Arc-flow Formulation for Vector Packing

FACULDADE DE CIÊNCIAS
UNIVERSIDADE DO PORTO

Filipe Brandão and João Pedro Pedroso

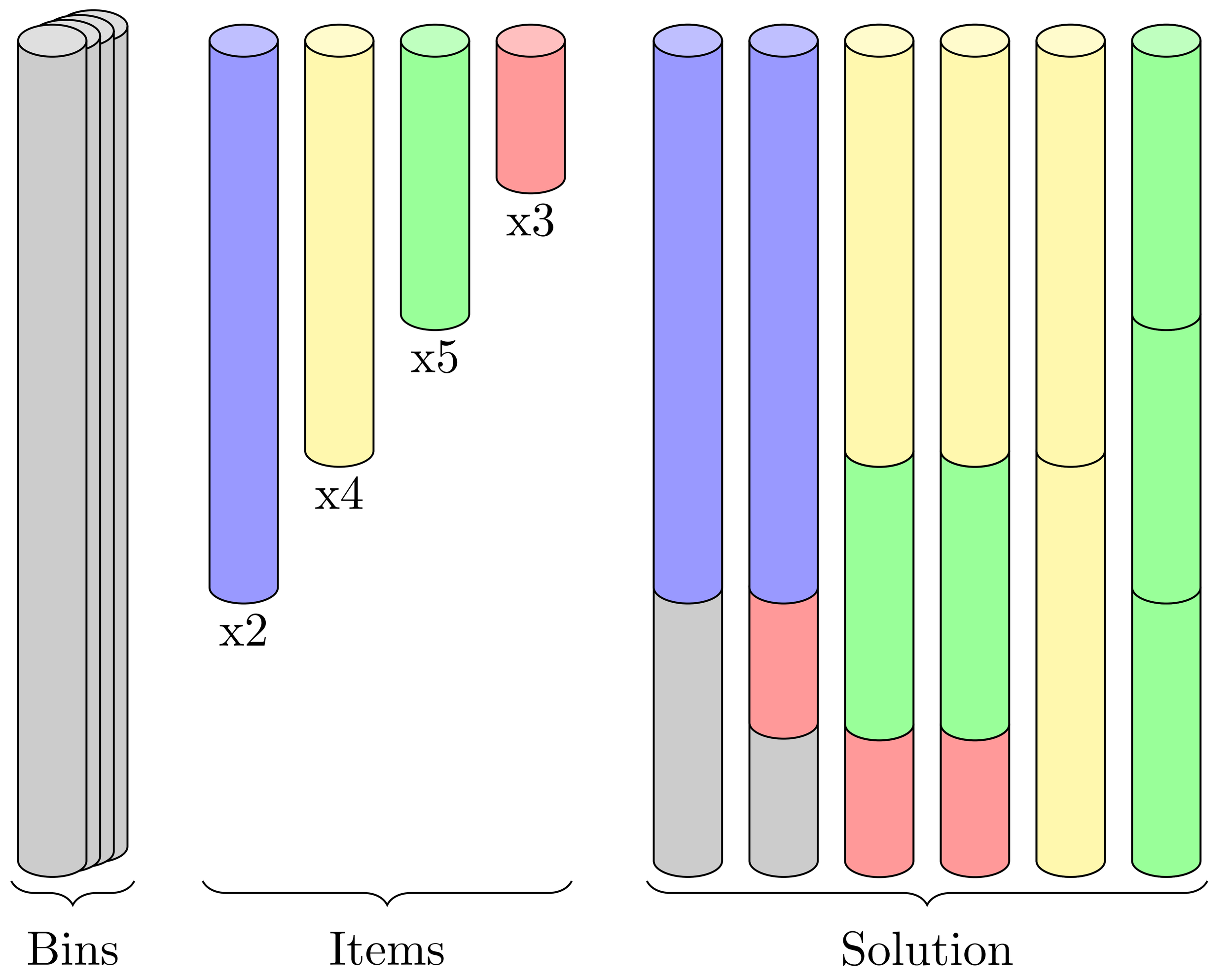
U.PORTO

I. Contribution

Exact method, based on an arc-flow formulation, for solving bin packing and cutting stock problems including multi-constraint variants.

II. Bin Packing/Cutting Stock

Objective: Pack a set of items into as few bins as possible



III. p-dimensional Vector packing

- **Bin packing with multiple constraints**
- Pack n items of m different types, represented by p-dimensional vectors, into as few bins as possible.

IV. Assignment-based model

$$\min \sum_{j=1}^{n} y_j$$

$$\text{s.t.} \quad \sum_{j=1}^{n} x_{ij} \geq b_i, \qquad i = 1..m,$$

$$\sum_{i=1}^{m} w_i^k x_{ij} \leq y_j W^k, \qquad j = 1..n,\ k = 1..p,$$

$$y_j \in \{0,1\}, \qquad j = 1..n,$$

$$x_{ij} \geq 0, \text{ integer}, \qquad i = 1..m,\ j = 1..n,$$

where w_i and b_i are the weight vector and demand of items of type i, and W is the capacity vector. The variables are y_j, which is 1 if bin j is used and 0 otherwise, and x_{ij}, the number of times item i is assigned to bin j.

- Highly symmetric
- Very weak linear relaxation

V. Gilmore-Gomory's model

Let column vectors $a^j = (a_1^j, \dots, a_m^j)^\top$ represent all possible cutting patterns j. The element a_i^j represents the number of items of type i in pattern j. Let x_j be a decision variable for the number of times pattern j is used.

$$\min \sum_{j \in J} x_j$$

$$\text{s.t.} \quad \sum_{j \in J} a_i^j x_j \geq b_i, \qquad i = 1..m,$$

$$x_j \geq 0, \text{ integer}, \qquad \forall j \in J,$$

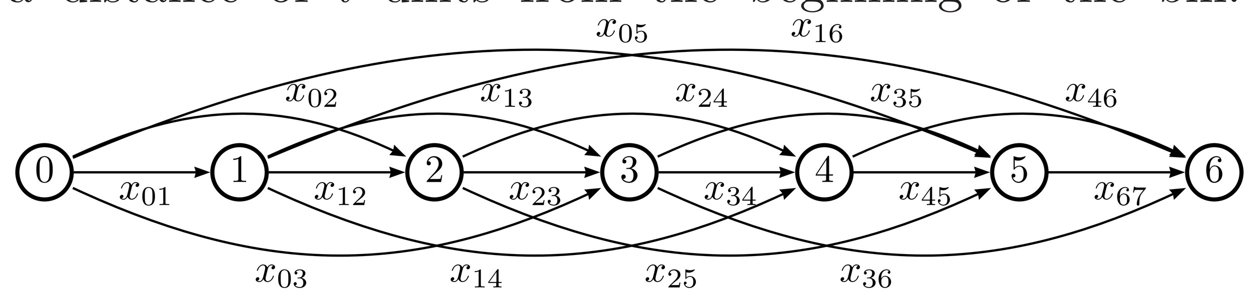
where J is the set of valid cutting patterns that satisfy:

$$\sum_{i=1}^{m} a_i^j w_i^k \leq W^k, \ k = 1..p, \ a_i^j \in \mathbb{N}_0.$$

- Very flexible
- Strong linear relaxation
- Exponential number of variables

VI. Valério de Carvalho's model

Consider decision variables x_{ij} corresponding to the number of items of size $j - i$ placed in any bin at a distance of i units from the beginning of the bin.

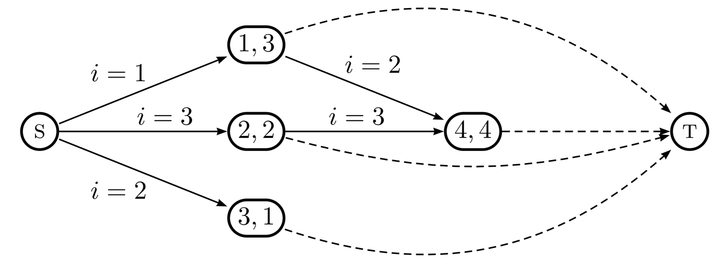


One-dimensional packing problems can be solved as a minimum flow between vertex 0 and vertex W with demand constraints.

- Strong linear relaxation
- Only models one-dimensional problems
- Large number of variables and constraints

VII. Vector Packing Graph

- For modeling p-dimensional problems, we use graphs with p-dimensional node labels.
- Every valid packing pattern is represented as a path from the source S to the target T.
- We only need to consider paths that respect a fixed order (permutations of items are redundant).



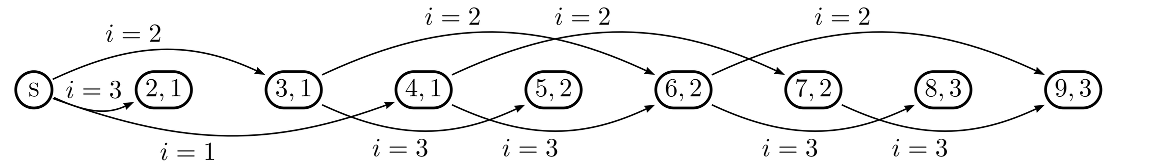
i	w_i^1	w_i^2	b_i
1	1	3	1
2	3	1	1
3	2	2	2
$W^1 = 7$	$W^2 = 3$		

The dashed arcs are loss arcs that represent unoccupied portions of the patterns.

VIII. Graph compression

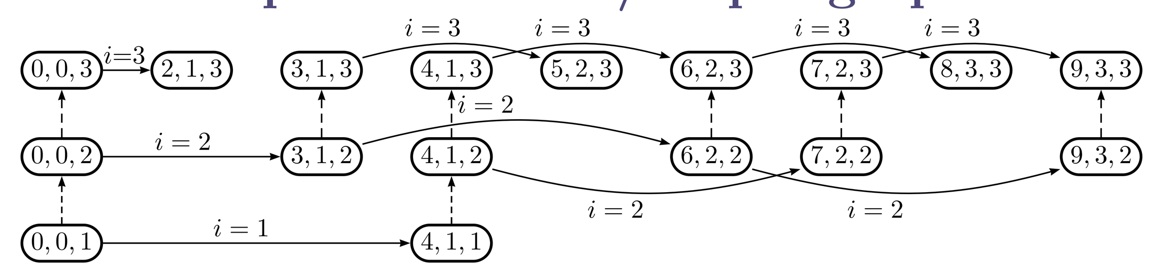
Consider an instance with bins of capacity $W = (9,3)$ and items of sizes (4,1), (3,1), (2,1) with demands 1, 3, 1, respectively.

Initial graph/Step-1 graph*



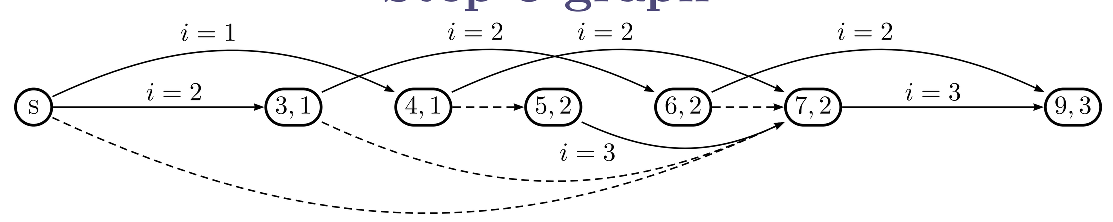
1) Break symmetry: we divide the graph into levels, one level for each different item.

Graph with levels/Step-2 graph*



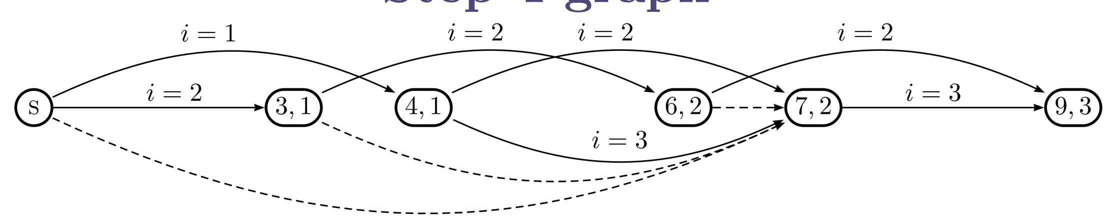
2) Main compression phase: we use the longest paths to the target in each dimension to relabel the nodes.

Step-3 graph*



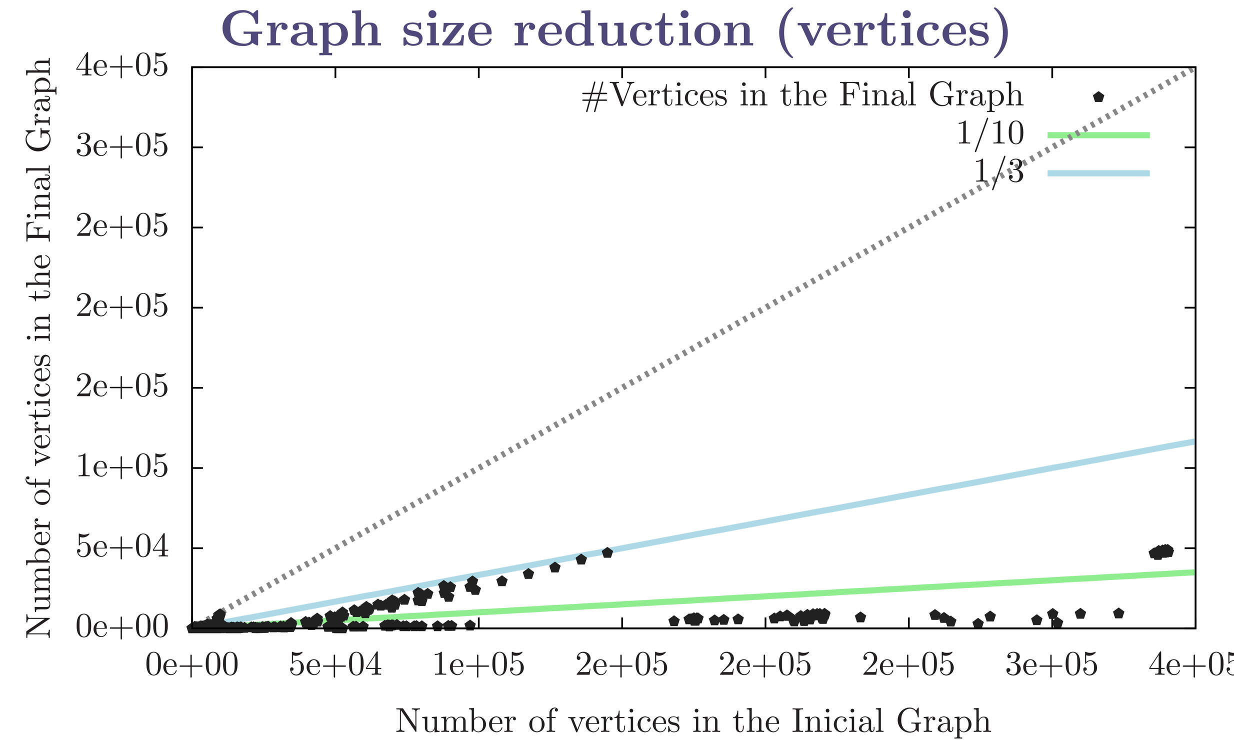
3) Final compression phase: we use the longest paths from the source in each dimension to relabel the nodes.

Step-4 graph*

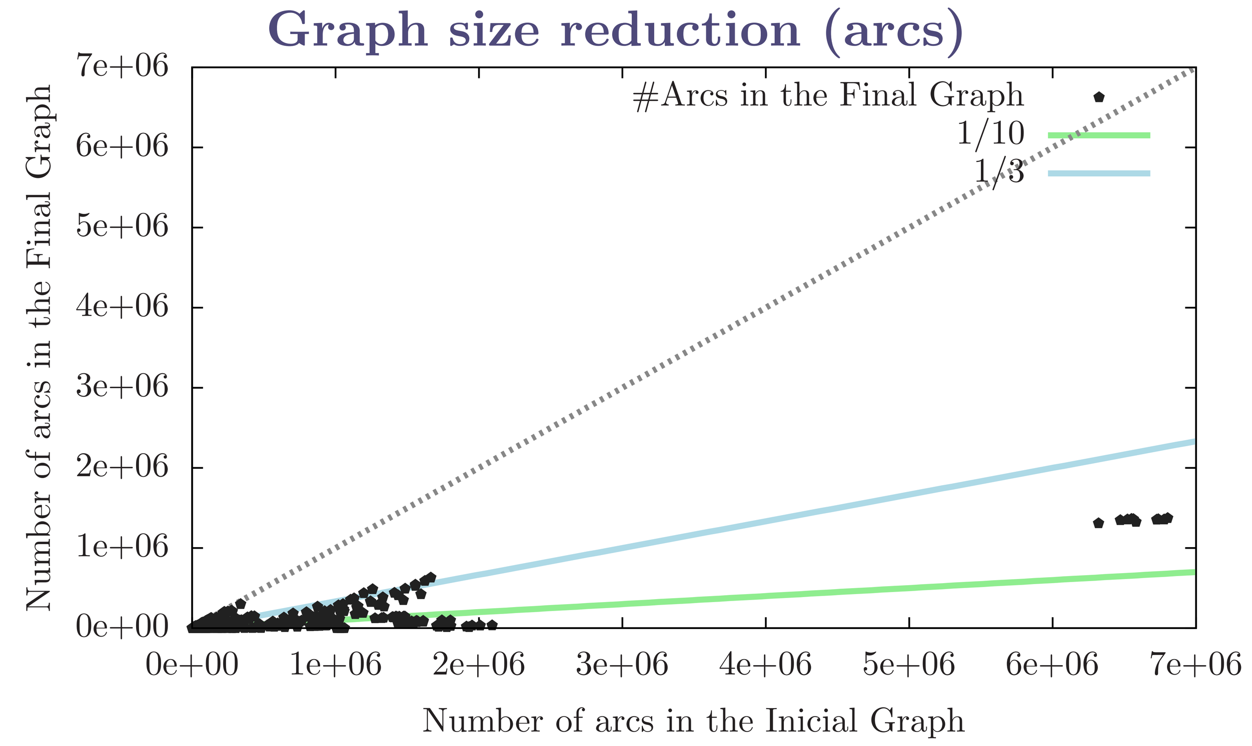


* - the target T and the loss arcs connecting every internal node to it were omitted for simplicity.

Graph size reduction (vertices)



Graph size reduction (arcs)



IX. General Arc-flow model

Our arc-flow model only requires a directed acyclic multigraph $G = (V, A)$ containing every valid packing pattern represented as a path from the source to the target in order to solve the corresponding cutting/packing problem.

$$\min z$$

$$\text{s.t.} \quad \sum_{(u,v,i) \in A: v=k} f_{uvi} - \sum_{(v,r,i) \in A: v=k} f_{vri} = \begin{cases} -z & \text{if } k = \text{S}, \\ z & \text{if } k = \text{T}, \\ 0 & \text{for } k \in V \setminus \{\text{S}, \text{T}\}, \end{cases}$$

$$\sum_{(u,v,i) \in A: i=j} f_{uvi} \geq b_j, \qquad j = 1..m,$$

$$f_{uvi} \geq 0, \text{ integer}, \qquad \forall (u,v,i) \in A,$$

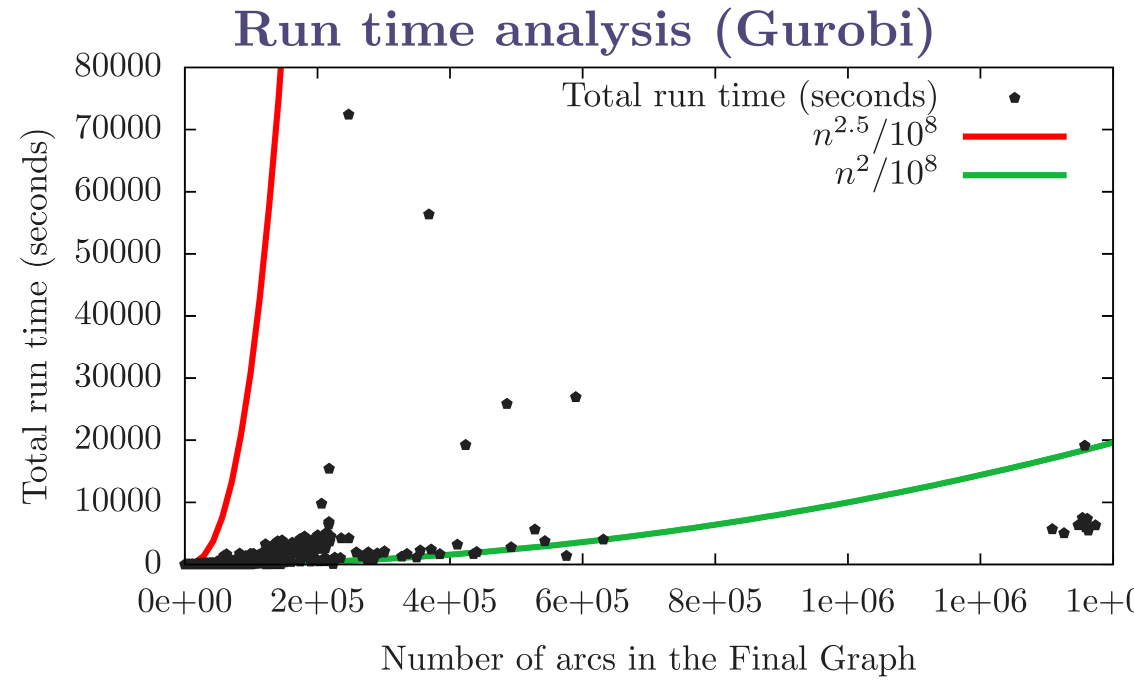
where (u,v,i) denotes an arc between u and v associated with items of type i, and arcs $(u,v,i=0)$ are loss arcs; and f_{uvi} is the amount of flow along the arc (u,v,i).

- Very flexible
- Strong linear relaxation
- Reasonably small models (graph compression!)

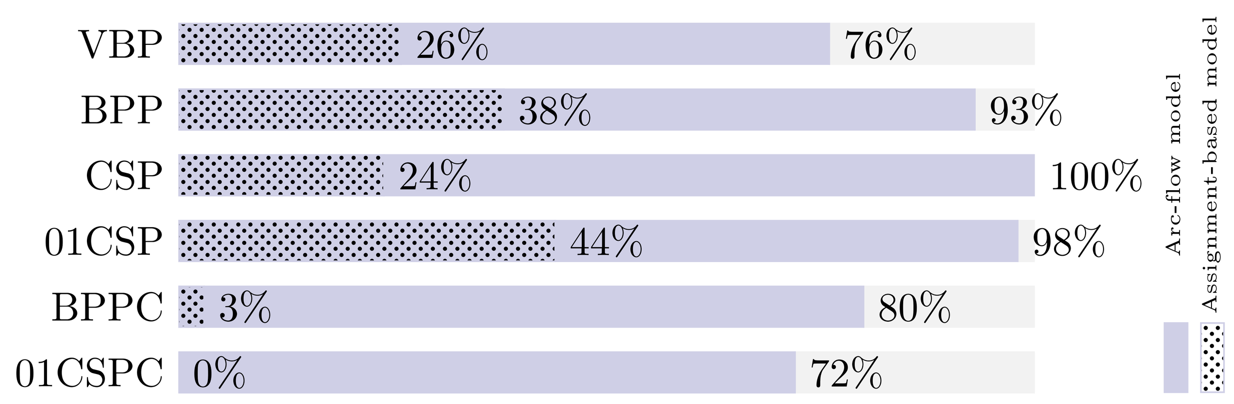
X. Results

Using the proposed method, we solved 23,153 benchmark instances on a desktop computer, spending 33 seconds per instance, on average.

Run time analysis (Gurobi)



These benchmark instances belong to several strongly NP-hard problems such as vector packing (VBP), bin packing (BPP), cutting stock (CSP), CSP with binary patterns (01CSP), BPP with conflicts (BPPC), and 01CSP with forbidden pairs (01CSPC).



Percentage of instances solved under a 60-second time limit

- We solved benchmark instances with up to millions of items of 1,000 different types and 1,000 dimensions.
- **Despite its simplicity and generality, the proposed method outperforms complex problem-specific approaches such as branch-and-price algorithms.**

References

[1] More information: http://www.dcc.fc.up.pt/~fdabrandao/research/vpsolver/
[2] Brandão, F. and Pedroso, J. P. (2013). Bin Packing and Related Problems: General Arc-flow Formulation with Graph Compression. Technical Report DCC-2013-08, Faculdade de Ciências da Universidade do Porto, Portugal.