# Vehicle Routing and MIP

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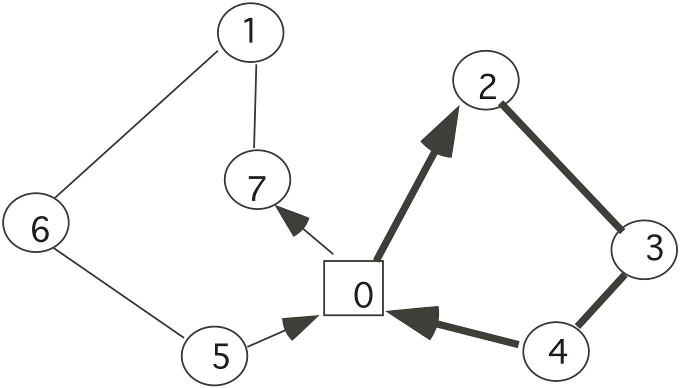
# The Capacitated Vehicle Routing Problem



- Single Depot. Single Period
- $K$ vehicles can make tours beginning and ending at the depot
- Client $i$ has a demand $q_i$ and is served by exactly one vehicle
- Each vehicle has capacity $Q$.
- The objective is to find subtours for the $K$ vehicles such that the amount delivered by each vehicle does not exceed $Q$, the demand of each client is satisfied and the total travel cost/distance is minimized.

# Modelling Connectivity: Spanning Trees

Graph $G = (V, E)$ with edge costs $c_e$ for $e \in E$.
Let $y_e = 1$ if $e$ is in the tree.

$$\min \sum_{e \in E} c_e y_e$$
$$\sum_{e \in E(S)} y_e \leq |S| - 1 \; \forall \; S \subset V$$
$$\sum_{e \in E} y_e = |V| - 1$$
$$y \in \mathbb{R}^E_+$$

where $E(S) = \{e = (i,j) \in E : i,j \in S\}$.
This linear program is tight. The subtour elimination constraints (SECs) describe the convex hull of the spanning trees, but there is an exponential number of constraints.

# Separation of SECs

Given a point $\hat{y} \in \mathbb{R}^E_+$, when does it violate one of the SECs?
We formulate this question as an IP.
Let $\alpha_i = 1$ if $i \in S$. We have to check whether there exists $\alpha \in \{0,1\}^V$ such that

$$\sum_{e=(i,j)\in E} \hat{y}_e \alpha_i \alpha_j - \sum_{i \in V} \alpha_i > -1.$$

This is a quadratic 0-1 integer program of the form:

$$\max_{\alpha \in \{0,1\}^V} \alpha^T Q \alpha - p\alpha.$$

It is well known that such IPs can be solved as max flow/min cut problems when $Q \geq 0$.

# An Extended “Multicommodity" Formulation

How else can one ensure connectivity?
One way: Construct a graph in which there is a path (one can send a flow) from one node ($r = 1$) to all the others.
$f_{ij}^k$ is the flow (commodity $k$) in arc $(i,j)$ on the path from $r$ to $k$.

$$\begin{aligned}
\sum_j f_{ij}^k - \sum_j f_{ji}^k &= 0 \quad i \neq k, r \\
\sum_j f_{ij}^k - \sum_j f_{ji}^k &= 1 \quad i = k \\
f_{ij}^k + f_{ji}^k &\leq y_e \quad \forall\, e \in E \\
f_{ij}^k &\geq 0 \ \forall\, i, j, k
\end{aligned}$$

This formulation is as strong as that with subtours, but it has $O(V^3)$ variables.

# The Traveling Salesman Problem: Formulation 1

Symmetric Version $c_{ij} = c_{ji} = c_e$
$y_e = 1$ if edge $e$ in the tour

$$\min \sum_{e \in E} c_e y_e$$
$$\sum_{e \in \delta(v)} y_e = 2 \ \forall \ i \in V$$
$$\sum_{e \in E(S)} y_e \leq |S| - 1 \ \forall \ S \subset V$$
$$y \in [0,1]^E$$

SECS can be replaced by “cut" inequalities

$$\sum_{e \in \delta(S)} y_e \geq 2 \ \forall \emptyset \subset S \subset V.$$

# The Traveling Salesman Problem: Formulation 2

Weak Extended Formulation (Directed):
$u_i$ is position of node $i$ in the tour starting with node 1 in position 0 ($u_1 = 0$).

$$\begin{aligned}
u_j - u_i \geq x_{ij} - (n-2)(1-x_{ij}) \; \forall \; (i,j), j \neq 1 \\
\sum_j x_{ij} = 1 \; \forall \; i \\
\sum_i x_{ij} = 1 \; \forall \; j \\
y_e = x_{ij} + x_{ji} \; \forall \; e \\
x \in \mathbb{Z}_+^A, u \in \mathbb{Z}_+^n
\end{aligned}$$

Useful if one wants a compact valid formulation.

# Formulations of CVRP

$x_{ij}^k = 1$ if arc $(i,j)$ in tour of vehicle $k$
$z_i^k = 1$if client $i$ visited by vehicle $k$

$$\min \sum_k \sum_{ij} c_{ij}^k x_{ij}^k$$

$$\sum_i x_{ij}^k = z_j^k \; \forall\, k, j, j \neq 0,$$

$$\sum_j x_{ij}^k = z_i^k \; \forall\, k, i, i \neq 0,$$

$$\sum_j x_{0j}^k = z_0^k \; \forall\, k$$

$$\sum_i d_i z_i^k \leq C z_0^k \; \forall\, k$$

$$\sum_k z_i^k = 1 \; \forall\, i$$

$$x_{ij}^k, z_i^k \in \{0, 1\}$$

# Formulations of CVRP with 2 Indices

$$\min \sum_{ij} c_{ij} x_{ij}$$

$$\sum_{i} x_{ij} = 1 \; \forall \, j \neq 0,$$

$$\sum_{j} x_{ij} = 1 \; \forall \, i \neq 0,$$

$$\sum_{j} x_{0j} = K$$

$$\sum_{ij \in \delta(\bar{S}, S)} x_{ij} \geq \lceil \frac{\sum_{i \in S} d_i}{C} \rceil \; \forall \, S \subseteq V$$

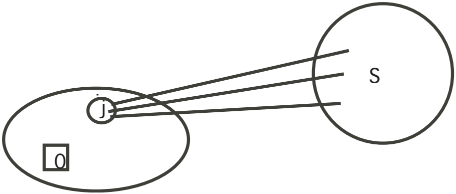
$$x_{ij} \in \{0, 1\}$$

1. Separation of $\sum_{ij \in \delta(\bar{S},S)} x_{ij} \geq \lceil \frac{\sum_{i \in S} d_i}{C} \rceil$?

Separation of $\sum_{ij \in \delta(\bar{S},S)} x_{ij} \geq \frac{\sum_{i \in S} d_i}{C}$ is polynomial.

2. Generalized Large Multistar

$$\sum_{e \in \delta(S)} y_e \geq \frac{2}{C}\left(d(S) + \sum_{j \notin S} d_j \left( \sum_{e \in \delta(S:\{j\})} y_e \right)\right)$$



# Column Generation

- A natural idea is to have columns corresponding to feasible subtours for a vehicle.
- The problem is then just to select a set of $K$ subtours such that each client is visited once.
- The trouble with this is that the column generation subproblem is a prize-collecting traveling salesman problem that is almost as hard as the original problem.
- So the typical approach is to look at a larger set of columns that includes all the feasible subtours, but for which the column generation problem is more tractable.

# q-routes

A q-route is a walk starting and ending at the depot visiting the clients 0,1, or more times but with total demand at most $C$. Note that each time a client $i$ is visited, his demand $d_i$ is counted again.

Suppose $n = 4$, $C = 13$ and $d = (2, 4, 5, 7)$.
A feasible subtour is for example $0 - 3 - 2 - 0$ with $\sum_i d_i = 9$.
A q-route, that is not a feasible subtour, is $0 - 1 - 2 - 1 - 0$ with $\sum_i d_i = 8$.
$0 - 1 - 2 - 3 - 1 - 0$ is a q-route without a 2-cycle as $\sum_i d_i = 13 \leq C$.
A minimum cost q-route without 2-cycles can be found by dynamic programming.

# Naive DP for q-routes

$f(i,j,k)$ is the min cost of a walk starting at the depot and ending with visits to $i$, then $j$ in which $k$ units are delivered.

$$f(i,j,k) = \min_{p:p\neq j}[f(p,i,k-d_j) + \bar{c}_{ij}]$$

Min Cost q-route is

$$\min_{i,j,k\leq C}[f(i,j,k) + \bar{c}_{0j}].$$

# Column Generation with $q$-routes without 2-cycles

Let $q_j^e$ be the coefficient of edge $e$ in $q$-route $j$ with variable $\lambda_j$.
One has

$$\begin{array}{rrcl}
\sum_{j=1}^p q_j^e \lambda_j & -x_e & = & 0 \; e \in E \\
\sum_{j=1}^p \lambda_j & & = & K \\
& \sum_{e \in \delta(i)} x_e & = & 2 \; i \in V \\
& x_e & \geq & 0 \; e \in E \\
\lambda_j & & \geq & 0 \; j = 1, \ldots, p
\end{array}$$

# The Master Problem

$$
\begin{array}{rcccl}
 & \sum_{e\in\delta(i)} x_e & = & 2 & i \in V \\
 & \sum_{e\in\delta(0)} x_e & = & 2K & \\
 & \sum_{e\in\delta(S)} x_e & \geq & 2k(S) & S \subseteq V \\
 & x_e & \leq & 1 & e \in E \setminus \delta(0) \\
\sum_{j=1}^{p} q_j^e \lambda_j & -x_e & = & 0 & e \in E \\
\sum_{j=1}^{p} \lambda_j & & = & K & \\
 & x_e & \geq & 0 & e \in E \\
\lambda_j & & \geq & 0 & j = 1, \ldots, p
\end{array}
$$

# The Master Problem

$$\min \sum_{j=1}^{p} \sum_{e \in E} l_e q_j^e \lambda_j$$

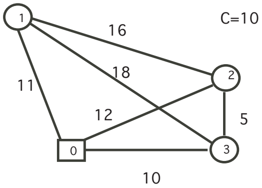
$$\begin{array}{llcl}
\mu & \sum_{j=1}^{p} \sum_{e \in \delta(i)} q_j^e \lambda_j & = & 2 \; i \in V \\
\nu & \sum_{j=1}^{p} \sum_{e \in \delta(0)} q_j^e \lambda_j & = & 2K \\
\pi & \sum_{j=1}^{p} \sum_{e \in \delta(S)} q_j^e \lambda_j & \geq & 2k(S) \; S \subseteq V \\
\omega & \sum_{j=1}^{p} q_j^e \lambda_j & \leq & 1 \; e \in E \setminus \delta(0) \\
 & \lambda_j & \geq & 0 \; j = 1, \ldots, p
\end{array}$$

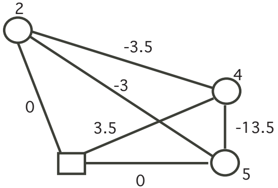
Reduced cost of $x_e$: $l_e - \mu_i - \mu_j - \sum_{e \in \delta(S)} \pi_S - \omega_e \quad e \in E \setminus \delta(0)$

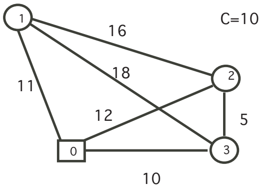
$l_e - \mu_j - \nu - \sum_{e \in \delta(S)} \pi_S \quad e \in \delta(0)$

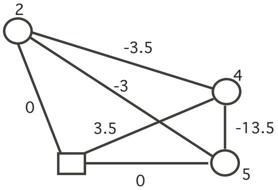
# $d = (2,4,5), C = 10$

| | $w_1$ | $w_2$ | $w_3$ | $w_{12}$ | | | $u$ |
|---|---|---|---|---|---|---|---|
| $cw =$ | 22 | 24 | 20 | 39 | | 59 | |
| $v = 1$ | 2 | 2 | | 2 | $=$ | 2 | 11 |
| $v = 2$ | | 2 | | 2 | $=$ | 2 | 8.5 |
| $v = 3$ | | | 2 | | $=$ | 2 | 10 |
| $v = 0$ | 2 | 2 | 2 | 2 | $=$ | 4 | |
| $1,2$ | 2 | 2 | | 2 | $\geq$ | 2 | |
| $1,3$ | 2 | | 2 | 2 | $\geq$ | 2 | |
| $2,3$ | | 2 | 2 | 2 | $\geq$ | 2 | |
| $1,2,3$ | 2 | 2 | 2 | 2 | $\geq$ | 4 | |
| $e = 12$ | | | | 1 | $\leq$ | 1 | |






min cost q-route: 0 – 2 – 3 – 0 with reduced cost of -10

alternative q-route: 0 – 1 – 2 – 1 – 0 with reduced cost of -7

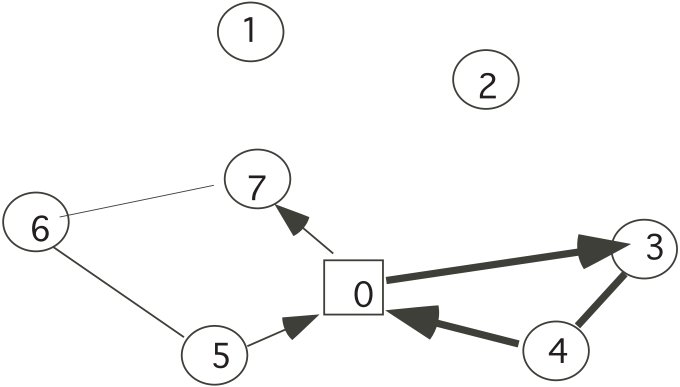
| | $w_1$ | $w_2$ | $w_3$ | $w_{12}$ | $w_{23}$ | | | $u$ |
|---|---|---|---|---|---|---|---|---|
| $cw =$ | 22 | 24 | 20 | 39 | 27 | | | 49 |
| $v = 1$ | 2 | 2 | | 2 | | $=$ | 2 | 7.5 |
| $v = 2$ | | 2 | | 2 | 2 | $=$ | 2 | 8.5 |
| $v = 3$ | | | 2 | | 2 | $=$ | 2 | 6.5 |
| $v = 0$ | 2 | 2 | 2 | 2 | 2 | $=$ | 4 | 3.5 |
| $1, 2$ | 2 | 2 | | 2 | 2 | $\geq$ | 2 | |
| $1, 3$ | 2 | | 2 | 2 | 2 | $\geq$ | 2 | |
| $2, 3$ | | 2 | 2 | 2 | 2 | $\geq$ | 2 | |
| $1, 2, 3$ | 2 | 2 | 2 | 2 | 2 | $\geq$ | 4 | |
| $e = 12$ | | | | 1 | | $\leq$ | 1 | |
| $e = 23$ | | | | | 1 | $\leq$ | 1 | $-10$ |

# Recent Developments

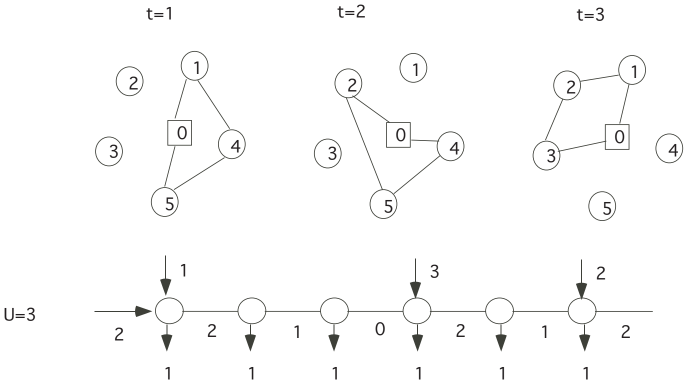
Recently $q$-routes replaced by ng-routes
Note that can add constraints $\alpha_e x_e \leq \alpha_0$ and approach still works.
Robust branch-and-cut-and-price.
If cuts on the q-route variables - column generation becomes more difficult

What about time-windows?
Similar BCP approach.
Column generation may be easier - less choice.
Cut generation more difficult - cuts based on infeasibility.

# The Inventory Routing Problem: One Period



# The Inventory Routing Problem



# Model 1: Single Period Inventory Routing Problem

$$\sum_{i \in I} x^i \leq Cz^0, \;\; x^i \leq W^i z^i, \quad z^0 \geq z^i, \quad i \in I$$

$$\sum_{j \in I0} y^{ij} = \sum_{j \in I0} y^{ji} = z^i, \quad j \in I0,$$

$$\sum_{i \in S \cup \{0\}} \sum_{j \in I \setminus S} y^{ij} \geq z^i, \;\; S \subseteq I,$$

*Capacitated Subtours*

$$x, \in R_+^T, z^i \in \{0,1\} \; i \in I, z^0 \in \{0,1,\ldots,K\}, y^{ij} \in \{0,1\}^A$$

# First Observations

Where do the capacities come from?

$$W_t^i = \min\{C, d_t^i + \bar{S}^i\}$$

What about the capacitated subtours?

$$C \sum \sum_{i \in I0 \setminus S} \sum_{j \in S} y^{ij} \geq \sum_{i \in S} x^i$$

where $S \subseteq I$ and $I0 = I \cup \{0\}$.

# 3-index formulation

Direct a flow of $x^k$ from the origin to each client $k \in I$:
$f^{ijk}$ is the flow i $(i, j)$ with destination $k \in I$

$$\sum_{i \in I0} f^{ijk} - \sum_{i \in I} f^{jik} = 0 \quad \forall\, j \neq k, \forall\, k$$

$$\sum_{i \in I0} f^{ijk} = x^k \quad k = j, \forall\, k$$

$$\sum_{k \in I} f^{ijk} \leq C y^{ij} \;\forall\, (i, j)$$

$$f^{ijk} \leq W^k y^{ij} \forall\, i, j, k$$

# Metric Inequalities

Suppose $\mu^{ij} \geq 0$ for all $(i,j) \in A$ and $S \subseteq I$, then

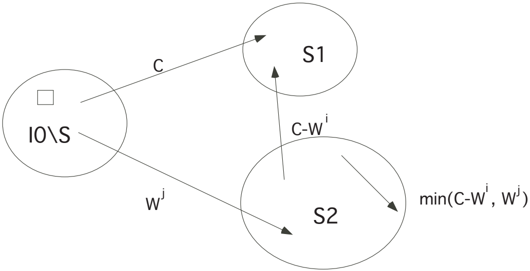
$$\sum_{(i,j)\in A} \mu^{ij} y^{ij} \geq \sum_{i\in S} x^i$$

is a valid inequality if

$$\sum_{(i,j)\in P} \mu^{ij} \geq \min(C, \sum_{i\in v(P)\cap S} W^i)$$

for every subtour $P$ beginning and ending at the depot, where $v(P) \subseteq I$ are the clients on subtour $P$.

# A Particular Metric Inequality




$S_1, S_2$ is a partition of $S$

$$\sum_{(I0\backslash S, S_1)} Cy^{ij} + \sum_{(I0\backslash S, S_2)} W^j y^{ij} + \sum_{(S_2, S_1)} (C - W^i) y^{ij}$$

$$+ \sum_{(S_2, S_2)} \min(C - W^i, W^j) y^{ij} \geq \sum_{i \in S} x^i$$

# Example of a Particular Metric Inequality

$$\sum_{(I0 \setminus S, S_1)} C y^{ij} + \sum_{(I0 \setminus S, S_2)} W^j y^{ij} + \sum_{(S_2, S_1)} (C - W^i) y^{ij}$$
$$+ \sum_{(S_2, S_2)} \min(C - W^i, W^j) y^{ij} \geq \sum_{i \in S} x^i$$

$|I| = 3$, $C = 300$, $W = (289, 123, 76)$, $S = \{1, 2, 3\}$,
$S_1 = \{1\}, S_2 = \{2, 3\}$

$$300y^{01} + 123y^{02} + 76y^{03} + (300 - 123)y^{21} + (300 - 76)y^{31}$$
$$+ \min(76, 300 - 123)y^{23} + \min(123, 300 - 76)y^{32} \geq x^1 + x^2 + x^3$$

# Multi-Period Metric Inequalities with Stock Upper Bounds: An Example

Add $s^i_{t-1} + x^i_t = D^i_t + s^i_t$ and IRP model for periods $1, \ldots, T$.

$$C \sum_{u=k}^{t} \sum_{(i,j)\in(I0\setminus S,S)} y^{ij}_u \geq \sum_{u=k}^{t} \sum_{i\in S} x^i_t = \sum_{i\in S} \sum_{u=k}^{t} x^i_u$$

$$\geq \sum_{i\in S} (D^i_{kt} - s^i_{k-1})$$

With $s^i_{k-1} \leq U^i$, above is of the form:

$$\sigma + Cv \geq b, 0 \leq \sigma \leq h, v \in Z.$$

This gives the inequalities:

$$v \geq \lceil \frac{b-h}{C} \rceil \text{ and } \sigma + \rho v \geq \rho \lceil \frac{b}{C} \rceil.$$

Table: VMIRP-SUB: computational results for the instances with $n = 50$ and $T_{max} = 6$

| | | BC | | | | | C&L | | |
|---|---|---|---|---|---|---|---|---|---|
| No | $LB_{Ini}$ | $LB_{LS}$ | $LB_{Cut}$ | BUB | Nodes | Time | BLB | BUB | Time |
| I1 | 8375 | 9754 | 9754 | **9966** | 15123 | 1485 | 9901 | 9976 | 86400 |
| I2 | 8952 | 10516 | 10523 | **10632** | 65 | 334 | 10632 | **10632** | 2536 |
| I3 | 8725 | 10376 | 10391 | **10511** | 3972 | 1876 | 10511 | **10511** | 1355 |
| I4 | 8628 | 10243 | 10243 | **10513** | 166667 | 18016 | 10513 | **10513** | 60289 |
| I5 | 8386 | 9860 | 9899 | **10113** | 2500 | 2327 | 10113 | **10113** | 2416 |
| I6 | 8417 | 9945 | 9948 | **10148** | 1900 | 2318 | 10114 | 1014x | 86400 |
| I7 | 8355 | 9776 | 9776 | **9982** | 284288 | 28195 | 9982 | **9982** | 14698 |
| I8 | 8385 | 10015 | 10066 | **10299** | 878 | 1360 | 10253 | 10229 | 86400 |
| I9 | 8484 | 9897 | 9904 | **10010** | 819 | 801 | 10010 | **10010** | 6326 |
| I10 | 8014 | 9546 | 9546 | **9659** | 2425 | 2081 | 9659 | **9659** | 3523 |
| h1 | 29508 | 29862 | 29906 | **30189** | 1235 | 645 | 30189 | **30189** | 3036 |
| h2 | 27983 | 29601 | 29615 | **29790** | 133 | 357 | 29790 | **29790** | 3334 |
| h3 | 27830 | 29634 | 29657 | **29791** | 219 | 809 | 29791 | **29791** | 4020 |
| h4 | 29517 | 31241 | 31241 | **31518** | 1424 | 1618 | 31518 | **31518** | 5737 |
| h5 | 27413 | 28993 | 29021 | **29240** | 199 | 565 | 29240 | **29240** | 684 |
| h6 | 30008 | 31621 | 31630 | **31903** | 367 | 1048 | 31903 | **31903** | 28320 |
| h7 | 27933 | 29397 | 29397 | **29734** | 7988 | 1703 | 29734 | **29734** | 13561 |
| h8 | 23923 | 25692 | 25692 | **25954** | 328 | 1202 | 25954 | **25954** | 21552 |
| h9 | 28467 | 29863 | 29884 | **30193** | 390 | 822 | 30193 | **30193** | 20581 |
| h10 | 29508 | 31101 | 31101 | **31338** | 83 | 488 | 31338 | **31338** | 1879 |

Table: Computational results for the instances 1n15T6 with $n = 15$ and $T_{max} = 6$

| | k | BC | | | | | C&L | |
|---|---|---|---|---|---|---|---|---|
| | k | Subtour | LS | Prop1 | Gen | Cover | BLB | BUB |
| l1 | 2 | 4880.7 | 5581.1 | 5681.8 | 5732.8 | 5803.6 | 5987.4 | 5987.4 |
| l1 | 3 | 5640.7 | 6222.0 | 6541.4 | 6580.9 | 6684.5 | 6861.07 | 6861.07 |
| l1 | 4 | 6527.8 | 7049.5 | 7484.6 | 7525.6 | 7622.3 | 7320.32 | 7767.75 |
| l1 | 5 | 7466.0 | 7697.3 | 8377.4 | 8480.9 | 8512.7 | 7574.4 | 8975.61 |
| h1 | 2 | 11388.5 | 12196.1 | 12305.6 | 12367.0 | 12369.2 | 12624.7 | 12624.7 |
| h1 | 3 | 12150.2 | 12820.6 | 13165.7 | 13248.6 | 13310.4 | 13517.6 | 13517.6 |
| h1 | 4 | 13039.4 | 13642.5 | 14111.8 | 14217.6 | 14240.1 | 13999.8 | 14515.3 |
| h1 | 5 | 13975.3 | 14540.1 | 15006.9 | 15109.4 | 15156.1 | 14262.2 | 15470.7 |

The MIP systems are amazing, but sometimes one can still help.

Thank you for your attention

CVRP

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