Parallel Algorithms - sorting

Fernando Silva

DCC-FCUP

(Some slides are based on those from the book “Parallel Programming Techniques & Applications Using Networked Workstations & Parallel Computers, 2nd ed.” de B. Wilkinson)

Sorting in Parallel

Why?

- it is a frequent operation in many applications

Goal?

- sorting a sequence of values in increasing order using n processors

Potential speedup?

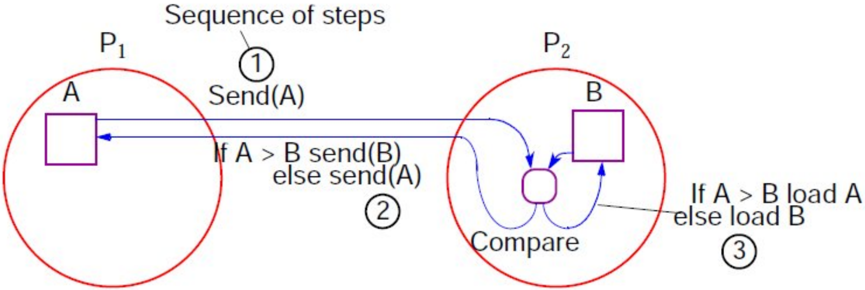
- best sequential algorithm has complexity $\mathcal{O}(n \log n)$
- the best we can aim with a parallel algorithm, using n processors is: optimal complexity of a parallel sorting algorithm: $\mathcal{O}(n \log n)/n = \mathcal{O}(\log n)$

Compare-and-swap with message exchange (1/2)

Sequential sorting requires the comparison of values and swapping in the positions they occupy in the sequence. And, if it is in parallel? And, if the memory is distributed?

version 1:

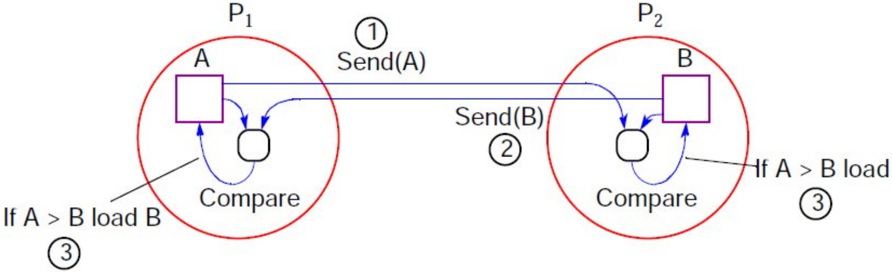
- P_1 send A to P_2, this compares B with A and sends to P_1 the $min(A, B)$.



Compare-and-swap with message exchange (2/2)

version 2:

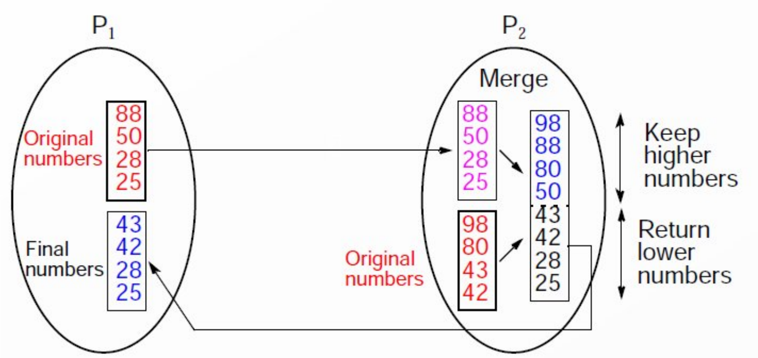
- P_1 sends A to P_2; P_2 sends B to P_1; P_1 does $A = min(A, B)$ and P_2 does $B = max(A, B)$.



Data partition

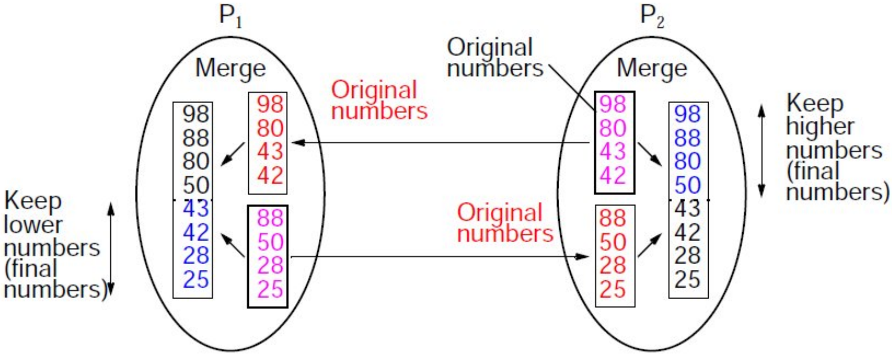
Version 1:

- n numbers and p processors
- n/p numbers assigned to each processor.



Merging two sub-lists

version 2:

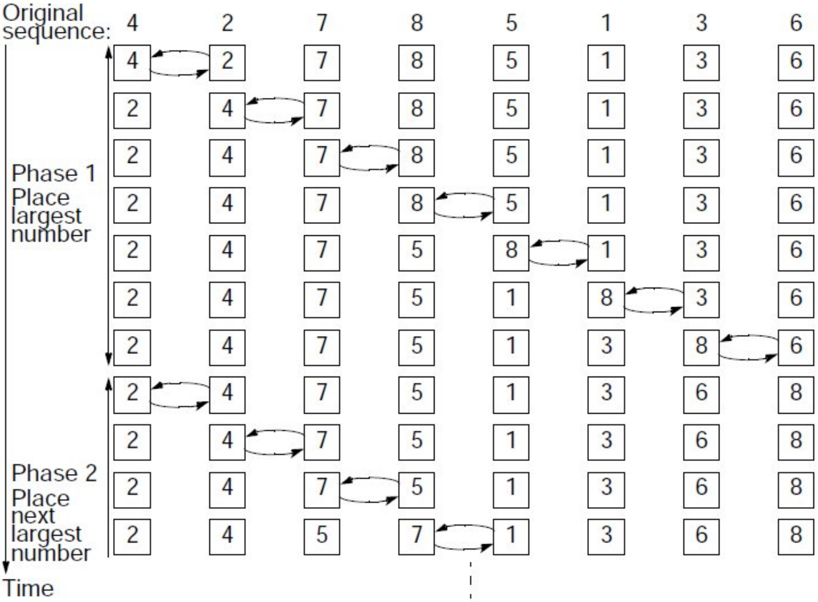


Bubble Sort

- compares two consecutive values at a time and swaps them if they are out of order.
- greater values are being moved towards the end of the list.
- number of comparisons and swaps: $\sum_{i=1}^{n-1} i = \frac{n(n-1)}{2}$

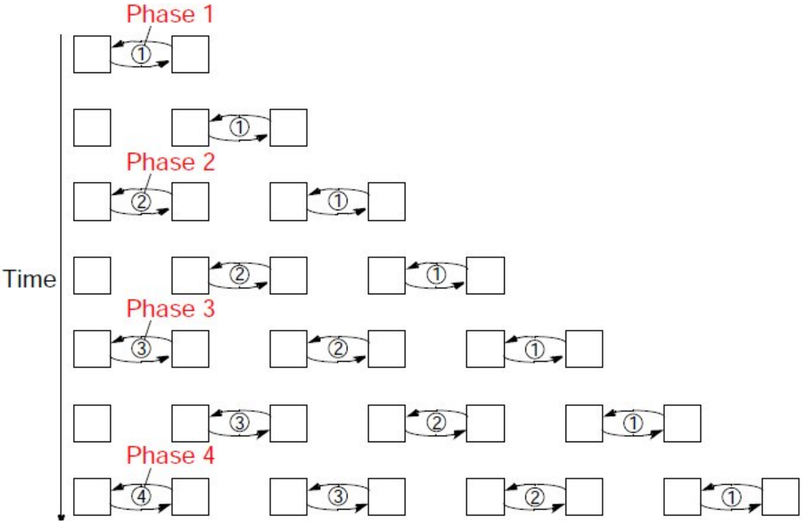
  which corresponds to a time complexity $\mathcal{O}(n^2)$.

Bubble-sort example



Parallel Bubble-sort

The idea is to run multiple iterations in parallel.



Odd-Even with transposition (1/2)

- it is a variant of the bubble-sort
- operates in two alternate phases:

  Phase-even:

  - even processes exchange values with right neighbors.

  Phase-odd:

  - odd processes exchange values with right neighbors.

Odd-Even with transposition (2/2)

Step	P_0	P_1	P_2	P_3	P_4	P_5	P_6	P_7
0	4	2	7	8	5	1	3	6
1	2	4	7	8	1	5	3	6
2	2	4	7	1	8	3	5	6
3	2	4	1	7	3	8	5	6
4	2	1	4	3	7	5	8	6
5	1	2	3	4	5	7	6	8
6	1	2	3	4	5	6	7	8
7	1	2	3	4	5	6	7	8

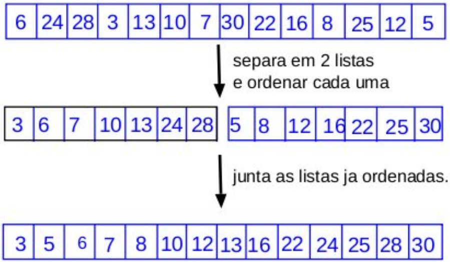
Time

Parallel algorithm - Odd-Even with transposition

```
void ODD-EVEN-PAR(n)
{
  id = process label
  for (i= 1; i<= n; i++) {
    if (i is odd)
        compare-and-exchange-min(id+1);
      else
        compare-and-exchange-max(id-1);
    if (i is even)
        compare-and-exchange-min(id+1);
      else
        compare-and-exchange-max(id-1);
  }
}
```

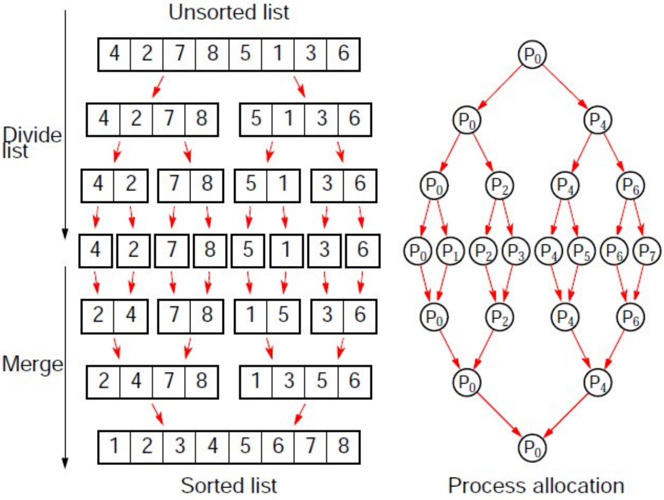
Mergesort (1/2)

- Example of a *divide-and-conquer* algorithm
- Sorting method to sort a vector; first subdivides it in two parts, applies again the same method to each part and when they are both sorted (2 sorted vectors/lists) with m and n elements, they are merged to produce a sorted vector that contains $m + n$ elements of the initial vector.
- The average complexidade is $\mathcal{O}(n \log n)$.



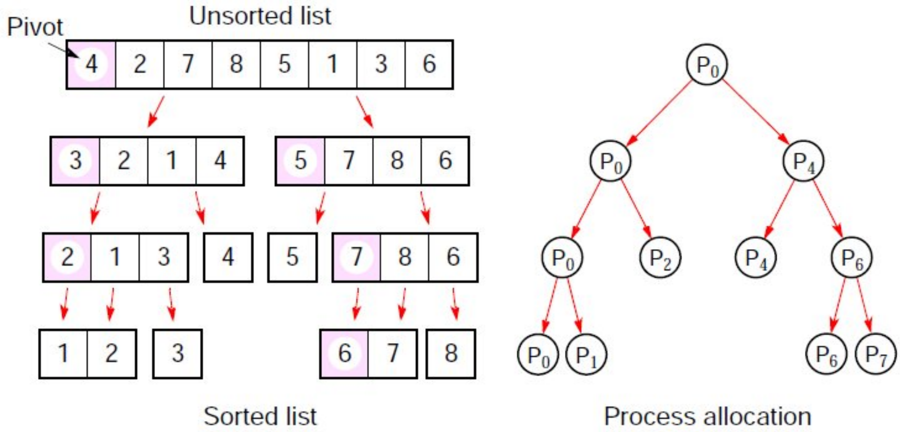
Parallel Mergesort (2/2)

Using a strategy to assign work to processors organized in a tree.



Parallel Quicksort

Using a strategy for work-assignment in a tree-fashion.

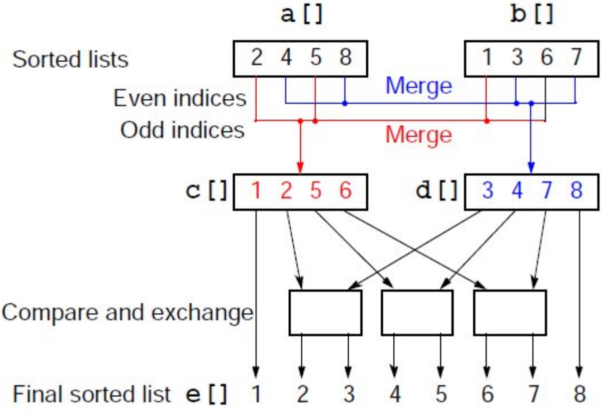


Difficulties with the allocation of processes organized in a tree

- the initial division starts with just one process, which is limitating.
- the search tree of quicksort is not, in general, balanced
- selecting the pivot is very important for efficiecy

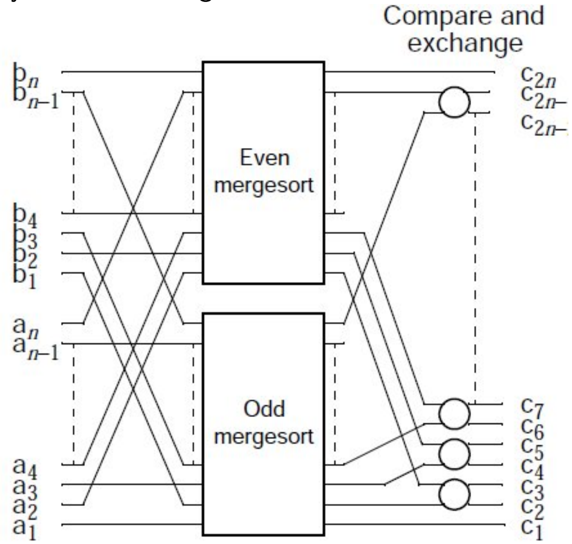
Odd-Even mergesort

- complexity: $\mathcal{O}(\log^2 n)$
- merging the two lists $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$, where n is a power of 2.



Odd-Even mergesort

Apply recursively odd-even merge:

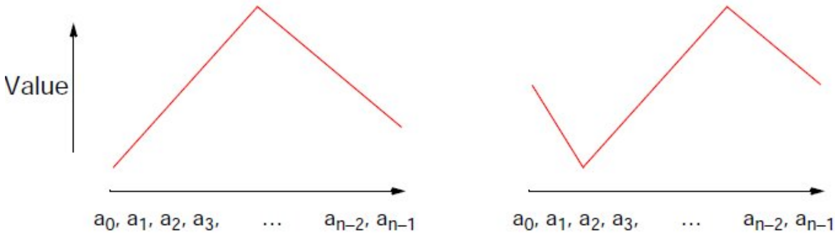


Bitonic Sort (1/7)

- complexity: $\mathcal{O}(\log^2 n)$
- a sequence is bitonic if it contains two sequences, one increasing and one decreasing, i.e.

  $a_1 < a_2 < \ldots < a_{i-1} < a_i > a_{i+1} > a_{i+2} > \ldots > a_n$

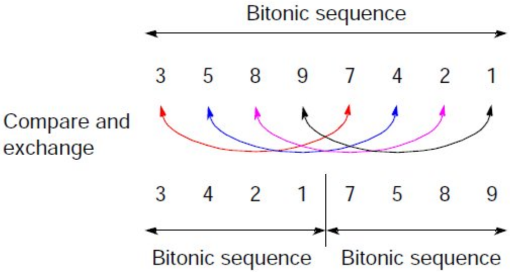
  for some i such that $(0 \leq i \leq n)$
- a sequence is bitonic if the property described is attained by a circular rotation to the right of its elements.
- Examples:



Bitonic Sort (2/7)

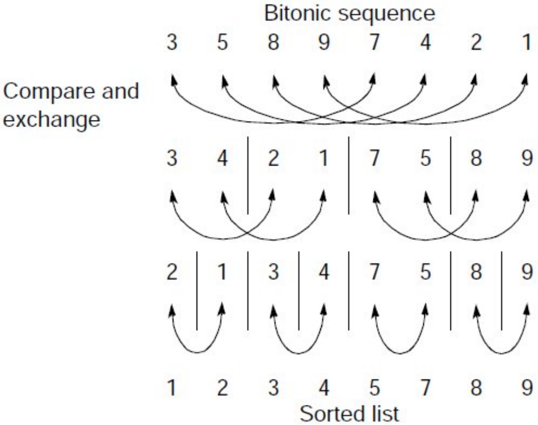
Special characteristic of bitonic sequences:

- if we do a compare-and-exchange operation with elements a_i and $a_{i+n/2}$, for all i, in a sequence of size n,
- we obtain *two bitonic sequences* in which all the values in one sequence are smaller then the values of the other.
- Example: start with sequence 3, 5, 8, 9, 7, 4, 2, 1 and we obtain:

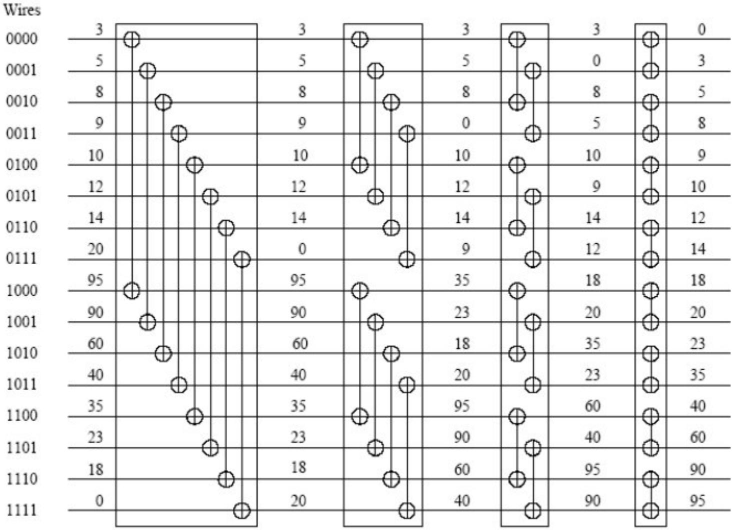


Bitonic Sort (3/7)

- the compare-and-exchange operation moves smaller values to the left and greater values to the right.
- given a bitonic sequence, if we apply recursively these operations we get a sorted sequence.



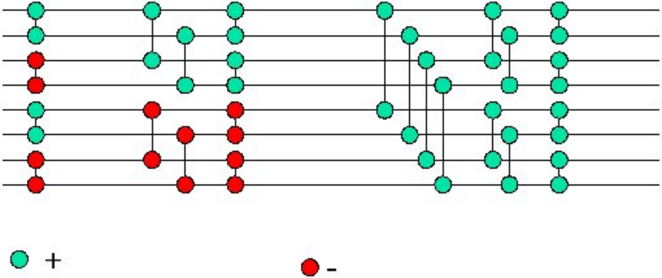
Bitonic Sort example (4/7)



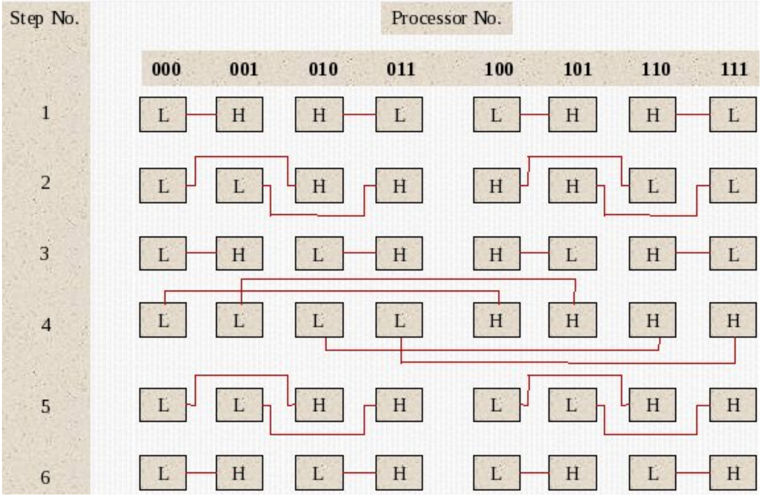
Bitonic Sort (5/7)

To sort an unsorted sequence

- merge sequences in larger bitonic sequences, starting with adjacent pairs, alternating monotonicity.
- in the end, the bitonic sequence becomes sorted in a unique increasing sequence.



Bitonic Sort (6/7)



Bitonic Sort (7/7)

Unsorted sequence ⇒ bitonic sequence ⇒ sorted sequence.