

Lot-Sizing and Mixed Integer Programming

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April 10th, 2014

Typical Multi-Item Single-Level Lot-Sizing Problem

x_t^i is production of i in t ,

s_t^i is stock of i at end of t

$y_t^i = 1$ if set up to produce i in t , $y_t^i = 0$ otherwise.

$$\min \sum_{i,t} (p_t^i x_t^i + h_t^i s_t^i + f_t^i y_t^i)$$

$$s_{t-1}^i + x_t^i = d_t^i + s_t^i \quad \forall i, t$$

$$x_t^i \leq C_t^i y_t^i \quad \forall i, t$$

$$\sum_i a^i x_t^i + \sum_i b^i y_t^i \leq L_t \quad \forall t$$

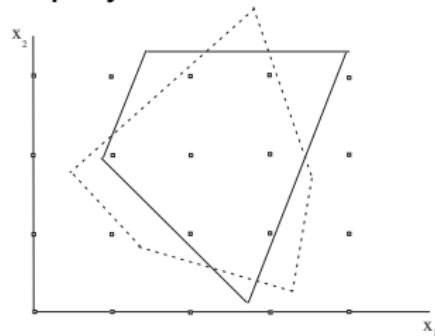
$$x, s \geq 0, y \in \{0, 1\}$$

- Some Basics and Decomposition of Mixed Integer Programs
- Uncapacitated Lot-Sizing: Polyhedral Viewpoint
- Extensions and Classification
- Production with Sequence-Dependent Set-Ups
- The Joint Replenishment Problem
- Multi-Level Lot-Sizing

Types of Reformulation

Think of the feasible region $X = \{x \in Z_+^n : Ax \leq b\}$ as the integer points in a *polyhedron* $P = \{x \in R_+^n : Ax \leq b\}$.

A polyhedron P is a *formulation* for X if $X = P \cap Z^n$.



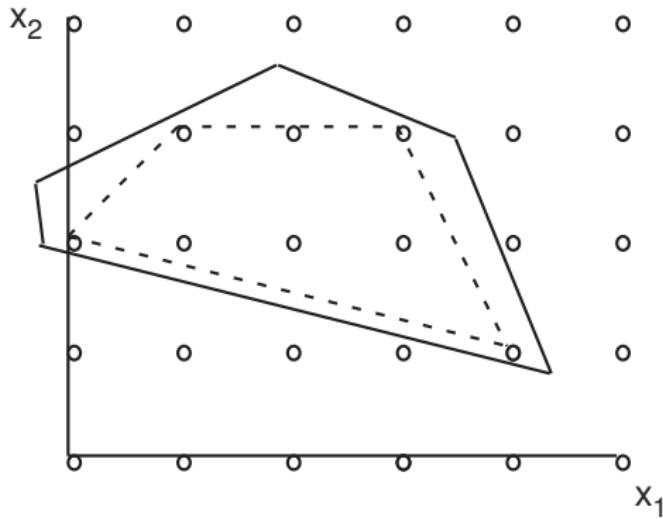
There are lots of formulations for X .

If P_1 and P_2 are two formulations for X with $P_1 \subset P_2$, we say that P_1 is *better* than P_2 , because the LP relaxation is tighter
 $\min\{cx : x \in P_1\} \geq \min\{cx : x \in P_2\}$ for all c

$P' = \text{conv}(X)$ is the *ideal* formulation for X as

$$z = \min\{cx : x \in \text{conv}(X)\} \geq \min\{cx : x \in P\}$$

for all formulations P of X



Improving IP Formulations

Feasible region $X = X^1 \cap X^2$ with formulation
 $P = P^1 \cap P^2$ where $P^i = \{x \in R_+^n : A^i x \leq b^i\}$

Let $\tilde{P}^i = \text{conv}(X^i)$ for $i = 1, 2$.

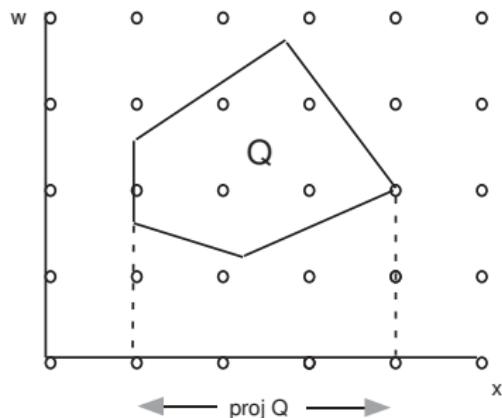
Then $\tilde{P} = \tilde{P}^1 \cap \tilde{P}^2$ is an improved formulation for X .

Extended Formulations: Adding Variables

Replace $X = P \cap Z^n$ with $P = \{x \in R_+^n : Ax \leq b\}$ by

$$Q = \{(x, z) \in R_+^n \times R^p : Cx + Dz \leq d\}$$

where $X = \{x \in Z^n : \text{there exists } z \text{ with } (x, z) \in Q\}$.



$P' = \text{proj}_x Q = \{x \in R^n : \text{there exists } z \text{ with } (x, z) \in Q\}$ is a formulation for X .

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$$s_{t-1}^i + x_t^i = d_t^i + s_t^i \quad \forall i, t$$

$$x_t^i \leq C_t^i y_t^i \quad \forall i, t$$

$$\sum_i a^i x_t^i + \sum_i b^i y_t^i \leq L_t \quad \forall t$$

$$x, s \geq 0, y \in \{0, 1\}$$

Decomposition Approach

X can be written as

$$X = \left(\prod_{i=1}^m \hat{Y}^i \right) \cap \left(\prod_{t=1}^n \hat{Z}^t \right)$$

The Decomposed Sets

$$\hat{Y}^i = \{(x^i, s^i, y^i) \in \mathbb{R}^n \times \mathbb{R}^n \times \{0, 1\}^n : \\ s_{t-1}^i + x_t^i = d_t^i + s_t^i \quad \forall t \\ x_t^i \leq C_t^i y_t^i \quad \forall t\}$$

is a single item lot-sizing region, and

$$\hat{Z}^t = \{(x_t, s_t, y_t) \in \mathbb{R}^m \times \mathbb{R}^m \times \{0, 1\}^m : \\ x_t^i \leq C_t^i y_t^i \quad \forall i \\ \sum_i a^i x_t^i + \sum_i b^i y_t^i \leq L_t\}$$

is a generalized single node flow region.

Algorithmic Options

$$X = (\cap_{i=1}^{NI} X^i).$$

- Inequality representation (approximate or exact) of $\text{conv}(X^i)$
- Add Valid Inequalities as Cutting Planes using a Separation Algorithm
- Extended Formulation for $\text{conv}(X^i)$
- Use Lagrangian Relaxation: $OPT(X^i)$ is easy
- Use (Dantzig-Wolfe) Column Generation: $OPT(X^i)$ is easy

- Basics
- Valid Inequalities and Convex Hull
- Use as Cutting Planes
- Tight and Compact Extended Formulation

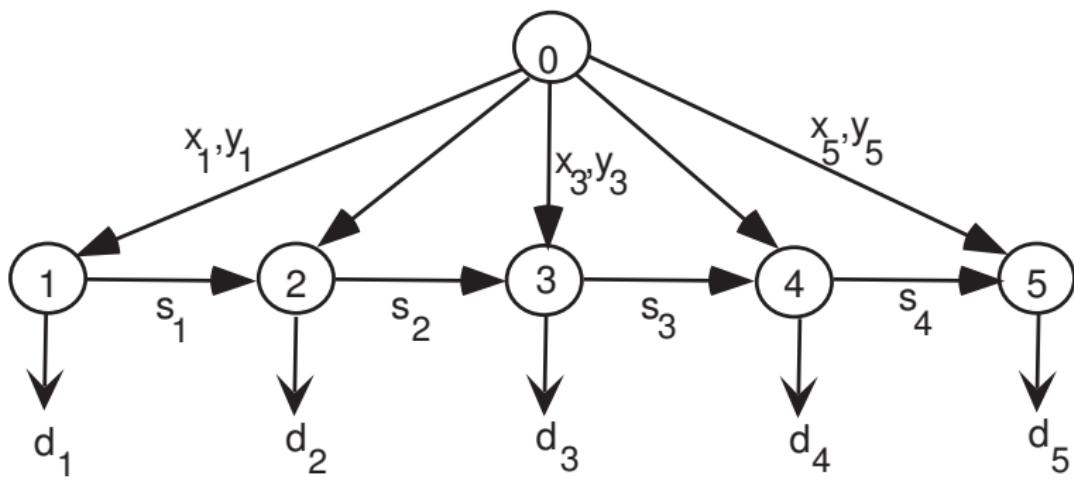
The Single-Item Lot-Sizing Set X^{LS-C}

$$\begin{aligned} s_{t-1} + x_t &= d_t + s_t \quad \forall t, \\ x_t &\leq C_t y_t \quad \forall t, \\ x, s &\geq 0, y \in \{0, 1\}^n. \end{aligned}$$

When $C_t = C$ for all t , we call it "Constant Capacity", denoted X^{LS-CC} .

When $C_t = M > d_{1t}$ for all t , we call it "Uncapacitated", denoted X^{LS-U} .

Notation. Throughout we use $d_{kl} \equiv \sum_{u=k}^l d_u$.



The Uncapacitated Lot-Sizing Set $LS-U$

We consider X^{LS-U}

$$\begin{aligned}s_{t-1} + x_t &= d_t + s_t \quad \forall t, \\x_t &\leq My_t \quad \forall t, \\x, s &\geq 0, y \in \{0, 1\}^n.\end{aligned}$$

Optimization is easy by Shortest Path/Dynamic Programming.
 $O(n^2)$ or even $O(n \log n)$

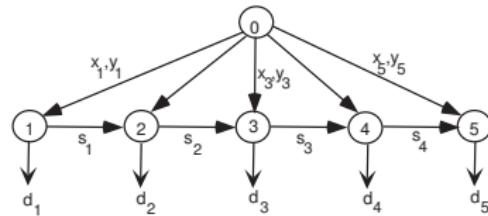
Thus by Grötschel, Lovasz and Schrijver, one can separate in polynomial time.

Thus there is some hope to find a nice description of $\text{conv}(X)!$

Valid Inequalities

A first inequality

$$s_{k-1} \geq d_k(1 - y_k).$$



Generalizing

$$s_{k-1} \geq \sum_{u=k}^t d_u(1 - y_k - \cdots - y_u) \quad \forall 1 \leq k \leq t \leq n.$$

Generalizing further

$$s_{k-1} + \sum_{j \in S} x_j \geq \sum_{u=k}^t d_u(1 - \sum_{j \in [k,u] \setminus S} y_j) \quad \forall 1 \leq k \leq t \leq n,$$

Equivalently with $L = \{1, \dots, I\}$ and $S \subseteq L$

$$\sum_{j \in L \setminus S} x_j + \sum_{j \in S} d_{jI} y_j \geq d_{1I}.$$

The Convex Hull of X^{LS-U}

$L = \{1, \dots, I\}$ and $S \subseteq L$: (I, S) inequalities

$$\sum_{j \in L \setminus S} x_j + \sum_{j \in S} d_{jI} y_j \geq d_{1I}.$$

Th. The initial inequalities plus the (I, S) inequalities give $\text{conv}(X^{LS-U})$.

Example of (I, S) Inequalities and Separation

Consider an instance with $n = 4$ and $d = (7, 2, 6, 4)$. For $I = 3$, the (I, S) inequalities are

$$\begin{array}{rccccc} x_1 & +x_2 & +x_3 & \geq & 15 \\ 15y_1 & x_2 & +x_3 & \geq & 15 \\ x_1 & +8y_2 & +x_3 & \geq & 15 \\ 15y_1 & +8y_2 & +x_3 & \geq & 15 \\ x_1 & +x_2 & +6y_3 & \geq & 15 \\ 15y_1 & x_2 & +6y_3 & \geq & 15 \\ x_1 & +8y_2 & +6y_3 & \geq & 15 \\ 15y_1 & +8y_2 & +6y_3 & \geq & 15. \end{array}$$

$$x^* = (7, 3, 5, 4), s^* = (0, 1, 0, 0), y^* = (1, \frac{1}{4}, \frac{1}{2}, 1)$$

$$I = 3 \cdot \min[7, 15 \times 1] + \min[3, 8 \times \frac{1}{4}] + \min[5, 6 \times \frac{1}{2}] = 12 < 15.$$

Violation of $x_1 + 8y_2 + 6y_3 \geq 15$.

Extended Formulation: Facility Location

Let w_{ut} with $u \leq t$ be the amount produced in period u to satisfy demand in period t . This leads to the formulation Q^{FL-U} .

$$\begin{aligned} & \min \sum_{u=1}^n p_u x_u + \sum_{t=1}^n q_t y_t \\ & \sum_{u=1}^t w_{ut} = d_t \quad \text{for } 1 \leq t \leq n \\ & w_{ut} \leq d_t y_u \quad \text{for } 1 \leq u \leq t \leq n \\ & x_u = \sum_{t=u}^n d_t w_{ut} \quad \text{for } 1 \leq u \leq n \\ & y \in [0, 1]^n, w_{ut} \in \mathbb{R}_+^1 \quad \text{for } 1 \leq u \leq t \leq n, \\ & y \in \mathbb{Z}^n. \end{aligned}$$

How Good is the Facility Location Extended Formulation?

The linear program

$$\min\{px + fy : (x, y, w) \in Q^{FL-U}\}$$

has an optimal solution with y integer, and thus it solves $LS-U$.

Theoretically this means:

$$\text{proj}_{x,y} Q^{FL-U} = \text{conv}(X^{LS-U}).$$

Wagner–Whitin Relaxation

The set X^{LS-C}

$$s_{t-1} + x_t = d_t + s_t \quad \forall t,$$

$$x_t \leq C_t y_t \quad \forall t,$$

$$x, s \geq 0, y \in \{0, 1\}^n.$$

Wagner-Whitin Relaxation: X^{WW-C}

$$\min \sum_t (h_t s_t + f_t y_t)$$

$$s_{t-1} + \sum_{u=t}^l C_u y_u \geq d_l \text{ for } 1 \leq t \leq l \leq n$$

$$s \geq 0, y \in \{0, 1\}^n$$

Wagner-Whitin Relaxation: Advantages

Th.

When $h_t = h'_t + p'_t - p'_{t+1} \geq 0$ for all t , (or p_t is non-increasing in t), the Wagner-Whitin relaxation solves $LS - U$.

Th.

The Uncapacitated Case $WW - U$:

$\text{conv}(X^{WW-U})$ is given by

$$s_{k-1} \geq \sum_{u=k}^t d_u(1 - y_k - \dots - y_u) \quad \forall 1 \leq k \leq t \leq n.$$

Classification

	<i>LS</i>	<i>WW</i>
FORM	<i>Cons</i> \times <i>Vars</i>	<i>Cons</i> \times <i>Vars</i>
<i>U</i>	$O(n) \times O(n^2)$	$O(n^2) \times O(n)$
<i>CC</i>	$O(n^3) \times O(n^3)$	$O(n^2) \times O(n^2)$
SEP		
<i>U</i>	$O(n \log n)$	$O(n)$
<i>CC</i>	*	$O(n^2 \log n)$
OPT		
<i>U</i>	$O(n \log n)$	$O(n)$
<i>CC</i>	$O(n^3)$	$O(n^2 \log n)$

Table: Models *PROB-[U, CC]*

* indicates that only a partial description of conv is known.

- Discrete Lot-Sizing *DLSI* (all or nothing production)
- Constant *CC* and Varying Capacity *C* Models
- Variants
 - Start-ups
 - Backlogging
 - Sales (variable demand)
 - Lower Bounds on Production
 - Production Time Windows
 - Safety Stocks
 - Concave Production, Convex Storage Costs
- Multi-Level Production
- Multi-item Models

Formulation of Lot-Sizing with Start-ups

Consider the sequence of set-up periods.

An item *starts up* in period t if $y_t = 1$ and $y_{t-1} = 0$, and an item is *switched off* in period t if $y_t = 1$ and $y_{t+1} = 0$.

$z_t = 1$ if i starts up at the beginning of period t , and
 $w_t = 1$ if i is switched off at the end of period t

$$z_t \geq y_t - y_{t-1}$$

$$z_t \leq y_t$$

$$z_t \leq 1 - y_{t-1}$$

$$z_t \in \{0, 1\}.$$

or in place of the first inequality

$$z_t - w_{t-1} = y_t - y_{t-1}.$$

Note that $y_k + y_{k+1} + \cdots + y_u = 0$ if and only if
 $y_k + z_{k+1} + \cdots + z_u = 0$

Th.

$\text{conv}(X^{WW-U})$ is given by

$$s_{k-1} \geq \sum_{u=k}^t d_u(1 - y_k - z_{k+1} - \cdots - z_u) \quad \forall 1 \leq k \leq t \leq n.$$

Production with Sequence Dependent Changeovers

- In this sequencing problem the time periods are short, so at most one item is produced per period.
- The order of production is important as there are significant changeover costs every time that one switches production from one item to another.
- There are also storage costs per item and period.
- The problem is to satisfy the demands while minimizing the sum of the storage and switch-over costs.

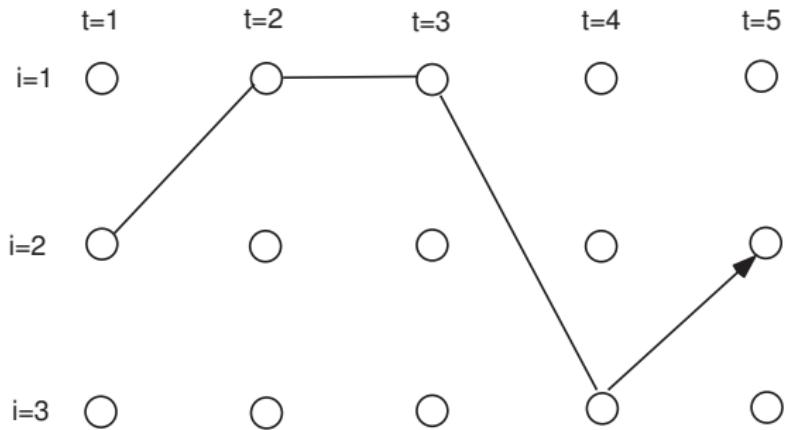
Modeling Changeovers

Define changeover variables $\chi_t^{ij} = 1$ if i is set up in period $t - 1$ and item j is set up in t , and $\chi_t^{ij} = 0$ otherwise.

Basic formulation:

$$\begin{aligned} X^{CH} = \{(\chi, y) \in \{0, 1\} : & \sum_i y_t^i = 1 \quad \forall t \\ & \chi_t^{ij} \geq y_{t-1}^i + y_t^j - 1 \quad \forall i, j, t \\ & \chi_t^{ij} \leq y_{t-1}^i, \quad \chi_t^{ij} \leq y_t^j \quad \forall i, j, t \} \end{aligned}$$

Improved Formulation



$$\sum_i y_1^i = 1$$

$$\sum_i \chi_t^{ij} = y_t^j \quad \forall j, t$$

$$\sum_j \chi_t^{ij} = y_{t-1}^i \quad \forall j, t > 1$$

$$0 \leq y_t^i, \chi_t^{ij} \quad \forall i, j, t.$$

Linking Changeovers and Lot-Sizing

Define the start-up and switch-off variables in terms of the changeover variables

$$z_t^j = \sum_{i:i \neq j} \chi_t^{ij} \text{ and } w_{t-1}^i = \sum_{j:j \neq i} \chi_t^{ij}.$$

Decomposition now suggests using $LS - CC - SC$.
Use the two relaxations $WW - U - SC$ and $WW - CC$ for which the convex hull is known and relatively compact.

Results for an Instance 5 items, 80 periods

- i) Default,
- ii) Path Reformulation,
- iii) Path + $WW - U - SC(15)$ + $WW - CC(15)$

LP	XLP	BIP	G(300)	G(60)
1131	16752	21327	20.4	26.9
11645	17978	20316	9.8	11.7
19450	19466	19792	0.6	2.3

Table: Sequence-Dependent Changeovers

The Joint Replenishment Problem

- Many different items can be produced within a period.
- Each item has its own storage costs and some have set-up costs.
- There is a aggregate production replenishment/batch capacity C and a fixed cost Q_t .
- The problem is to satisfy all the demands while minimizing the total costs.

Let Y_t be the number of batches, or in the case of replenishment $Y_t = 1$ if there is replenishment and $Y_t = 0$ otherwise. Now a basic formulation is

$$\min \sum_{i,t} (p_t^i x_t^i + h_t^i s_t^i + q_t^i y_t^i) + \sum_t Q_t Y_t$$
$$s_{t-1}^i + x_t^i = d_t^i + s_t^i \quad \forall i, t \quad (1)$$

$$x_t^i \leq M y_t^i \quad \forall i, t \quad (2)$$

$$\sum_i x_t^i \leq C Y_t \quad \forall t \quad (3)$$

$$x, s \geq 0, y \in \{0, 1\}, Y \in \mathbb{Z}_+^n \quad (4)$$

where (3) counts the number of batches (or forces a replenishment if any item is produced) in period t .

Consider a surrogate item consisting of the aggregation of all the items in $V \subseteq \{1, \dots, m\}$.

We associate with it the aggregate variables $X_t^V = \sum_{i \in V} x_t^i$, $S_t^V = \sum_{i \in V} s_t^i$ and the aggregate demands $D_t^V = \sum_{i \in V} d_t^i$.

Now

- i) $(X^V, S^V, Y) \in X^{LS-CC}$ for all V , and
- ii) $(x^i, s^i, y^i) \in X^{LS-CC}$ for all i .

Add $\text{conv}(X^{WW-CC})$ for the m surrogate items

$$V = \{1\}, \{1, 2\}, \dots, \{1, 2, \dots, m\}$$

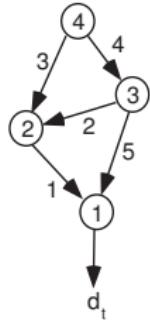
Results: 30 items, 50 periods

- i) Default,
- ii) $WW - CC$ for m surrogate items,
- iii) $WW - CC$ for m surrogate items + $WW - U$ for individual items

LP	XLP	BIP	G(300)	G(60)
2612.3	3315.2	4171.1	18.8	21.7
3389.1	3911.9	4134.8	5.4	22.7
3914.7	3918.1	4127.8	4.6	31.2

Table: Joint Replenishment Problem

Multi-level Lot-Sizing



$$s_{t-1}^4 + x_t^4 = 3x_t^2 + 4x_t^3 + s_t^4, x_t^4 \leq C_t^4 y_t^4, x^4, s^4 \geq 0, y^4 \in \{0, 1\}$$

$$s_{t-1}^3 + x_t^3 = 2x_t^2 + 5x_t^1 + s_t^3, x_t^3 \leq C_t^3 y_t^3, x^3, s^3 \geq 0, y^3 \in \{0, 1\}$$

$$s_{t-1}^2 + x_t^2 = 1x_t^1 + s_t^2, x_t^2 \leq C_t^2 y_t^2, x^2, s^2 \geq 0, y^2 \in \{0, 1\}$$

$$s_{t-1}^1 + x_t^1 = d_t^1 + s_t^1, x_t^1 \leq C_t^1 y_t^1, x^1, s^1 \geq 0, y^1 \in \{0, 1\}$$

ECHELON STOCK: e_t^i is TOTAL STOCK of i in System at t .

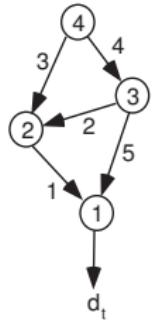
$$e_t^1 \equiv s_t^1$$

$$e_t^2 \equiv s_t^2 + e_t^1 = s_t^2 + s_t^1$$

$$e_t^3 \equiv s_t^3 + 2e_t^2 + 5e_t^1$$

$$e_t^4 \equiv s_t^4 + 3e_t^2 + 4e_t^3 = s_t^4 + 4s_t^3 + 11s_t^2 + 31s_t^1.$$

Echelon Stock Reformulation



$$e_{t-1}^4 + x_t^4 = 31d_t^1 + e_t^4, x_t^4 \leq C_t^4 y_t^4, x^4, e^4 \geq 0, y^4 \in \{0, 1\}$$

$$e_{t-1}^3 + x_t^3 = 7d_t^1 + e_t^3, x_t^3 \leq C_t^3 y_t^3, x^3, e^3 \geq 0, y^3 \in \{0, 1\}$$

$$e_{t-1}^2 + x_t^2 = 1d_t^1 x_t^1, x_t^2 \leq C_t^2 y_t^2, x^2, e^2 \geq 0, y^2 \in \{0, 1\}$$

$$e_{t-1}^1 + x_t^1 = d_t^1 + e_t^1, x_t^1 \leq C_t^1 y_t^1, x^1, e^1 \geq 0, y^1 \in \{0, 1\}$$

$$e_t^1 \geq 0, e_t^2 \geq e_t^1, e_t^3 \geq 2e_t^2 + 5e_t^1, e_t^4 \geq 3e_t^2 + 4e_t^3$$



Y. Pochet and L.A. Wolsey, Production Planning by Mixed Integer Programming,
Springer, New York, 2006.