4 *Let $\mathcal{B}'$ be a subset of the columns of $\mathbf{P}'$ and $\mathcal{B}$ the set of the corresponding columns in $\mathbf{P}$. If some column $\mathbf{p}_{ij} \notin \mathcal{B}$ is written in an unique way as a combination of the columns in $\mathcal{B}$ by $\mathbf{p}_{ij} = \mathbf{p}_{ij_1} - \mathbf{p}_{i_1j_1} + \mathbf{p}_{i_1j_2} - \cdots - \mathbf{p}_{i_kj_k} + \mathbf{p}_{i_kj}$ then $\mathbf{p}'_{ij}$ is free relatively to $\mathcal{B}'$ if and only if*

$$\sigma_{ij} \neq \sigma_{ij_1} - \sigma_{i_1j_1} + \sigma_{i_1j_2} - \cdots - \sigma_{i_kj_k} + \sigma_{i_kj}$$

Thus, if $\mathcal{B}$ is not a free set, then $\mathcal{B}'$ is free if and only there exists some $\mathbf{p}_{ij}$ such that $\mathcal{B} \setminus \{\mathbf{p}_{ij}\}$ is free and the equality of the form $\mathbf{p}_{ij} = \mathbf{p}_{ij_1} - \mathbf{p}_{i_1j_1} + \mathbf{p}_{i_1j_2} - \cdots - \mathbf{p}_{i_kj_k} + \mathbf{p}_{i_kj}$ that expresses $\mathbf{p}_{ij}$ as a combination of the columns in $\mathcal{B} \setminus \{\mathbf{p}_{ij}\}$, does not hold for the corresponding columns in $\mathcal{B}'$. This means that the graph G associated to $\mathcal{B}$ can have at most a cycle when $\mathcal{B}'$ is free.

The method. Property 4 states the main result that supports our method. We shall now explain the fundamental ideas of that method by solving the problem for roundabout DDDSE. In this case $\mathcal{O} = \{1,2,3,5\}$ and $\mathcal{D} = \{1,2,3,4\}$, being $e = s = 4$. The tableau is shown below on the left, whereas on the right we give the coefficients σ_{ij}, assuming that the last equation in the subsystem is the one defining F_1. The positions of the variables $q_{i\,i+1}$ are marked with $\star$.

	1	2	3	4
1	$\mathbf{p}_{11}$	$\mathbf{p}_{12}$	$\mathbf{p}_{13}$	$\mathbf{p}_{14}$
2	$\mathbf{p}_{21}$	$\mathbf{p}_{22}$	$\mathbf{p}_{23}$	$\mathbf{p}_{24}$
3	$\mathbf{p}_{31}$	$\mathbf{p}_{32}$	$\mathbf{p}_{33}$	$\mathbf{p}_{34}$
5	$\mathbf{p}_{51}$	$\mathbf{p}_{52}$	$\mathbf{p}_{53}$	$\mathbf{p}_{54}$

	1	2	3	4
1	0	$\star$	0	0
2	0	1	$\star$	0
3	0	1	1	$\star$
5	$\star$	1	1	1

In this case, the rank of the system matrix is $e+s$, that is eight, which means that we can avoid counting eight of the q_{ij}'s. In order to discard the registration at entry i (respectively, exit j) we need to select all the flows entering from road i (respectively, exiting at road j), with the possible exception of those marked with $\star$, whose observation is supposed of null cost.

Thus, the problem may be seen as that of placing eight $\circ$'s in the given tableau, so as to cover, with $\circ$'s and $\star$'s, a maximum number of its columns and rows, while ensuring the linear independence of selected columns $\mathbf{p}'_{ij}$. One may conclude that there are just two optimal solutions, the optimal cost being $(e-2)+(s-2)+4^\star$.

	1	2	3	4
1	$\circ$	$\star$	$\circ$	$\circ$
2	$\circ$		$\star$	
3	$\circ$	$\circ$	$\circ$	$\star$
5	$\star$		$\circ$	

	1	2	3	4
1		$\star$		$\circ$
2	$\circ$	$\circ$	$\star$	$\circ$
3		$\circ$		$\star$
5	$\star$	$\circ$	$\circ$	$\circ$

In fact, we cannot cover exit 1 and exit 2 at the same time (neither both the entries 2 and 3), because of the following dependencies.

$$
\begin{array}{cccc}
\circ & \star & \mathbf{p}_{13} & \mathbf{p}_{14} \\
\circ & \circ & \mathbf{p}_{23} & \mathbf{p}_{24} \\
\circ & \circ & \mathbf{p}_{33} & \mathbf{p}_{34} \\
\star & \circ & \mathbf{p}_{53} & \mathbf{p}_{54}
\end{array}
\qquad
\begin{array}{ccccc}
\mathbf{p}_{11} & & \mathbf{p}_{12} & \mathbf{p}_{13} & \mathbf{p}_{14} \\
\mathbf{p}_{21} & - & \mathbf{p}_{22} & \mathbf{p}_{23} & \mathbf{p}_{24} \\
| & & | & & \\
\mathbf{p}_{31} & - & \mathbf{p}_{32} & \mathbf{p}_{33} & \mathbf{p}_{34} \\
\mathbf{p}_{51} & & \mathbf{p}_{52} & \mathbf{p}_{53} & \mathbf{p}_{54}
\end{array}
\qquad
\begin{array}{ccccc}
0 & & \star & 0 & 0 \\
0 & - & 1 & \star & 0 \\
| & & | & & \\
0 & - & 1 & 1 & \star \\
\star & & 1 & 1 & 1
\end{array}
$$

Neither can we cover both the exits 1 and 4 (nor both the entries 1 and 2), because of the dependencies shown below.

$$
\begin{array}{cccc}
\circ & \mathbf{p}_{12} & \mathbf{p}_{13} & \circ \\
\circ & \mathbf{p}_{22} & \mathbf{p}_{23} & \circ \\
\circ & \mathbf{p}_{32} & \mathbf{p}_{33} & \star \\
\star & \mathbf{p}_{52} & \mathbf{p}_{53} & \circ
\end{array}
\qquad
\begin{array}{cccc}
\mathbf{p}_{11} & -\cancel{\mathbf{p}_{12}}- & -\cancel{\mathbf{p}_{13}}- & \mathbf{p}_{14} \\
| & & & | \\
\mathbf{p}_{21} & -\cancel{\mathbf{p}_{22}}- & -\cancel{\mathbf{p}_{23}}- & \mathbf{p}_{24} \\
\mathbf{p}_{31} & \mathbf{p}_{32} & \mathbf{p}_{33} & \mathbf{p}_{34} \\
\mathbf{p}_{51} & \mathbf{p}_{52} & \mathbf{p}_{53} & \mathbf{p}_{54}
\end{array}
\qquad
\begin{array}{cccc}
0 & -\cancel{\star}- & -\cancel{0}- & 0 \\
| & & & | \\
0 & -\cancel{1}- & -\cancel{\star}- & 0 \\
0 & 1 & 1 & \star \\
\star & 1 & 1 & 1
\end{array}
$$

The same can be said about the exits 2 and 3 (and entries 3 and 5). By contrast, we can cover both the entries 1 and 3, because although $\mathbf{p}_{11} = \mathbf{p}_{13} - \mathbf{p}_{33} + \mathbf{p}_{31}$, we do have $\sigma_{11} \neq \sigma_{13} - \sigma_{33} + \sigma_{31}$.

$$
\begin{array}{cccc}
\mathbf{p}_{11} & -\cancel{\mathbf{p}_{12}}- & \mathbf{p}_{13} & \mathbf{p}_{14} \\
\cancel{\mathbf{p}_{21}} & \mathbf{p}_{22} & \cancel{\mathbf{p}_{23}} & \mathbf{p}_{24} \\
\mathbf{p}_{31} & -\cancel{\mathbf{p}_{32}}- & \mathbf{p}_{33} & \mathbf{p}_{34} \\
\mathbf{p}_{51} & \mathbf{p}_{52} & \mathbf{p}_{53} & \mathbf{p}_{54}
\end{array}
\qquad
\begin{array}{cccc}
0 & -\cancel{\star}- & 0 & 0 \\
\cancel{0} & 1 & \cancel{\star} & 0 \\
0 & -\cancel{1}- & 1 & \star \\
\star & 1 & 1 & 1
\end{array}
$$

By proceeding in a similar way, we see that it is possible to cover both the entries 1 and 3 (or, 2 and 5), and in addition both the exits 1 and 3 (or, 2 and 4), but only two of these four solutions are optimal.

Symmetry between solutions to r and $\Psi(r)$. By definition of Ψ, for each roundabout $r = R_1R_2 \ldots R_{n-1}R_n$, we have $\Psi(\Psi(r)) = r$, since

$$\Psi(R_1R_2 \ldots R_{n-1}R_n) = \Psi(R_n)\Psi(R_{n-1}) \ldots \Psi(R_2)\Psi(R_1) = R'_1R'_2 \ldots R'_{n-1}R'_n$$

with $R'_k = \Psi(R_{n-k+1})$, for all k. Therefore, the first exit of $\Psi(r)$ is $\Psi(R_n)$, and R_n is the last entry in r. Clearly, $\mathcal{O}_{\Psi(r)} = \{n - j + 1 \mid j \in \mathcal{D}_r\}$ and $\mathcal{D}_{\Psi(r)} = \{n - i + 1 \mid i \in \mathcal{O}_r\}$, where $\mathcal{O}_r$ and $\mathcal{D}_r$ denote the sets of entries and exits in the roundabout r, respectively.

As suggested by Figure 2, to each variable q_{ij} in the mathematical model of r, there exists a variable $q_{n-j+1\, n-i+1}$ in the model of $\Psi(r)$, and reciprocally. In particular, to each variable $q_{i\, i+1}$, there exists a variable $q_{n-i\, n-i+1}$ in the model of $\Psi(r)$, which means that for both models determine exactly the same number of $\star$'s.

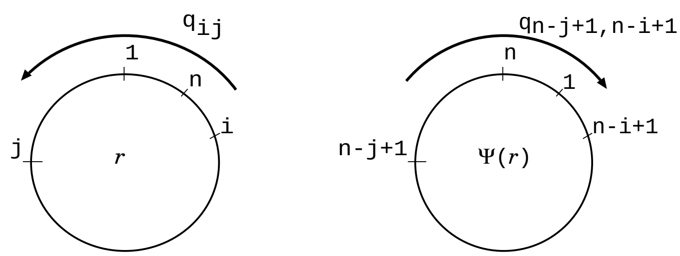


Figure 2: Aspects of the symmetries between r and $\Psi(r)$.

Naturally this symmetry extends to the constraints. For each $i \in \mathcal{O}$, the equation

$$\sum_{j \in \mathcal{D}} q_{ij} \quad = \quad O_i$$

corresponds to

$$\sum_{j \in \mathcal{D}} q_{n-j+1\, n-i+1} \quad = \quad D'_{n-i+1}$$

whereas, for each $j \in \mathcal{D}$, the equation

$$\sum_{i \in \mathcal{O}} q_{ij} \quad = \quad D_j$$

corresponds to

$$\sum_{i \in \mathcal{O}} q_{n-j+1\, n-i+1} \quad = \quad O'_{n-j+1}$$

which define the traffic volumes D'_{n-j+1} and O'_{n-i+1} exiting $\Psi(r)$ at road $n-i+1$ and entering from road $n - j + 1$, respectively. Similarly, for the frontal cross-sections, the equation

$$\sum_{i \in \mathcal{O}\setminus\{k\}} \quad \sum_{j \in \mathcal{D},\, k \prec j \preceq i} \quad q_{ij} \quad = \quad F_k,$$

gives that defining F'_{n-k+1} wrt $\Psi(r)$, that is

$$\sum_{i \in \mathcal{O}\setminus\{k\}} \quad \sum_{j \in \mathcal{D},\, k \prec j \preceq i} \quad q_{n-j+1\, n-i+1} \quad = \quad F'_{n-k+1},$$

meaning that the equations defining F_1 and F'_n are related. To illustrate, we analyse the roundabouts $SDESEE$ and $SSESDE$.

$SDESEE$

	1	2	4
2	0	1	0
3	0	1	$\star$
5	0	1	1
6	$\star$	1	1

wrt F_1

$SSESDE = \Psi(SDESEE)$

	1	2	4	5
3	1	1	$\star$	0
5	1	1	1	1
6	$\star$	0	0	0

wrt F'_n

$\xrightarrow{\text{that is}}$

5	0	1	0
4	0	1	$\star$
2	0	1	1
1	$\star$	1	1
	6	5	3

Proposition 1 states this natural relationship between the optimal solutions to the roundabouts r and $\Psi(r)$, which is just a consequence of r and $\Psi(r)$ being modelled by the same system of equations.

Proposition 1 *Let $\mathbf{P}'_r$ and $\mathbf{P}'_{\Psi(r)}$ be the matrices of the systems of equations defining the given mathematical models to the roundabouts r and $\Psi(r)$. Then, whichever $\mathcal{I}$ is, $\{\mathbf{p}'_{r,ij} \mid (i,j) \in \mathcal{I}\}$ is free if and only if $\{\mathbf{p}'_{\Psi(r),n-j+1\,n-i+1} \mid (i,j) \in \mathcal{I}\}$ is free.*

Each optimal solution to the roundabout r, for which recording is discarded at some subset $\mathcal{E} \subseteq \mathcal{O}_r$ and some subset $\mathcal{S} \subseteq \mathcal{D}_r$, determines an optimal solution to $\Psi(r)$, in which recording is discarded at $\{n-i+1 \mid i \in \mathcal{E}\} \subseteq \mathcal{D}_{\Psi(r)}$ and $\{n-j+1 \mid j \in \mathcal{S}\} \subseteq \mathcal{O}_{\Psi(r)}$. If the optimal cost wrt r is $(e_r - n_1) + (s_r - n_2) + n_3^\star$, the optimal cost for $\Psi(r)$ is given by $(e_{\Psi(r)} - n_2) + (s_{\Psi(r)} - n_1) + n_3^\star$. Moreover, if the flows $q_{i\,i+1}$, for i in a given set $\mathcal{M}$, are the n_3 directional flows that have to be measured to achieve that cost wrt r, then $q_{n-i\,n-i+1}$, with $i \in \mathcal{M}$, are the ones to measure in $\Psi(r)$.

4 Finding Patterns

By reasoning about the possible locations of the $\mathbf{p}_{ij}$'s in the tableau to get bases, we found an exact characterize of the optimal cost for all the roundabouts. An important feature is that when both $e \geq 5$ and $s \geq 5$, the optimal cost is necessarily $(e-1)+(s-1)$. Only in that way cycles as those illustrated in Fig. 3 are avoided, since there is at most one $q_{i\,i+1}$ in each line of the tableau. This

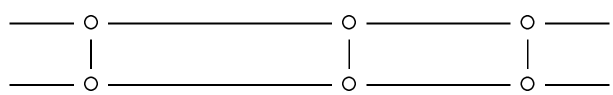
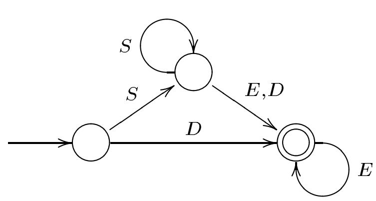


Figure 3: Example of linear dependence.

makes quite small the number of the relevant cases that we need to analyse. Distinguishing features are the numbers of entries, exits and flows $q_{i\,i+1}$, as well as the relative places of the $\mathbf{p}_{i\,i+1}$'s in the tableau. In order to obtain a coarse description of the classes by means of *regular expressions* we first design some *finite automata* (see e.g. [3]).

4.1 One $q_{i\,i+1}$

$S^\star(D+SE)E^\star$

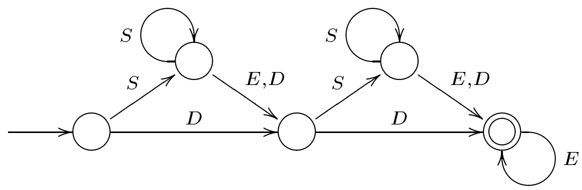


4.2 Two $\mathbf{q_{i\,i+1}}$'s

4.2.1 First $\star$ in the first entry

Number $q_{i\,i+1} = 2$, with $\imath_1 + 1 \in \mathcal{D}$

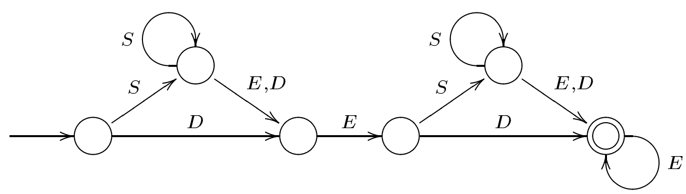
$$S^\star(D+SE)S^\star(D+SE)E^\star$$



4.2.2 First $\star$ in the second entry

Number $q_{i\,i+1} = 2$, with $\imath_1 + 1 \notin \mathcal{D}$, $\imath_2 + 1 \in \mathcal{D}$

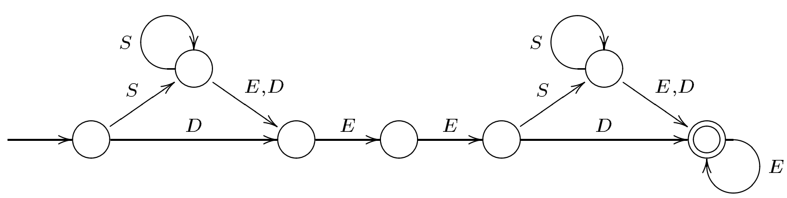
$$S^\star(D+SE)ES^\star(D+SE)E^\star$$



4.2.3 First $\star$ in the third entry

Number $q_{i\,i+1} = 2$, with $\imath_1 + 1 \notin \mathcal{D}$, $\imath_2 + 1 \notin \mathcal{D}$, $\imath_3 + 1 \in \mathcal{D}$

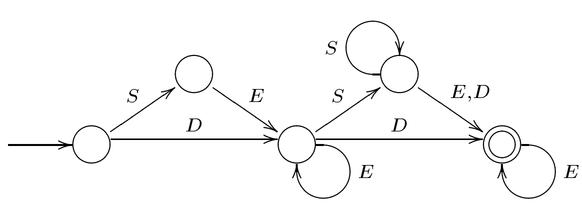
$$S^\star(D+SE)EES^\star(D+SE)E^\star$$



4.2.4 First $\star$ in the second exit

Number $q_{i\,i+1} = 2$, with $\jmath_2 - 1 \in \mathcal{O}$

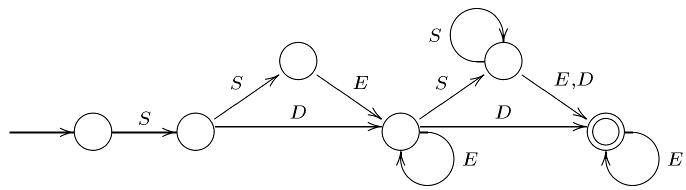
$$(D+SE)E^\star S^\star(D+SE)E^\star$$



4.2.5 First $\star$ in the third exit

Number $q_{i\,i+1} = 2$, with $\jmath_2 - 1 \notin \mathcal{O}$, $\jmath_3 - 1 \in \mathcal{O}$

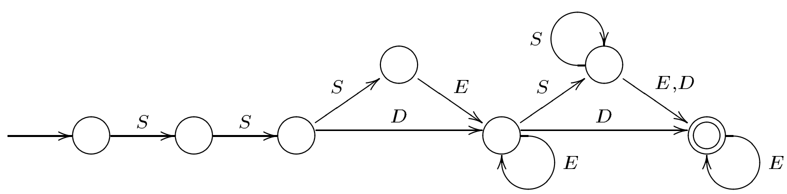
$$S(D+SE)E^\star S^\star(D+SE)E^\star$$



4.2.6 First $\star$ in the fourth exit

Number $q_{i\,i+1} = 2$, with $\jmath_2 - 1 \notin \mathcal{O}$, $\jmath_3 - 1 \notin \mathcal{O}$, $\jmath_4 - 1 \in \mathcal{O}$

$$SS(D+SE)E^\star S^\star(D+SE)E^\star$$

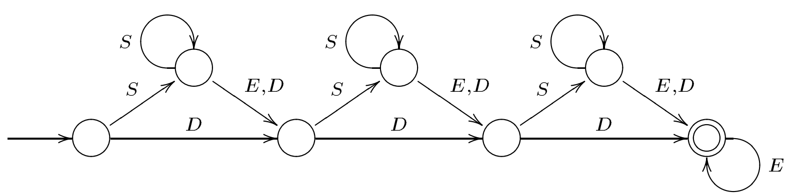


4.3 Three $q_{i\,i+1}$'s

4.3.1 First two $\star$ in the first and second entries

Number $q_{i\,i+1} = 3$, with $\imath_1 + 1 \in \mathcal{D}$ and $\imath_2 + 1 \in \mathcal{D}$

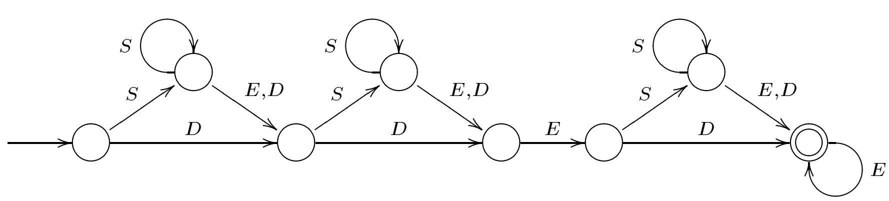
$$S^\star(D+SE)S^\star(D+SE)S^\star(D+SE)E^\star$$



4.3.2 First two $\star$ in the first and third entries

Number $q_{i\,i+1} = 3$, with $\imath_1 + 1 \in \mathcal{D}$, $\imath_2 + 1 \notin \mathcal{D}$ and $\imath_3 + 1 \in \mathcal{D}$

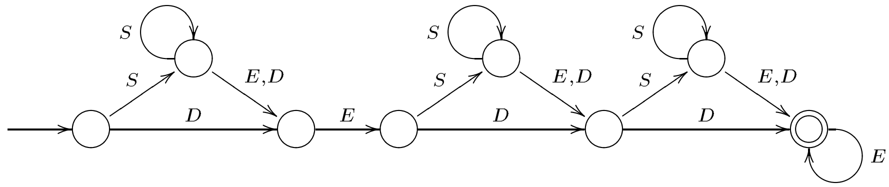
$$S^\star(D+SE)S^\star(D+SE)ES^\star(D+SE)E^\star$$



4.3.3 First two $\star$ in the second and third entries

Number $q_{i\,i+1} = 3$, with $\imath_1 + 1 \notin \mathcal{D}$, $\imath_2 + 1 \in \mathcal{D}$ and $\imath_3 + 1 \in \mathcal{D}$

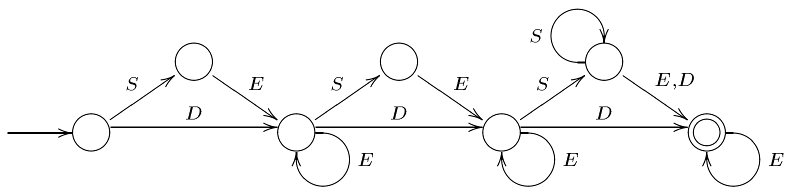
$$S^\star(D+SE)ES^\star(D+SE)S^\star(D+SE)E^\star$$



4.3.4 First two $\star$ in the second and third exits

Number $q_{i\,i+1} = 3$ with $\jmath_2 - 1 \notin \mathcal{O}$, $\jmath_3 - 1 \in \mathcal{O}$

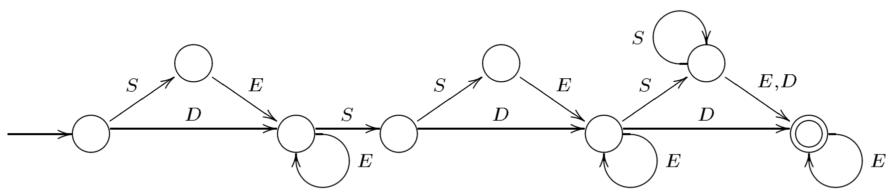
$$(D+SE)E^\star(D+SE)E^\star S^\star(D+SE)E^\star$$



4.3.5 First two $\star$ in the second and fourth exits

Number $q_{i\,i+1} = 3$ with $\jmath_2 - 1 \in \mathcal{O}$, $\jmath_3 - 1 \notin \mathcal{O}$, $\jmath_4 - 1 \in \mathcal{O}$

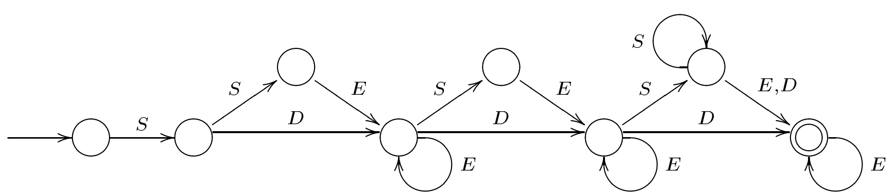
$$(D+SE)E^\star S(D+SE)E^\star S^\star(D+SE)E^\star$$



4.3.6 First two $\star$ in the third and fourth exits

Number $q_{i\,i+1} = 3$ with $\jmath_2 - 1 \notin \mathcal{O}$, $\jmath_3 - 1 \in \mathcal{O}$, $\jmath_4 - 1 \in \mathcal{O}$

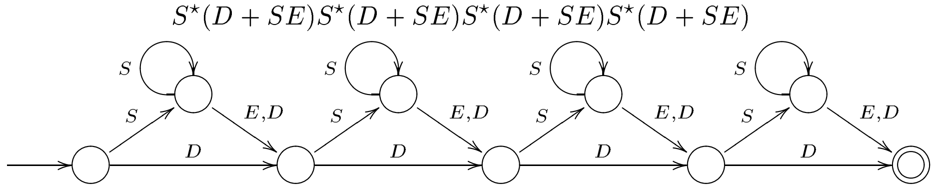
$$S(D+SE)E^\star(D+SE)E^\star S^\star(D+SE)E^\star$$



4.4 Four $q_{i\,i+1}$'s

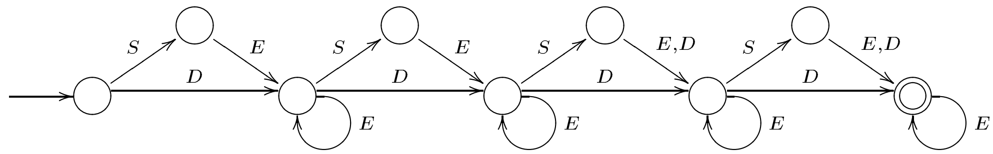
4.4.1 Four entries

$$S^\star(D+SE)S^\star(D+SE)S^\star(D+SE)S^\star(D+SE)$$



4.4.2 Four exits

$$(D + SE)E^\star(D + SE)E^\star(D + SE)E^\star(D + SE)E^\star$$



5 Cost Analysis

We now proceed by case analysis to find the optimal cost as well as optimal solutions for each class of roundabouts.

5.1 One $q_{i\,i+1}$

The rank of the system matrix is $e + s - 1$.

5.1.1 Case $e = 2$ and $s \geq 5$

$$S^{s-1}(D + SE)E, \;\; s \geq 5$$

	1		$\cdots$		$\jmath_s{-}1$	$\jmath_s$
$\jmath_s \leq \imath_1$	**0**	**1**	$\cdots$	1	1	1
$\imath_1{+}1$	$\star$	**1**	$\cdots$	1	1	1

cover at most one

	1		$\cdots$		$\jmath_s{-}1$	$\jmath_s$
$\imath_1$	$\circ$	$\circ$	$\cdots$	$\circ$	$\circ$	$\circ$
$\imath_2$	$\star$	$\circ$				

The cost is $(e - 1) + (s - 2) + 1^\star$.

Case $s = 2$ and $e \geq 5$. Described by $S(D+SE)E^{e-1}$ with $e \geq 5$. The cost is $(e-2)+(s-1)+1^\star$.

5.1.2 Case $e > 2$ and $s > 2$

$$S^{s-1}(D + SE)E^{e-1}, \;\; e > 2, \; s > 2$$

The cost is $(e - 1) + (s - 1)$.

	1		$\cdots$	$\jmath_s{-}1$	$\jmath_s$
$\jmath_s \leq \imath_1$	**0**	1	$\cdots$	1	1
$\imath_1{+}1$	**0**	1	$\cdots$	1	1
$\vdots$	$\vdots$				
$\imath_e$	$\star$	1	$\cdots$	1	1

only a row and a column

	1		$\cdots$	$\jmath_s - 1$	$\jmath_s$
$\imath_1$	$\circ$	$\circ$	$\cdots$	$\circ$	$\circ$
$\imath_2$	$\circ$				
$\vdots$	$\vdots$				
$\imath_e$	$\circ$				

5.2 Two $q_{i\,i+1}$'s

5.2.1 Case $e = 2$ and $s \geq 5$

$$S^{k_1}(D + SE)S^{k_2}(D + SE), \;\; k_1 + k_2 \geq 3$$

$SS^{k_1}(D+SE)SS^{k_2}(D+SE)$
where $k_1+k_2 \geq 1$

	1	$\cdots$	$\cdots$		$\imath_1{+}1$	$\cdots$	$\cdots$	$\jmath_s{-}1$	$\jmath_s$
$\imath_1$	**0**	**1**	1	1	$\star$	**0**	0	0	0
n	$\star$	**1**	1	1	**1**	**1**	1	1	1

	1	$\cdots$	$\cdots$		$\imath_1{+}1$	$\cdots$	$\cdots$	$\jmath_s{-}1$	$\jmath_s$
$\imath_1$	$\circ$	$\circ$	$\circ$	$\circ$	$\star$	$\circ$	$\circ$	$\circ$	$\circ$
n	$\star$	$\circ$			$\circ$	$\circ$			

Cost: $(e-1)+(s-4)+2^\star$

$S^{s-2}(D+SE)(D+SE)$
where $s \geq 5$

	1	$\cdots$	$\cdots$		$\imath_1{+}1$
$\imath_1$	**0**	**1**	1	1	$\star$
n	$\star$	**1**	1	1	**1**

	1	$\cdots$	$\cdots$		$\imath_1{+}1$
$\imath_1$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$
n	$\star$	$\circ$			$\circ$

Cost: $(e-1)+(s-3)+1^\star$

Case s = 2 and e ≥ 5. Described by

$$(D+SE)E^{k_1}(D+SE)E^{k_2}, \quad k_1+k_2 \geq 3$$

being

$$(D+SE)EE^\star(D+SE)EE^\star \;\rightarrow\; \text{Cost: } (e-4)+(s-1)+2^\star$$
$$(D+SE)(D+SE)E^\star \;\rightarrow\; \text{Cost: } (e-3)+(s-1)+1^\star$$

5.2.2 Case e = 3 and s ≥ 5

$$\left.\begin{array}{r} S^\star(D+SE)S^\star(D+SE)E \\ S^\star(D+SE)ES^\star(D+SE) \end{array}\right\} \text{ equivalent modulo rotation}$$

Let the pattern be $S^{k_1}(D+SE)S^{k_2}(D+SE)E$, with $k_1+k_2 \geq 3$.

	1	$\cdots$	$\cdots$		$\imath_1{+}1$	$\cdots$	$\cdots$	$\jmath_s$
$\imath_1$	**0**	?	$\cdots$	?	$\star$	0	$\cdots$	0
$\jmath_s \leq \imath_2$	**0**	1	1	1	**1**	1	$\cdots$	1
$\imath_2+1$	$\star$	1	1	1	**1**	1	$\cdots$	1

	1	$\cdots$	$\cdots$		$\imath_1{+}1$	$\cdots$	$\cdots$	$\jmath_s$
$\imath_1$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\cdots$	$\circ$
$\jmath_s \leq \imath_2$	$\circ$				$\circ$			
$\imath_2+1$	$\star$				$\circ$			

Cost: $(e-1)+(s-2)+1^\star$

In the table above, ? stands for 0 when $\imath_1 = 1$, being 1, otherwise. The same convention applies in the sequel.

Case s = 3 and e ≥ 5. Applies to $S(D+SE)E^{k_1}(D+SE)E^{k_2}$, with $k_1+k_2 \geq 3$, the cost being $(e-2)+(s-1)+1^\star$.

5.2.3 Case e = 4 and s ≥ 5

$$\left.\begin{array}{r} S^\star(D+SE)S^\star(D+SE)EE \\ S^\star(D+SE)EES^\star(D+SE) \end{array}\right\} \text{ equivalent}$$
$$S^\star(D+SE)ES^\star(D+SE)E$$

$S^{k_1}(D+SE)S^{k_2}(D+SE)EE$
with $k_1+k_2 \geq 3$ $\quad \rightarrow$ Cost: $(e-1)+(s-1)-1^{\circ}$

	1		$\cdots$		$\imath_1{+}1$		$\cdots$	$\jmath_s$
$\imath_1$	$\mathbf{0}$	?	$\cdots$	?	$\star$	0	$\cdots$	0
$\jmath_s \leq \imath_2$	$\mathbf{0}$	1	$\cdots$	1	1	1	$\cdots$	1
$\imath_2+1$	$\mathbf{0}$	1	$\cdots$	1	1	1	$\cdots$	1
$\imath_2+2$	$\star$	1	$\cdots$	1	1	1	$\cdots$	1

	1		$\cdots$		$\imath_1{+}1$		$\cdots$	$\jmath_s$
$\imath_1$	$\circ$	$\circ$	$\cdots$	$\circ$	$\circ$	$\circ$	$\cdots$	$\circ$
$\imath_2$	$\circ$							
$\imath_2{+}1$	$\circ$							
$\imath_2{+}2$	$\circ$							$\circ$

$S^{k_1}(D+SE)ES^{k_2}(D+SE)E$
with $k_1+k_2 \geq 3$ $\quad \rightarrow$ Cost: $(e-1)+(s-2)+2^{\star}$

	1		$\cdots$		$\imath_2{+}1$		$\cdots$	$\jmath_s$
$\imath_1$	$\mathbf{0}$	?	$\cdots$	?	$\mathbf{0}$	0	$\cdots$	0
$\imath_1{+}1 = \imath_2$	$\mathbf{0}$	1	$\cdots$	1	$\star$	0	$\cdots$	0
$\jmath_s \leq \imath_3$	$\mathbf{0}$	1	$\cdots$	1	$\mathbf{1}$	1	$\cdots$	1
$\imath_3{+}1 = \imath_4$	$\star$	1	$\cdots$	1	$\mathbf{1}$	1	$\cdots$	1

	1		$\cdots$		$\imath_2{+}1$		$\cdots$	$\jmath_s$
$\imath_1$	$\circ$				$\circ$			
$\imath_2$	$\circ$	$\circ$	$\cdots$	$\circ$	$\star$	$\circ$	$\cdots$	$\circ$
$\imath_3$	$\circ$				$\circ$			
$\imath_4$	$\star$				$\circ$			

Case s = 4 and e ≥ 5.

$$\left.\begin{array}{r} (D+SE)E^{\star}SS(D+SE)E^{\star} \\ SS(D+SE)E^{\star}(D+SE)E^{\star} \end{array}\right\} \text{equivalent}$$
$$S(D+SE)E^{\star}S(D+SE)E^{\star}$$

$SS(D+SE)E^{k_1}(D+SE)E^{k_2}$
with $k_1+k_2 \geq 3$ $\quad \rightarrow$ Cost: $(e-1)+(s-1)-1^{\circ}$

$S(D+SE)E^{k_1}S(D+SE)E^{k_2}$
with $k_1+k_2 \geq 3$ $\quad \rightarrow$ Cost: $(e-2)+(s-1)+2^{\star}$

5.2.4 Case e = 2 and s = 2

$$(SE+D)(SE+D)$$

The rank of the system matrix is 4, which equals the number of variables q_{ij}. The cost is 0.

5.2.5 Case e = 3 and s = 2

$$(SE+D)(SE+D)E$$

	1	$\imath_1{+}1$
$\imath_1$	$\mathbf{0}$	$\star$
$\imath_1{+}1$	$\mathbf{0}$	$\mathbf{1}$
$\imath_1{+}2$	$\star$	$\mathbf{1}$

Cover two columns

	1	$\imath_1{+}1$
$\imath_1$	$\circ$	$\circ$
$\imath_1{+}1$	$\circ$	$\circ$
$\imath_1{+}2$	$\star$	$\circ$

Single optimal solution

The optimal cost is $1^{\star}$.

Case e = 2 and s = 3. Described by $S(SE+D)(SE+D)$, being

$$\Psi(S(D+SE)(D+SE)) = (D+SE)(D+SE)E$$

and consequently, the optimal cost is $1^\star$.

5.2.6 Case e = 3 and s = 3

Described by the following expressions.

$$S(SE+D)(SE+D)E$$
$$(SE+D)S(SE+D)E$$

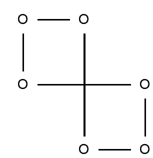
$S(SE+D)(SE+D)E$

	1	2	$\imath_1{+}1$
$2 \leq \imath_1$	**0**	1	$\star$
$\imath_2$	**0**	1	**1**
$\imath_3$	$\star$	1	**1**

	1	2	$\imath_1{+}1$
$\imath_1$	$\circ$	$\circ$	$\circ$
$\imath_2$	$\circ$		$\circ$
$\imath_3$	$\star$		$\circ$

The cost is $(e-2)+(s-1)+1^\star$. It is also, $(e-1)+(s-2)+1^\star$, since $S(SE+D)(SE+D)E$ is invariant under Ψ. Note that we cannot cover two rows and two columns at the same time.

$$\begin{array}{ccc} \mathbf{0} & \mathbf{1} & \star \\ \mathbf{0} & & \mathbf{1} \\ \star & \mathbf{1} & \mathbf{1} \end{array}$$



$$1 = 1 - 1 + 0 - 0 + 1$$

In $(SE+D)S(SE+D)E$, two rows and two columns can be covered.

	1	$\imath_1{+}1$	$\imath_1{+}2$
$\imath_1$	**0**	$\star$	**0**
$\imath_2$	**0**	**1**	1
$\imath_3$	$\star$	**1**	**1**

	1	$\imath_1{+}1$	$\imath_1{+}2$
$\imath_1$	$\circ$	$\star$	$\circ$
$\imath_2$	$\circ$	$\circ$	
$\imath_3$	$\star$	$\circ$	$\circ$

The cost is $(e-2)+(s-2)+2^\star$. Also $S(SE+D)(SE+D)E$ is invariant under Ψ.

5.2.7 Case e = 4 and s = 4

Described by the following expressions.

$$SS(SE+D)(SE+D)EE$$
$$S(SE+D)S(SE+D)EE$$
$$(SE+D)SS(SE+D)EE$$
$$S(SE+D)ES(SE+D)E$$
$$SS(SE+D)E(SE+D)E, \quad \text{which is equivalent to } (SE+D)ESS(SE+D)E$$

$$SS(SE+D)(SE+D)EE \quad \rightarrow \quad (e-1)+(s-1)-1^\circ$$

	1	2	3	$\imath_1{+}1$
$\imath_1$	**0**	**1**	**1**	$\star$
$\imath_1{+}1 \leq \imath_2$	**0**	1	1	1
$\imath_3$	**0**	1	1	1
$\imath_4$	$\star$	1	1	1

	1	2	3	$\imath_1{+}1$
$\imath_1$	$\circ$	$\circ$	$\circ$	$\circ$
$\imath_1{+}1 \leq \imath_2$	$\circ$			
$\imath_3$	$\circ$			
$\imath_4$	$\circ$			$\circ$

$$S(SE+D)S(SE+D)EE \quad \rightarrow \quad (e-2)+(s-1)+2^{\star}$$

	1	2	$\imath_1{+}1$	$\imath_1{+}2$
$\imath_1$	**0**	**1**	$\star$	**0**
$\imath_1{+}2 \leq \imath_2$	**0**	1	1	1
$\imath_3$	**0**	1	1	1
$\imath_4$	$\star$	**1**	**1**	**1**

	1	2	3	$\imath_1{+}1$
$\imath_1$	$\circ$	$\circ$	$\star$	$\circ$
$\imath_1{+}1 \leq \imath_2$	$\circ$			
$\imath_3$	$\circ$			
$\imath_4$	$\star$	$\circ$	$\circ$	$\circ$

$$(SE+D)SS(SE+D)EE \quad \rightarrow \quad (e-1)+(s-1)-1^{\circ}$$

	1	$\imath_1{+}1$	$\jmath_3$	$\jmath_4$
$\imath_1$	**0**	$\star$	**0**	**0**
$\jmath_4 \leq \imath_2$	**0**	1	1	1
$\imath_3$	**0**	1	1	1
$\imath_4$	$\star$	1	1	1

	1	$\imath_1{+}1$	$\jmath_3$	$\jmath_4$
$\imath_1$	$\circ$	$\circ$	$\circ$	$\circ$
$\imath_1{+}1 \leq \imath_2$	$\circ$			
$\imath_3$	$\circ$			
$\imath_4$	$\circ$			$\circ$

As concerns $S(SE+D)ES(SE+D)E)$, which is invariant under Ψ, the cost is given by $(e-2)+(s-1)+2^{\star}$ and $(e-1)+(s-2)+2^{\star}$.

	1	2	$\imath_2{+}1$	$\imath_2{+}2$
$2 \leq \imath_1$	0	1	0	0
$\imath_2$	0	1	$\star$	0
$\imath_3$	0	1	1	1
$\imath_4$	$\star$	1	1	1

	1	$\jmath_2$	$\jmath_3$	$\jmath_4$
$\imath_1$	$\circ$			
$\imath_2$	$\circ$	$\circ$	$\star$	$\circ$
$\imath_3$	$\circ$			
$\imath_4$	$\star$	$\circ$	$\circ$	$\circ$

	1	$\jmath_2$	$\jmath_3$	$\jmath_4$
$\imath_1$	$\circ$		$\circ$	
$\imath_2$	$\circ$		$\star$	
$\imath_3$	$\circ$		$\circ$	
$\imath_4$	$\star$	$\circ$	$\circ$	$\circ$

Either 2 columns or 2 rows $\quad (e-2)+(s-1)+2^{\star} \quad (e-1)+(s-2)+2^{\star}$

Finally, $SS(SE+D)E(SE+D)E$ is $\Psi(S(SE+D)S(SE+D)EE)$, and thus, the cost is given by $(e-1)+(s-2)+2^{\star}$.

5.2.8 Case e = 4 and s = 3

Described by the following expressions.

$$S(SE+D)(SE+D)EE$$
$$(SE+D)S(SE+D)EE$$
$$S(SE+D)E(SE+D)E, \quad \text{which is equivalent to } (SE+D)ES(SE+D)E$$

$$S(SE+D)(SE+D)EE \quad \rightarrow \quad \text{COST: } (e-2)+(s-1)+1^{\star}$$

	1	2	$\imath_1{+}1$
$\imath_1$	**0**	**1**	$\star$
$\imath_2$	0	1	1
$\imath_3$	0	1	1
$\imath_4$	$\star$	**1**	**1**

	1	2	$\imath_1{+}1$
$\imath_1$	$\circ$	$\circ$	$\circ$
$\imath_2$	$\circ$		
$\imath_3$	$\circ$		
$\imath_4$	$\star$	$\circ$	$\circ$

$$(SE+D)S(SE+D)EE \quad \rightarrow \quad \text{COST: } (e-2)+(s-1)+1^{\star}$$

	1	$\imath_1{+}1$	$\imath_1+2$
$\imath_1$	0	$\star$	0
$\imath_2$	0	1	1
$\imath_3$	0	1	1
$\imath_4$	$\star$	1	1

	1	$\imath_1{+}1$	$\imath_1+2$
$\imath_1$	$\circ$	$\star$	$\circ$
$\imath_2$	$\circ$		
$\imath_3$	$\circ$		
$\imath_4$	$\circ$	$\circ$	$\circ$

As concerns $S(SE+D)E(SE+D)E$, the cost is given by $(e-2)+(s-1)+1^{\star}$.

	1	2	$\imath_2$+1
$2 \le \imath_1$	0	1	0
$\imath_2$	0	1	$\star$
$\imath_3$	0	1	1
$\imath_4$	$\star$	1	1

	1	2	$\imath_2$+1
$\imath_1$	$\circ$		$\circ$
$\imath_2$	$\circ$	$\circ$	$\star$
$\imath_3$	$\circ$		$\circ$
$\imath_4$	$\star$		$\circ$

$(e-1)+(s-2)+2^{\star}$

	1	2	$\imath_2$+1
$\imath_1$			$\circ$
$\imath_2$	$\circ$	$\circ$	$\circ$
$\imath_3$			$\circ$
$\imath_4$	$\star$	$\circ$	$\circ$

$(e-2)+(s-1)+1^{\star}$

Case e = 3 and s = 4. The possible patterns and corresponding costs are as follows.

$$
\begin{array}{lcl}
SS(SE+D)(SE+D)E & \to & (e-1)+(s-2)+1^{\star} \\
(SE+D)SS(SE+D)E & \to & (e-1)+(s-2)+1^{\star} \\
S(SE+D)S(SE+D)E & \to & (e-1)+(s-2)+1^{\star}
\end{array}
$$

5.2.9 Case e = 4 and s = 2

Described by $(SE+D)(SE+D)EE$ and $(SE+D)E(SE+D)E$.

$$(SE+D)(SE+D)EE \quad \to \quad (e-3)+(s-1)+1^{\star}$$

	1	$\imath_1$+1
$\imath_1$	**0**	$\star$
$\imath_2$	**0**	**1**
$\imath_3$	0	1
$\imath_4$	$\star$	**1**

at most a column

	1	$\imath_1$+1
$\imath_1$	$\circ$	$\circ$
$\imath_2$	$\circ$	$\circ$
$\imath_3$	$\circ$	
$\imath_4$	$\star$	$\circ$

$$(SE+D)E(SE+D)E \quad \to \quad 2^{\star}$$

	1	$\imath_2$+1
$\imath_1$	**0**	**0**
$\imath_2$	**0**	$\star$
$\imath_3$	**0**	**1**
$\imath_4$	$\star$	**1**

two columns

	1	$\imath_1$+1
$\imath_1$	$\circ$	$\circ$
$\imath_2$	$\circ$	$\star$
$\imath_3$	$\circ$	$\circ$
$\imath_4$	$\star$	$\circ$

Case e = 2 and s = 4. The possibilities are as follows.

$$
\begin{array}{lcl}
SS(SE+D)(SE+D) & \to & (e-1)+(s-3)+1^{\star} \\
S(SE+D)S(SE+D) & \to & 2^{\star}
\end{array}
$$

5.3 Three $\mathbf{q_{i\,i+1}}$'s

5.3.1 Case $e=3$ and $s \ge 5$

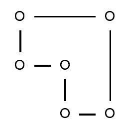
$$S^{k_1}(D+SE)S^{k_2}(D+SE)S^{k_3}(D+SE), \;\; k_1+k_2+k_3 \ge 2$$

	1		$\cdots$		$\imath_1{+}1$		$\cdots$		$\imath_2{+}1$		$\cdots$	$\jmath_s$
$\imath_1$	$\mathbf{0}$	?	$\cdots$	?	$\star$	0	$\cdots$	0	$\mathbf{0}$	0	$\cdots$	0
$\imath_2$	$\mathbf{0}$	1	$\cdots$	1	$\mathbf{1}$	1	$\cdots$	1	$\star$	0	$\cdots$	0
$\imath_3$	$\star$	1	$\cdots$	1	$\mathbf{1}$	1	$\cdots$	1	$\mathbf{1}$	1	$\cdots$	1

only one (columns 2 to $\imath_1$) — only one (columns $\imath_1+2$ to $\jmath_s$)

It is possible to cover three exits (columns) because

$$\begin{array}{ccc} 0 & \star & 0 \\ 0 & 1 & \star \\ \star & 1 & 1 \end{array}$$



$$0 \neq 0 - 0 + 1 - 1 + 1$$

The cost is $(e-1)+(s-3)+3^\star$, corresponding to the solution

	1		$\cdots$		$\imath_1{+}1$		$\cdots$		$\imath_2{+}1$		$\cdots$	$\jmath_s$
$\imath_1$	$\circ$	$\circ$	$\circ$	$\circ$	$\star$	$\circ$	$\cdots$	$\circ$	$\circ$	$\circ$	$\cdots$	$\circ$
$\imath_2$	$\circ$				$\circ$				$\star$			
$\imath_3$	$\star$				$\circ$				$\circ$			

Case $s=3$ and $e \geq 5$. Described by

$$(D+SE)E^{k_1}(D+SE)E^{k_2}(D+SE)E^{k_3}, \;\; k_1+k_2+k_3 \geq 2$$

being the cost given by $(e-3)+(s-1)+3^\star$.

5.3.2 Case $e=4$ and $s \geq 5$

$$\left.\begin{array}{l} S^\star(D+SE)S^\star(D+SE)S^\star(D+SE)E \\ S^\star(D+SE)S^\star(D+SE)ES^\star(D+SE) \\ S^\star(D+SE)ES^\star(D+SE)S^\star(D+SE) \end{array}\right\} \text{ equivalent}$$

Let the pattern be $S^{k_1}(D+SE)S^{k_2}(D+SE)S^{k_3}(D+SE)E$, with $k_1+k_2+k_3 \geq 2$.

	1		$\cdots$		$\imath_1{+}1$		$\cdots$		$\imath_2{+}1$		$\cdots$	$\jmath_s$
$\imath_1$	$\mathbf{0}$	?	$\cdots$	?	$\star$	0	$\cdots$	0	$\mathbf{0}$	0	$\cdots$	0
$\imath_2$	$\mathbf{0}$	1	$\cdots$	1	1	1	$\cdots$	1	$\star$	0	$\cdots$	0
$\jmath_s \leq \imath_3$	$\mathbf{0}$	1	$\cdots$	1	1	1	$\cdots$	1	$\mathbf{1}$	1	$\cdots$	1
$\imath_4$	$\star$	1	$\cdots$	1	1	1	$\cdots$	1	$\mathbf{1}$	1	$\cdots$	1

can cover only one (columns 2 to $\jmath_s$)

The cost is $(e-1)+(s-2)+2^\star$.

	1		$\cdots$		$\imath_1{+}1$		$\cdots$		$\imath_2{+}1$		$\cdots$	$\jmath_s$
$\imath_1$	$\circ$	$\circ$	$\cdots$	$\circ$	$\circ$	$\circ$	$\cdots$	$\circ$	$\circ$	$\circ$	$\cdots$	$\circ$
$\imath_2$	$\circ$								$\star$			
$\jmath_s \leq \imath_3$	$\circ$								$\circ$			
$\imath_4$	$\star$								$\circ$			

Case $s = 4$ and $e \geq 5$. The cost is $(e-2)+(s-1)+2^\star$, the possible patterns being as follows.

$$\left.\begin{array}{l} (D+SE)E^\star(D+SE)E^\star S(D+SE)E^\star \\ (D+SE)E^\star S(D+SE)E^\star(D+SE)E^\star \\ S(D+SE)E^\star(D+SE)E^\star(D+SE)E^\star \end{array}\right\}\ \text{equivalent}$$

5.3.3 Case e = 4 and s = 3

$$(D+SE)(D+SE)(D+SE)E$$

	1	$\imath_1{+}1$	$\imath_2{+}1$
$\imath_1$	$\mathbf{0}$	$\star$	$\mathbf{0}$
$\imath_2$	$\mathbf{0}$	$\mathbf{1}$	$\star$
$\imath_2{+}1 \leq \imath_3$	0	1	1
$\imath_3{+}1$	$\star$	$\mathbf{1}$	$\mathbf{1}$

	1	$\imath_1{+}1$	$\imath_2{+}1$
$\imath_1$	$\circ$	$\star$	$\circ$
$\imath_2$	$\circ$	$\circ$	$\star$
$\imath_2{+}1 \leq \imath_3$		$\circ$	
$\imath_3{+}1$	$\star$	$\circ$	$\circ$

The optimal cost is $(e-3)+(s-1)+3^\star$.

Case e = 3 and s = 4. Defined by

$$S(D+SE)(D+SE)(D+SE)$$

with

$$\Psi(S(D+SE)(D+SE)(D+SE)) = (D+SE)(D+SE)(D+SE)E$$

so that, the optimal cost is $(e-1)+(s-3)+3^\star$.

5.3.4 Case e = 4 and s = 4

The expression describing the case of three $q_{i\,i+1}$'s and four entries is

$$S^\star(D+SE)S^\star(D+SE)S^\star(D+SE)E$$

so that, when $s = 4$, we have

$$\begin{array}{c} S(D+SE)(D+SE)(D+SE)E \\ (D+SE)S(D+SE)(D+SE)E \\ (D+SE)(D+SE)S(D+SE)E \end{array}$$

$S(D+SE)(D+SE)(D+SE)E$

	1	$\jmath_2$	$\imath_1{+}1$	$\imath_2{+}1$
$\jmath_2{\leq}\imath_1$	$\mathbf{0}$	1	$\star$	$\mathbf{0}$
$\imath_2$	$\mathbf{0}$	1	1	$\star$
$\imath_3$	$\mathbf{0}$	1	1	$\mathbf{1}$
$\imath_3{+}1$	$\star$	1	1	$\mathbf{1}$

	1	$\jmath_2$	$\imath_1{+}1$	$\imath_2{+}1$
$\jmath_2{\leq}\imath_1$	$\mathbf{0}$	$\mathbf{1}$	$\star$	$\mathbf{0}$
$\imath_2$	0	1	1	$\star$
$\imath_3$	0	1	1	1
$\imath_3{+}1$	$\star$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$

cover two columns

	1	$\jmath_2$	$\imath_1{+}1$	$\imath_2{+}1$
$\jmath_2{\leq}\imath_1$	$\circ$	$\circ$	$\circ$	$\circ$
$\imath_2$	$\circ$			$\star$
$\imath_3$	$\circ$			$\circ$
$\imath_3{+}1$	$\star$			$\circ$

cover two rows

	1	$\jmath_2$	$\imath_1{+}1$	$\imath_2{+}1$
$\jmath_2{\leq}\imath_1$	$\circ$	$\circ$	$\star$	$\circ$
$\imath_2$	$\circ$			
$\imath_3$	$\circ$			
$\imath_3{+}1$	$\star$	$\circ$	$\circ$	$\circ$

The optimal cost is $(e-1)+(s-2)+2^\star$ or also $(e-2)+(s-1)+2^\star$. Being

$$\Psi(S(D+SE)(D+SE)(D+SE)E) = S(D+SE)(D+SE)(D+SE)E$$

this pattern is invariant under Ψ.

$$(D+SE)S(D+SE)(D+SE)E$$

	1	$\imath_1$+1	$\imath_1$+2	$\imath_2$+1
$\imath_1$	**0**	$\star$	0	**0**
$\imath_2$	**0**	1	1	$\star$
$\imath_3$	**0**	1	1	**1**
$\imath_3$+1	$\star$	1	1	**1**

	1	$\imath_1$+1	$\imath_1$+2	$\imath_2$+1
$\imath_1$	**0**	$\star$	**0**	**0**
$\imath_2$	**0**	**1**	**1**	$\star$
$\imath_3$	0	1	1	1
$\imath_3$+1	$\star$	1	1	1

cover two columns

	1	$\jmath_2$	$\imath_1$+1	$\imath_2$+1
$\jmath_2\leq\imath_1$	$\circ$	$\circ$	$\circ$	$\circ$
$\imath_2$	$\circ$			$\star$
$\imath_3$	$\circ$			$\circ$
$\imath_3$+1	$\star$			$\circ$

cover two rows

	1	$\jmath_2$	$\imath_1$+1	$\imath_2$+1
$\jmath_2\leq\imath_1$	$\circ$	$\star$	$\circ$	$\circ$
$\imath_2$	$\circ$	$\circ$	$\circ$	$\star$
$\imath_3$	$\circ$			
$\imath_3$+1	$\circ$			

The optimal cost is $(e-1)+(s-2)+2^\star$ or also $(e-2)+(s-1)+2^\star$. Note that this pattern is invariant under Ψ.

$$\begin{aligned}\Psi((D+SE)S(D+SE)(D+SE)E) &= S(D+SE)(D+SE)E(D+SE)\\ &\equiv (D+SE)S(D+SE)(D+SE)E\end{aligned}$$

For the last case $(D+SE)(D+SE)S(D+SE)E$, the cost is similar: $(e-1)+(s-2)+2^\star$ or also $(e-2)+(s-1)+2^\star$.

$$(D+SE)(D+SE)S(D+SE)E$$

	1	$\imath_1$+1	$\imath_2$+1	$\imath_2$+2
$\imath_1$	0	$\star$	0	0
$\imath_2$	0	1	$\star$	0
$\imath_2+2\leq\imath_3$	0	1	1	1
$\imath_3$+1	$\star$	1	1	1

$$\begin{aligned}\Psi((D+SE)(D+SE)S(D+SE)E) &= S(D+SE)E(D+SE)(D+SE)\\ &\equiv (D+SE)(D+SE)S(D+SE)E\end{aligned}$$

5.4 Four $\mathbf{q_{i\,i+1}}$'s

5.4.1 Case $\mathbf{e=4}$ and $\mathbf{s\geq 5}$

$$S^{k_1}(D+SE)S^{k_2}(D+SE)S^{k_3}(D+SE)S^{k_4}(D+SE),\quad k_1+k_2+k_3+k_4\geq 1$$

	1		$\cdots$		$\imath_1$+1		$\cdots$		$\imath_2$+1				$\imath_3$+1		$\cdots$	$\jmath_s$
$\imath_1$	**0**	?	$\cdots$	?	$\star$	0	$\cdots$	0	**0**	0	$\cdots$	0	0	0	$\cdots$	0
$\imath_2$	**0**	1	$\cdots$	1	1	1	$\cdots$	1	$\star$	0	$\cdots$	0	0	0	$\cdots$	0
$\imath_3$	**0**	1	$\cdots$	1	1	1	$\cdots$	1	**1**	1	$\cdots$	1	$\star$	0	$\cdots$	0
$\jmath_s\leq\imath_4$	$\star$	1	$\cdots$	1	1	1	$\cdots$	1	**1**	1	$\cdots$	1	1	1	$\cdots$	1

can cover only one　　　can cover only one

The following is an optimal solution (there is another one).

	1		$\cdots$		$\imath_1+1$		$\cdots$		$\imath_2+1$				$\imath_3+1$		$\cdots$	$\jmath_s$
$\imath_1$	$\circ$	$\circ$	$\cdots$	$\circ$	$\circ$	$\circ$	$\cdots$	$\circ$	$\circ$	$\circ$	$\cdots$	$\circ$	$\circ$	$\circ$	$\cdots$	$\circ$
$\imath_2$	$\circ$								$\star$							
$\imath_3$	$\circ$								$\circ$							
$\jmath_s \leq \imath_4$	$\star$								$\circ$							

Optimal cost: $(e-1)+(s-2)+2^{\star}$.

Case $\mathbf{s = 4}$ and $\mathbf{e \geq 5}$. Given by

$$(D+SE)E^{k_1}(D+SE)E^{k_2}(D+SE)E^{k_3}(D+SE)E^{k_4}, \quad k_1+k_2+k_3+k_4 \geq 1$$

the optimal cost being $(e-2)+(s-1)+2^{\star}$.

5.4.2 Case $\mathbf{e = 4}$ and $\mathbf{s = 4}$

$$(SE+D)(SE+D)(SE+D)(SE+D)$$

	1	$\imath_1+1$	$\imath_2+1$	$\imath_3+1$
$\imath_1$	0	$\star$	0	0
$\imath_2$	0	1	$\star$	0
$\imath_3$	0	1	1	$\star$
$\imath_4$	$\star$	1	1	1

Optimal cost: $(e-2)+(s-2)+4^{\star}$. Can select either $(\imath_1, \imath_3)$ or $(\imath_2, \imath_4)$, and either $(\jmath_1, \jmath_3)$ or $(\jmath_2, \jmath_4)$. The rank is 8, which equals the number of $\circ$ that can be used to cover rows or columns. There exist just two optimal solutions.

	$\jmath_1$	$\jmath_2$	$\jmath_3$	$\jmath_4$
$\imath_1$	$\circ$	$\star$	$\circ$	$\circ$
$\imath_2$	$\circ$		$\star$	
$\imath_3$	$\circ$	$\circ$	$\circ$	$\star$
$\imath_4$	$\star$		$\circ$	

	$\jmath_1$	$\jmath_2$	$\jmath_3$	$\jmath_4$
$\imath_1$		$\star$		$\circ$
$\imath_2$	$\circ$	$\circ$	$\star$	$\circ$
$\imath_3$		$\circ$		$\star$
$\imath_4$	$\star$	$\circ$	$\circ$	$\circ$

6 Conclusions

Work proceeds to model a similar but rather more general problem, that of solving minimal cost sensor location when some other movements and possibly not all the $q_{i\,i+1}$'s are thought of negligible cost. Our aim is to model it as a Constraint Satisfaction Problem in Finite Domains (e.g. [4] for some bibliographic references).

The fact that we had spent some time showing that the minimal cost in this specific case admits an exact characterization gave us a deep insight on the problem structure which is proving of great help to proceed our work.

References

[1] Andrade M.: *Métodos e Técnicas de Recolha de Dados de Tráfego – Algoritmo para a Definição da Matriz Origem-Destino.* Msc. Thesis, Faculdade de Engenharia da Universidade do Porto, 2000.

[2] Hadley, G.: *Linear Programming.* Addison-Wesley, 1969.

[3] Hopcroft J. E., Ullman J. D.: *Introduction to Automata Theory, Languages, and Computation*, Addison-Wesley, 1979.

[4] Marriott K., and Stuckey P.: *Programming with Constraints – An Introduction*, The MIT Press, 1998.

[5] Tomás A. P., Andrade M., and Pires da Costa A.: Obtaining Origin-Destination Data at Optimal Cost at Urban Roundabouts, Internal Report, DCC-FC & LIACC, Universidade do Porto, 2001 (in preparation).