

# Basic notions of Mixed Integer Non-Linear Programming

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# Outline

## 1 Intro to MINLP

## 2 What is a convex MINLP?

## 3 “Basic” subproblems

## 4 Convex MINLP Algorithms

- Branch-and-Bound
- Outer-Approximation
- Generalized Benders Decomposition
- Extended Cutting Plane
- LP/NLP-based Branch-and-Bound
- Hybrid Algorithms

## 5 Practical Tools

- Convex MINLP Solvers
- Modeling Languages
- Neos
- MINLP Libraries
- Solvers' performance

Mixed Integer NonLinear Programming (MINLP).

$$\min f(x, y)$$

$$g(x, y) \leq 0$$

$$x \in X = \{x \mid x \in \mathbb{R}^p, Dx \leq d, x^L \leq x \leq x^U\}$$

$$y \in Y = \{y \mid y \in \mathbb{Z}^q, Ay \leq a, y^L \leq y \leq y^U\}$$

with  $f(x) : \mathbb{R}^{p+q} \rightarrow \mathbb{R}$  and  $g(x) : \mathbb{R}^{p+q} \rightarrow \mathbb{R}^m$  are

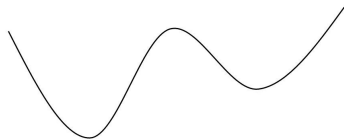
- \* continuous
- \* twice differentiable

functions.

- **Undecidable/NP-hard** problem.
- Local optima might not be also **global optima** .

Sources of non-convexities:

i. non-convex functions



ii. integrality requirement  $\rightarrow$  worst-case: **exponential number** of possible values of integer variables, e.g., if binary,  $2^{|I|}$  cases.

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# What is a convex MINLP?

**Convex** Mixed Integer NonLinear Programming (MINLP).

$$\min f(x, y)$$

$$g(x, y) \leq 0$$

$$x \in X = \{x \mid x \in \mathbb{R}^p, Dx \leq d, x^L \leq x \leq x^U\}$$

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with  $f(x) : \mathbb{R}^{p+q} \rightarrow \mathbb{R}$  and  $g(x) : \mathbb{R}^{p+q} \rightarrow \mathbb{R}^m$  are

- \* continuous
- \* twice differentiable
- \* convex

functions.

- Local optima are also **global optima** .
- Continuous relaxation: convex NLP.

# Convex vs. Non-Convex MINLP

## Convex

- **Convex** function involved
- **Convex** continuous relaxation  $\rightarrow$  **polynomially** solvable
- Local optima = Global optima

## Non-Convex

- **Non-Convex** function involved
- **Non-Convex** continuous relaxation  $\rightarrow$  **NP-hard** problem
- Local optima  $\neq$  Global optima

Focus on

## Convex MINLPs

Non-convex MINLPs  $\rightarrow$  A. Gleixner

$$\begin{aligned} \min f(x, y) \\ g(x, y) &\leq 0 \\ x &\in X \\ y &\in \{y \mid Ay \leq a\} \\ y_j &\leq \alpha_j^k && j \in \{1, 2, \dots, q\} \\ y_j &\geq \beta_j^k && j \in \{1, 2, \dots, q\} \end{aligned}$$

$k$ : current step of a Branch-and-Bound procedure;

$\alpha^k$ : current lower bound on  $y$  ( $\alpha^k \geq y^L$ );

$\beta^k$ : current upper bound on  $y$  ( $\beta^k \leq y^U$ ).



# NLP restriction and Feasibility subproblem

NLP restriction for a fixed  $y^k$ :

$$\begin{aligned} \min f(x, y^k) \\ g(x, y^k) &\leq 0 \\ x &\in X. \end{aligned}$$

Feasibility subproblem for a fixed  $y^k$ :

$$\begin{aligned} \min u \\ g(x, y^k) &\leq u \\ x &\in X \\ u &\in \mathbb{R}. \end{aligned}$$

$$\begin{aligned} \min \gamma \\ f(x^k, y^k) + \nabla f(x^k, y^k)^T \begin{pmatrix} x - x^k \\ y - y^k \end{pmatrix} &\leq \gamma & \forall k \\ g_i(x^k, y^k) + \nabla g_i(x^k, y^k)^T \begin{pmatrix} x - x^k \\ y - y^k \end{pmatrix} &\leq 0 & \forall k \forall i \in I^k \\ x &\in X \\ y &\in Y. \end{aligned}$$

where  $I^k \subseteq \{1, 2, \dots, m\}$ . Two “classical” choices:

- $I^k = \{1, 2, \dots, m\}$
- $I^k = \{i \mid g_i(x^k, y^k) > 0, 1 \leq i \leq m\}$

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- Branch-and-Bound (BB).
- Outer-Approximation (OA).
- Generalized Benders Decomposition (GBD).
- Extended Cutting Plane (ECP).
- LP/NLP-based Branch-and-Bound (QG).
- Hybrid Algorithms (Hyb).

# Branch-and-Bound (BB)

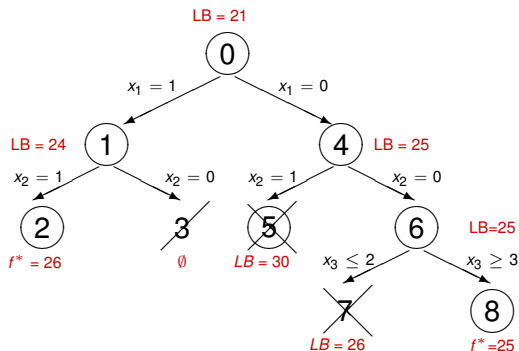
Gupta and Ravindran, 1985. Link BB for MILPs.

- 1:  $f^* = +\infty$ ,  $\Pi = \{P^0\}$  where  $P^0 = \text{NLP relaxation}$ .
- 2: **while**  $\Pi \neq \emptyset$  **do**
- 3:   Choose the current subproblem  $P \in \Pi$ ,  $\Pi = \Pi \setminus \{P\}$ .
- 4:   Solve  $P$  obtaining  $(\bar{x}, \bar{y})$ .
- 5:   **if**  $P$  infeasible  $\vee f(\bar{x}, \bar{y}) \geq f^*$  **then**
- 6:     **break**
- 7:   **end if**
- 8:   **if**  $\bar{y} \in \mathbb{Z}^q$  **then**
- 9:      $f^* = f(\bar{x}, \bar{y})$ ,  $(x^*, y^*) = (\bar{x}, \bar{y})$ .
- 10:    Update  $\Pi$  potentially fathoming subproblems.
- 11:    **else**
- 12:     Take a fractional value  $\bar{y}_j$  and obtain two subproblems  $P^1 = P$  with  $\alpha_j^1 = \lfloor \bar{y}_j \rfloor$  and  $P^2 = P$  with  $\beta_j^2 = \lfloor \bar{y}_j \rfloor + 1$ .
- 13:      $LB(P^1) = LB(P^2) = f(\bar{x}, \bar{y})$ .
- 14:      $\Pi = \Pi \cup \{P^1, P^2\}$ .
- 15:    **end if**
- 16: **end while** **RETURN**  $(x^*, y^*)$ .

Fathoming is performed when:

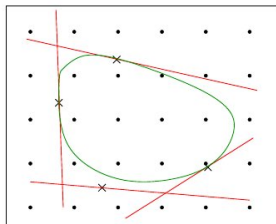
- The subproblem solution is MINLP feasible ( $f^*$ ).
- The subproblem is infeasible.
- The subproblem  $P^k$  has  $LB(P^k) \geq f^*$ .

# Branch-and-Bound (BB)



# Outer-Approximation (OA)

Duran and Grossmann, 1986.



$$\begin{aligned} & \min \gamma \\ & f(x^k, y^k) + \nabla f(x^k, y^k)^T \begin{pmatrix} x - x^k \\ y - y^k \end{pmatrix} \leq \gamma \quad \forall k \\ & g_i(x^k, y^k) + \nabla g_i(x^k, y^k)^T \begin{pmatrix} x - x^k \\ y - y^k \end{pmatrix} \leq 0 \quad \forall k \forall i \in I^k \\ & x \in X \\ & y \in Y. \end{aligned}$$

$I^k = \{1, 2, \dots, m\} \quad \forall k = 1, \dots, K.$

NB. The linearization constraints of MILP relaxation are not valid for non-convex MINLPs.

# Outer-Approximation (OA)

Initialization:  $K = 1$ , define an initial MILP relaxation,  $f^* = +\infty$ ,  
LB =  $-\infty$ .

- 1: **while**  $f^* \neq \text{LB}$  **do**
- 2:     Solve the current MILP relaxation (obtaining  $(\hat{x}^K, \hat{y}^K)$ ) and  
      update LB.
- 3:     Solve the current NLP restriction for  $\hat{y}^K$ .
- 4:     **if** NLP restriction for  $\hat{y}^K$  infeasible **then**
- 5:         Solve the infeasibility subproblem for  $\hat{y}^K$ .
- 6:     **else**
- 7:         Update  $f^*$ .
- 8:     **end if**
- 9:     Generate new linearization cuts and update MILP relaxation.
- 10:     $K = K + 1$ .
- 11: **end while**



# Generalized Benders Decomposition (GBD)

Geoffrion, 1972.

Similar to OA, but with a different MILP relaxation, i.e.,

- $x \in X$  is relaxed.
- $I^k = \{i \mid g(x^k, y^k) = 0, 1 \leq i \leq m\} \forall k = 1, \dots, K.$

## Proposition

*Given the same set of  $K$  subproblems, the LB provided by the MILP relaxation of OA is  $\geq$  of the one provided by the MILP relaxation of GDB.*

## Proof.

(Sketch of) It can be shown that the constraints of GDB MILP relaxation are surrogate of the ones of OA MILP relaxation (see, Quesada and Grossmann, 1992). □

# Extended Cutting Plane (ECP)

Westerlund and Pettersson, 1995.

- 1:  $k = 1$ , obtain an initial MILP relaxation.
- 2: Solve the MILP relaxation obtaining  $(x^k, y^k)$ .
- 3: **if** no constraint is violated by  $(x^k, y^k)$  **then**
- 4:   Return  $(x^k, y^k)$  (optimal solution).
- 5: **else**
- 6:   Generate (some) new linearization constraints from  $(x^k, y^k)$  and update MILP relaxation.
- 7: **end if**
- 8:  $k = k + 1$ .
- 9: Go to step 1.

More iterations needed wrt OA.

Quesada and Grossmann, 1992.

- 1: Obtain an initial MILP relaxation.
- 2: Solve the MILP relaxation through BB for MILP, but, anytime a MILP feasible solution is found
  - Solve NLP restriction.
  - Generate new linearization constraints.
  - Update open MILP relaxation subproblems.

Link OA, but only 1 MILP relaxation is solved, and updated in the tree search.

# Hybrid Algorithms (Hyb)

For example, Bonami et al., 2008 (**BONMIN**).

Very similar to Quesada and Grossmann, 1992, but NLP solved not only when the node is integer feasible but also, for example, any 10 nodes.

**Pros** : more “nonlinear” information added to the MILP relaxation.

**Cons** : More NLP solved.

---

Alternative,

Abhishek et al., 2010 (**FILMINT**).

Very similar to Quesada and Grossmann, 1992, but add linearization cuts not only when the node is integer feasible (different strategies).

**Pros** : more “nonlinear” information added to the MILP relaxation.

**Cons** : MILP relaxation more difficult to solve.

	# MILP	# NLP	note
BB	0	N	
OA	I	I	
GBD	I	I	weaker lower bound w.r.t. OA
ECP	I	0	
QG	1	1 + E	
Hyb (Abhishek et al.)	1	1 + E	stronger lower bound w.r.t. QG
Hyb (Bonami et al.)	1	[E,N]	

**Table:** Number of MILP and NLP subproblems solved by each algorithm.

where I is the # of iterations, N is the # of nodes, E is the # of explored MILP solutions

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# Convex MINLP Solvers

- **ALPHA-ECP:** <http://www.gams.com/dd/docs/solvers/alphaecp.pdf>
- **BONMIN:** <https://projects.coin-or.org/Bonmin>
- **DICOPT:** [www.gams.com/dd/docs/solvers/dicopt.pdf](http://www.gams.com/dd/docs/solvers/dicopt.pdf)
- **FILMINT:** [www.mcs.anl.gov/~leyffer/papers/fm.pdf](http://www.mcs.anl.gov/~leyffer/papers/fm.pdf)
- **KNITRO:** <http://www.ziena.com/knitro.htm>
- **MINLPBB:** [www-unix.mcs.anl.gov/~leyffer/solvers.htm](http://www-unix.mcs.anl.gov/~leyffer/solvers.htm)
- **SBB:** <http://www.gams.com/solvers/solvers.htm#SBB>
- **SCIP:** <http://scip.zib.de/>

Problem: need value of the function, its first and its second derivative at a given point  $(x^*, y^*)$ .

Possible source of errors!

Solution? **Modeling Languages!**

Modeling languages, e.g., AMPL and GAMS.

Example:

```
param pi := 3.142;
param N;
set VARS ordered := {1..N};
param U {j in VARS};
param a {j in VARS};
param b {j in VARS};
param c {j in VARS};
param d {j in VARS};
param w{VARS};
param C;
var x {j in VARS} >= 0, <= U[j], integer;

maximize Total_Profit:
    sum {j in VARS} c[j]/(1+b[j]*exp(-a[j]*(x[j]+d[j])));
subject to KP_constraint: sum{j in VARS} w[j]*x[j] <= C;
```



NEOS: <http://www.neos-server.org/neos/>.

The screenshot shows a web browser window titled "Optimization Tree - NEOS - Mozilla Firefox". The address bar contains the URL [http://www.neos-server.org/NEOS/index.php/Optimization\\_Tree](http://www.neos-server.org/NEOS/index.php/Optimization_Tree). The browser's taskbar shows several open windows, including "openSUSE", "Getting Started", "Latest Headlines", "Gmail - Priority Inbox (6) - c...", "CMU-IBM Open Source MINL...", and "Optimization Tree - NEOS".

The website content includes:

- Navigation:**
  - NEOS Wiki
  - NEOS Server
  - Optimization Tree
  - Software Guide
  - Optimization FAQs
  - Algorithms
  - Case Studies
  - Test Problems
  - Applications
  - Views and News
  - Contributing Authors
  - Recent changes
  - Help
- Search:** A search box with "Go" and "Search" buttons.
- Toolbox:**
  - What links here
  - Related changes
  - Special pages
  - Printable version
  - Permanent link
- Optimization Tree:**
  - Introduction to Optimization**
  - Taxonomy of Optimization Tree**
  - Continuous Optimization**
    - Unconstrained Optimization
    - Bound Constrained Optimization
    - Derivative-Free Optimization
    - Global Optimization
    - Linear Programming
    - Network Flow Problems
    - Nondifferentiable Optimization
    - Nonlinear Programming
    - Optimization of Dynamic Systems
    - Quadratic Constrained Quadratic Programming
    - Quadratic Programming
    - Second Order Cone Programming
    - Semidefinite Programming
    - Semifinite Programming
  - Discrete and Integer Optimization**
    - Combinatorial Optimization
    - Traveling Salesman Problem
    - Integer Programming
      - Mixed Integer Linear Programming
      - Mixed Integer Nonlinear Programming
  - Optimization Under Uncertainty**
    - Robust Optimization
    - Stochastic Programming
      - Chance Constrained Optimization
    - Simulation/Noisy Optimization
    - Stochastic Algorithms
  - Complementarity Constraints and Variational Inequalities**
    - Complementarity Constraints
    - Game Theory
    - Linear Complementarity Problems
    - Mathematical Programs with Complementarity Constraints
    - Nonlinear Complementarity Problems
  - Systems of Equations and Inequalities**
    - Data Fitting/Robust Estimation
    - Nonlinear Equations
    - Nonlinear Least Squares
  - Multiobjective Programming**

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- About NEOS
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- CMU/IBM: 42 different kind of MINLP problems

<http://www.minlp.org>

- MacMINLP:  $\simeq$ 50 instances

<http://wiki.mcs.anl.gov/leyffer/index.php/MacMINLP>

- MINLPlib: 274 instances

<http://www.gamsworld.org/minlp/minlplib.htm>

The MINLPlib 2.0 is under construction:  
**call for contribution** (S. Vigerske)!

# Solvers' performance

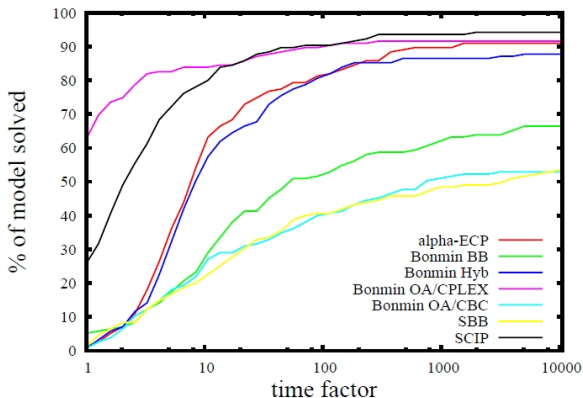


Figure: Comparison of solvers in GAMS [Vigerske, 2012]

Best Algorithms are OA based!

# Modeling Phase Importance

Thanks to P. Bonami for the picture

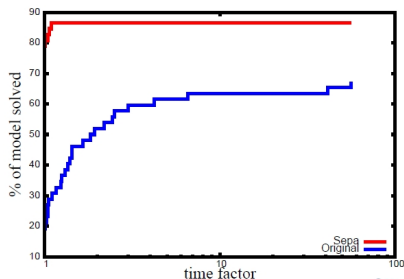


Figure: OA on 2 different formulations

Example:

$$\begin{aligned} \min c^T x \\ \sum_{i=1}^n (x_i - 0.5)^2 \leq (n-1)/4 \\ x \in \mathbb{Z}^n \end{aligned}$$

$$\begin{aligned} \min c^T x \\ \sum_{i=1}^n y_i \leq (n-1)/4 \\ (x_i - 0.5)^2 \leq y_i \quad \forall i = 1, \dots, n \\ x \in \mathbb{Z}^n \end{aligned}$$

## MINLP modeling and real-world applications!!