Equilibria on a Game with Discrete Variables

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Abstract. Equilibrium in Economics has been seldom addressed in a situation where some variables are discrete. This work introduces a problem related to lot-sizing with several players, and analyses some strategies which are likely to be found in real world games. An illustration with a simple example is presented, with concerns about the difficulty of the problem and computation possibilities.

1 Introduction

Market equilibrium is a classical problem arising in Economics, with applications ranging from the analysis of market power to the simulation of new types of regulation. The most studied versions involve agents that choose a (continuous) variable; a Nash equilibrium occurs when no firm can do better by unilaterally changing its strategy. When the variable being played is the quantity, this is usually called a Cournot equilibrium when all firms play simultaneously; when a leader firm moves first, and follower firms move afterwords, this is called a Stackelberg equilibrium (see e.g. [1] for an introduction).

Much less attention has been drawn to those games where some variables in this competition are discrete. In this case, the computation of equilibria is a much harder task; actually, determining the optimal strategy for a single player is itself an NP-complete problem in many situations.

Let us start with the description of the market we will deal with in this paper. The demand is forecast with complete certainty for a number of coming periods. A firm has the possibility of producing in a given period or not; demand is met in each period with goods either produced or existing in inventory. There are fixed (setup) and inventory costs, and the production capacity is limited. The variables for each firm are, thus:

- decide whether to produce or not, for each period in the planning horizon;
- choose the quantity to place in the market in each period.

If the demand is independent of the price (i.e. it is a fixed amount) and there is only one firm, this leads to the well known lot-sizing problem [2]. We are more interested in the case where demand is dependent of the price; this dependence will be captured by modelling demand as a linear function of the price (or, equivalently, price as a linear function of the quantity put in the market). This leads to a lot-sizing variant where the firm, instead of simply meeting the demand, must decide what quantity to put in the market. Price is, thus, a function of the total quantity, i.e., the quantity played summed for all the firms.

2 Some interesting markets

2.1 Example

For the sake of clarity, we will use an example throughout this paper.

Demand is different from period to period, and is modelled by:

$$P_t(Q_t) = \max(a_t - b_t Q_t, 0), \quad \text{for } t = 1, \ldots, T, \tag{1}$$

where t is the period, T is the total number of periods, a_t, b_t are parameters of the model, and $Q_t \geq 0$ is the total quantity placed in the market in period t.

The decision variables concern producing or not in each period, as well as the amount to produce, and the amount to place in the market. Let y be the vector of setup, binary variables, where y_t is 1 if there is production during period t, and 0 otherwise. Variables x_t and q_t are, respectively, the corresponding manufactured amount and the quantity placed in the market in t, and the quantity held in inventory at the end of the period t is h_t; these are non-negative, continuous variables. The bill of materials can be written as:

$$h_{t-1} + x_t = q_t + h_t, \quad \text{for } t = 1, \ldots, T. \tag{2}$$

We assume that there is a limit K on the capacity available on each period, and production can only occur if machines have been setup; this implies that

$$x_t \leq K y_t, \quad \text{for } t = 1, \ldots, T. \tag{3}$$

Let us denote the fixed production costs by F and the unit inventory costs by H. For a given production plan the total costs are:

$$C(y, h) = \sum_{t=1}^{T} (F y_t + H h_t).$$

We will provide a numerical example, allowing to draw conclusions for simple situations. Demand is enough for at least one firm to be able to produce with profit on the first half of the horizon, and larger for the second half. Inventory costs are such that it is worthy to produce some periods in advance (when demand justifies it), but too early production is discouraged by them.

We assume throughout this paper that firms know each others' costs and capacities (i.e., technology is known).

2.2 Monopoly

In the monopoly case, all the variables are under the control of the firm. In this case, market quantities are those decided by the monopolist, $Q_t = q_t^M$, and the corresponding price is given by Equation 1.

The profit is therefore given by

$$\Pi = \sum_{t=1}^{T} \left[q_t^M P_t(q_t^M) - (F y_t + H h_t) \right]. \tag{4}$$

For a single period, the optimal result for this model is well known; maximum profit is obtained when the derivatives with respect to the quantity are zero, which for linear demand leads to optimal quantities

$$q^{*M} = \frac{a}{2b}.$$

The firm will produce if the corresponding profit is positive, i.e., if the revenue is larger than the fixed cost F.

When there are several periods, the situation is more complex; an illustration is provided for our example in Table 1. The firm produces a small quantity for the initial periods, and produces at full capacity after the demand raises. Notice that this problem is NP-hard even when the quantities are fixed [3] (this is the "standard" lot-sizing problem, i.e., all that the firm has to do is to meet demand at minimal cost), so for more realistic examples the computation of the optimum is not trivial. For obtaining the results presented in this paper we used the software Couenne [4], which is based on the latest developments in mixed-integer nonlinear programming [5].

			$F = 10, H = 1, K = 10$				$F = 10, H = 1, K = 25$			
Period (t)	a_t	b_t	y_t	x_t	h_t	q_t	y_t	x_t	h_t	q_t
1	10	1	1	5.00	0.00	5.00	1	13.49	8.49	4.99
2	10	1	1	9.50	4.50	5.00	0	0.00	4.00	4.50
3	10	1	0	0.00	0.00	4.50	0	0.00	0.00	4.00
4	10	0.5	1	10.00	0.00	10.00	1	10.00	0.00	10.00
5	10	0.5	1	10.00	0.00	10.00	1	19.00	9.00	10.00
6	10	0.5	1	10.00	0.00	10.00	0	0.00	0.00	9.00
			$\Pi = 170.25$				$\Pi = 171.75$			
1	10	0.25	1	10.00	0.00	10.00	1	20.00	0.00	20.00
2	10	0.25	1	10.00	0.00	10.00	1	23.01	3.01	20.00
3	10	0.25	1	10.00	2.00	8.00	1	25.00	10.01	18.00
4	10	0.125	1	10.00	0.00	12.00	1	25.00	3.00	32.01
5	10	0.125	1	10.00	0.00	10.00	1	25.00	0.00	28.00
6	10	0.125	1	10.00	0.00	10.00	1	25.00	0.00	25.00
			$\Pi = 429.00$				$\Pi = 768.875$			

Table 1. Optimal results for the monopoly situation, under four parameter sets.

2.3 Oligopoly

Let us now consider the case of several firms operating in the market, an oligopoly. The difference between the monopoly and the current situation is that now the total quantity put in the market is no longer the decision of a single firm. This can be written in the profit for each firm i as:

$$\Pi^i = \sum_{t=1}^{T} \left[q_t^i P_t(\sum_{j=1}^{N} q_t^j) - (F^i y_t^i + H^i h_t^i) \right]. \tag{5}$$

where N is the number of firms operating.

One-period games. For a single period, an equilibrium for this model can again be derived analytically, and is well known: it is the Cournot equilibrium. Assuming that the fixed costs are null or low enough, maximum profit is obtained when the partial derivatives of the profit with respect to the quantity are zero, for each firm, leading to the system of equations

$$q^i = \frac{a + b \sum_{j \neq i} q^j}{2b}.$$

At this point, no firm has incentive to deviate; any unilateral variation will lead to a smaller profit.

The solution is more complex in the presence of fixed costs. The previous equilibrium may be a solution with positive profit for all firms in this case too. When not all firms can produce with profit, i.e., fixed costs F^i are larger than the revenue at the Cournot equilibrium, there may be the case that one firm can produce with profit, but if any other enters the market, all will have losses.

Another potential equilibrium, seldom analysed, occurs when one firm plays a large quantity, in order to try to put the others out of the market; this "aggressive" firm plays a large quantity, in such a way that the other companies' profit is zero (or negative, if they produce a positive amount). Clearly, if all firms play this large amount, they all will be worse off; but if one of them succeeds imposing this quantity, as in a Stackelberg equilibrium, the others' optimal strategy is no production.

Yet another possibility in this game occurs when the firms coalesce and maximise the sum of the profits of all firms, deciding in another stance how to share them. This situation is quite similar to the monopoly case.

The multi-period situation is considerably more complex. We present next some results for equilibria with a single move for all periods; an *iterated* version will be developed in section 3.

Multi-period duopoly: a case study. For illustrating the duopoly case we take the same production and demand parameters used for the monopoly example, and analyse the behaviour of a market with two firms.

For the following results we use fixed-point iteration, and the software Couenne for optimisation. In each iteration, Firm 1 optimises its quantities based on the previous output of Firm 2, and vice-versa; details are available in Algorithm 1, where $\bar{q}^i$ are initial values for the quantities played, $\bar{q}^{*i}$ are the corresponding values optimised for the competitor's current value, and q^i the decision variables in each optimization problem; ϵ is a convergence criterion. Initial quantities for Firm 2 are zero for all periods, except if otherwise stated.

Algorithm 1: A fix-point iteration for the duopoly equilibrium

initialise $\bar{q}^1$ and $\bar{q}^2$
repeat
  $\bar{q}^{*1} \leftarrow \text{argmax}(\Pi^1(q^1, \bar{q}^2))$
  $\bar{q}^{*2} \leftarrow \text{argmax}(\Pi^2(\bar{q}^{*1}, q^2))$
  $\Delta \leftarrow |\bar{q}^{*1} - \bar{q}^1| + |\bar{q}^{*2} - \bar{q}^2|$
  $\bar{q}^1 \leftarrow \bar{q}^{*1}$
  $\bar{q}^2 \leftarrow \bar{q}^{*2}$
until $\Delta < \epsilon$

Let us first analyse the case where firms are symmetric, as in the results presented in Table 2. The first observation is that when the setup decisions are important, even though the profits at equilibrium are roughly the same, the quantities played in each period by each of the firms may be considerably different.

Notice that for small demand, when capacities increase, both firms may become worse off even if the costs are unchanged; this has occurred from the topmost to the second situations in Table 2. When demand is large enough, this no longer occurs (third and fourth entries in the table).

It is common to have several equilibria on games with discrete variables; an illustration with our example is presented in Table 3. This tables shows two different outcomes of the fixed-point iteration, obtained using different starting solutions. Both firms are better off in the top scenario, even though no one has incentive to deviate from the situation in the bottom.

There may also be the case that the capacities are asymmetric; in this case the firm with larger capacity has a competitive advantage, as shown in Table 4.

Let us now turn to the case where one firm decides to play a quantity such that the other is put out of the market, as in a Stackelberg equilibrium. The problem of optimally determining that (leader's) quantity is not trivial, as a sub-problem of this is to determine the point where profit becomes non-null in a minimum cost production plan (this is the follower's problem). Empirically, this equilibrium can be determined by increasing the quantity put into the market by one firm, until the optimal response of the opponents be to produce zero. Results for this case are presented in Table 5. Note that we are unsure if this is an equilibrium; we just verified that any slight reduction in Firm 2's quantities

			Firm 1				Firm 2			
			$F = 10, H = 1, K = 10$				$F = 10, H = 1, K = 10$			
Period (t)	a_t	b_t	y_t	x_t	h_t	q_t	y_t	x_t	h_t	q_t
1	10	1	1	8.37	5.04	3.33	1	6.33	3.00	3.34
2	10	1	0	0.00	2.04	3.00	0	0.00	0.00	3.00
3	10	1	0	0.00	0.00	2.04	1	10.00	6.08	3.92
4	10	0.5	1	10.00	4.41	5.59	0	0.00	0.00	6.08
5	10	0.5	0	0.00	0.00	4.41	1	10.00	3.54	6.46
6	10	0.5	1	8.22	0.00	8.22	0	0.00	0.00	3.54
			$\Pi^1 = 67.13$				$\Pi^2 = 65.72$			
			$F = 10, H = 1, K = 25$				$F = 10, H = 1, K = 25$			
Period (t)	a_t	b_t	y_t	x_t	h_t	q_t	y_t	x_t	h_t	q_t
1	10	1	1	8.33	5.00	3.34	1	6.33	3.00	3.34
2	10	1	0	0.00	2.00	3.00	0	0.00	0.00	3.00
3	10	1	0	0.00	0.00	2.00	1	9.33	5.33	4.00
4	10	0.5	1	12.67	5.33	7.34	0	0.00	0.00	5.33
5	10	0.5	0	0.00	0.00	5.33	1	12.66	5.33	7.33
6	10	0.5	1	7.34	0.00	7.34	0	0.00	0.00	5.33
			$\Pi^1 = 62.15$				$\Pi^2 = 61.43$			
			$F = 10, H = 1, K = 10$				$F = 10, H = 1, K = 10$			
Period (t)	a_t	b_t	y_t	x_t	h_t	q_t	y_t	x_t	h_t	q_t
1	10	0.25	1	10.00	0.00	10.00	1	10.00	0.00	10.00
2	10	0.25	1	10.00	0.89	9.11	1	10.00	0.90	9.10
3	10	0.25	1	10.00	3.11	7.78	1	10.00	3.11	7.78
4	10	0.125	1	10.00	0.22	12.89	1	10.00	0.23	12.89
5	10	0.125	1	10.00	0.00	10.22	1	10.00	0.00	10.23
6	10	0.125	1	10.00	0.00	10.00	1	10.00	0.00	10.00
			$\Pi^1 = 321.375$				$\Pi^2 = 321.368$			
			$F = 10, H = 1, K = 25$				$F = 10, H = 1, K = 25$			
Period (t)	a_t	b_t	y_t	x_t	h_t	q_t	y_t	x_t	h_t	q_t
1	10	0.25	1	13.34	0.00	13.34	1	13.34	0.00	13.34
2	10	0.25	1	13.34	0.00	13.34	1	13.34	0.00	13.34
3	10	0.25	1	13.34	0.00	13.34	1	13.34	0.00	13.34
4	10	0.125	1	25.00	0.00	25.00	1	25.00	0.00	25.00
5	10	0.125	1	25.00	0.00	25.00	1	25.00	0.00	25.00
6	10	0.125	1	25.00	0.00	25.00	1	25.00	0.00	25.00
			$\Pi^1 = 354.558$				$\Pi^2 = 354.55$			

Table 2. Results for a duopoly: Cournot equilibria for the lot-sizing problem.

			Firm 1				Firm 2			
			$F = 10, H = 1, K = 10$				$F = 10, H = 1, K = 10$			
Period (t)	a_t	b_t	y_t	x_t	h_t	q_t	y_t	x_t	h_t	q_t
1	10	1	1	8.37	5.04	3.33	1	6.33	3.00	3.34
2	10	1	0	0.00	2.04	3.00	0	0.00	0.00	3.00
3	10	1	0	0.00	0.00	2.04	1	10.00	6.08	3.92
4	10	0.5	1	10.00	4.41	5.59	0	0.00	0.00	6.08
5	10	0.5	0	0.00	0.00	4.41	1	10.00	3.54	6.46
6	10	0.5	1	8.22	0.00	8.22	0	0.00	0.00	3.54
			$\Pi^1 = 67.13$				$\Pi^2 = 65.72$			
1	10	1	1	9.00	5.67	3.33	1	9.01	5.67	3.34
2	10	1	0	0.00	2.67	3.00	0	0.00	2.67	3.00
3	10	1	0	0.00	0.00	2.67	0	0.00	0.00	2.67
4	10	0.5	1	8.00	0.66	7.34	1	10.00	4.67	5.33
5	10	0.5	1	10.00	4.00	6.67	0	0.00	0.00	4.67
6	10	0.5	0	0.00	0.00	4.00	1	8.00	0.00	8.00
			$\Pi^1 = 64.30$				$\Pi^2 = 64.34$			

Table 3. Different equilibria obtained for the same parameter set.

			Firm 1				Firm 2			
			$F=10, H=1, K=10$				$F=10, H=1, K=25$			
Period (t)	a_t	b_t	y_t	x_t	h_t	q_t	y_t	x_t	h_t	q_t
1	10	1	1	8.33	5.00	3.33	1	6.33	3.00	3.33
2	10	1	0	0.00	2.00	3.00	0	0.00	0.00	3.00
3	10	1	0	0.00	0.00	2.00	1	9.34	5.34	3.99
4	10	0.5	1	7.33	0.00	7.32	0	0.00	0.00	5.34
5	10	0.5	1	10.00	4.67	5.34	1	13.99	6.67	7.32
6	10	0.5	0	0.00	0.00	4.67	0	0.00	0.00	6.67
			$\Pi^1 = 56.11$				$\Pi^2 = 69.46$			

Table 4. Equilibrium with asymmetric capacities.

induce Firm 1 to play, resulting in a large decrease on Firm 2's profit. In the situation presented, Firm 1 has no incentive to play a positive amount, as it would result in losses.

			Firm 1				Firm 2			
			$F=10, H=1, K=25$				$F=10, H=1, K=25$			
Period (t)	a_t	b_t	y_t	x_t	h_t	q_t	y_t	x_t	h_t	q_t
1	10	1	0	0.000	0.000	0.000	1	10.888	5.449	5.439
2	10	1	0	0.000	0.000	0.000	0	0.000	0.000	5.449
3	10	1	0	0.000	0.000	0.000	1	5.469	0.000	5.469
4	10	0.5	0	0.000	0.000	0.000	1	13.409	0.000	13.409
5	10	0.5	0	0.000	0.000	0.000	1	12.889	0.000	12.889
6	10	0.5	0	0.000	0.000	0.000	1	12.769	0.000	12.769
			$\Pi^1 = 0$				$\Pi^2 = 155.119$			

Table 5. Equilibrium when firm 2 defects (plays a quantity such that firm 1's optimal response is to play zero).

Clearly, if Firm 1 would play the same quantities as a Firm 2, it would result in large losses for both firms. This example serves as an introduction to a different type of game, where firms may collaborate (play Cournot's quantities) or defect. This makes a bridge between games in market equilibria and 2-person games like the Prisoner's Dilemma. In this context, due to the existence of several periods, repeated games are particularly interesting; we will turn to their analysis in the next section.

3 Iterated games

One may consider that the decisions for all the periods have to be taken in advance, and the corresponding quantities fixed for all the planning horizon. However, it is much more natural to consider that at the begin of each period there is a commitment only regarding the quantity to produce on that period, and that the moves concerning later periods remain open. This situation leads to an interesting game between the firms, where in each period a firm may decide to cooperate with the others, or to defect.

In many situations, the computation of a Nash equilibrium in the presence of discrete variables is NP-hard [6, 7]. As for our problem, even the monopoly case is NP-hard. The equilibria computed in the previous section for the duopoly case are based on the assumption that each firm knows the production decisions of the other for the whole planning horizon. If that information is not available (as usual in real-world cases), the problem becomes more complicated. As we have seen, the solutions are many times asymmetric; there is no easy strategy for deciding on the role of each firm if the future periods' moves are not known. Indeed, we are not aware of an optimal strategy for this general case.

In order to complete the illustration, we go back to the example and provide results for a simple strategy, on a two-firm game, equivalent of a well known strategy in the iterated Prisoners' Dilemma [8]:

1. On the first period cooperate: play an optimal (non-defecting) quantity given by equation 5 if it leads to a positive profit, or null quantity otherwise.
2. On the subsequent periods:
   (a) if in the previous period the opponent cooperated, cooperate too: play a Cournot's quantity;
   (b) otherwise, retaliate: play a large quantity, such that the other firms' optimal reaction would be producing zero quantity.

Notice that, as in some instances there are several possibilities for Cournot equilibria, the above strategy may be ambiguous. This can be observed on Table 6. Even though in the best situation the Cournot equilibrium with future information is obtained (top), other (inferior) situations may also arise (middle and bottom); firms have no information to decide which plays what in asymmetric moves.

			Firm 1				Firm 2			
			$F = 10, H = 1, K = 25$				$F = 10, H = 1, K = 25$			
Period (t)	a_t	b_t	y_t	x_t	h_t	q_t	y_t	x_t	h_t	q_t
1	10	1	1	8.33	5.00	3.34	1	6.33	3.00	3.34
2	10	1	0	0.00	2.00	3.00	0	0.00	0.00	3.00
3	10	1	0	0.00	0.00	2.00	1	9.33	5.33	4.00
4	10	0.5	1	12.67	5.33	7.34	0	0.00	0.00	5.33
5	10	0.5	0	0.00	0.00	5.33	1	12.66	5.33	7.33
6	10	0.5	1	7.34	0.00	7.34	0	0.00	0.00	5.33
			$\Pi^1 = 62.15$				$\Pi^2 = 61.43$			
1	10	1	1	6.33	3.00	3.34	1	6.33	3.00	3.34
2	10	1	0	0.00	0.00	3.00	0	0.00	0.00	3.00
3	10	1	1	9.33	5.33	4.00	1	9.33	5.33	4.00
4	10	0.5	0	0.00	0.00	5.33	0	0.00	0.00	5.33
5	10	0.5	1	12.66	5.33	7.33	1	12.66	5.33	7.33
6	10	0.5	0	0.00	0.00	5.33	0	0.00	0.00	5.33
			$\Pi^1 = 56.70$				$\Pi^2 = 56.70$			
1	10	1	1	8.33	5.00	3.34	1	8.33	5.00	3.34
2	10	1	0	0.00	2.00	3.00	0	0.00	2.00	3.00
3	10	1	0	0.00	0.00	2.00	0	0.00	0.00	2.00
4	10	0.5	1	12.67	5.33	7.34	1	12.67	5.33	7.34
5	10	0.5	0	0.00	0.00	5.33	0	0.00	0.00	5.33
6	10	0.5	1	7.34	0.00	7.34	1	7.34	0.00	7.34
			$\Pi^1 = 56.78$				$\Pi^2 = 56.78$			

Table 6. Results for repeated games: both firms cooperate.

Much inferior outcomes are obtained if one of the firms decides to defect. A firm might be tempted to impose a large move; but, as the other firm has no information about this, it will play either the "cooperate" or the "defect" move, resulting in large losses for both. Table 7 presents these situations. It should be noted that, in the cases presented, the last firm defecting is worse off at the end. This encourages firms to defect, trying to make the other firms drop out of the market, and have the large profits of Table 5.

			Firm 1				Firm 2			
			$F=10, H=1, K=25$				$F=10, H=1, K=25$			
Period (t)	a_t	b_t	y_t	x_t	h_t	q_t	y_t	x_t	h_t	q_t
1	10	1	1	10.88	5.44	5.44	1	10.88	5.44	5.44
2	10	1	1	0.00	0.00	5.44	0	0.00	0.00	5.44
3	10	1	0	5.47	0.00	5.47	1	5.47	0.00	5.47
4	10	0.5	1	13.41	0.00	13.41	1	13.41	0.00	13.41
5	10	0.5	1	12.89	0.00	12.89	1	12.89	0.00	12.89
6	10	0.5	1	12.78	0.00	12.78	1	12.78	0.00	12.78
			$\Pi^1 = -55.44$				$\Pi^2 = -55.44$			
1	10	1	1	6.33	3.00	3.34	1	10.88	5.44	5.44
2	10	1	0	0.00	0.00	3.00	0	0.00	0.00	5.44
3	10	1	1	9.33	5.33	4.00	1	5.47	0.00	5.47
4	10	0.5	0	0.00	0.00	5.33	1	13.41	0.00	13.41
5	10	0.5	1	12.66	5.33	7.33	1	12.89	0.00	12.89
6	10	0.5	0	0.00	0.00	5.33	1	12.78	0.00	12.78
			$\Pi^1 = -23.77$				$\Pi^2 = -15.01$			
1	10	1	1	8.33	4.99	3.34	1	10.88	5.44	5.44
2	10	1	1	5.89	5.44	5.44	0	0.00	2.10	3.34
3	10	1	0	0.00	2.10	3.34	1	3.37	0.00	5.47
4	10	0.5	1	11.31	0.00	13.41	1	12.67	5.33	7.34
5	10	0.5	1	12.67	5.33	7.34	1	7.56	0.00	12.89
6	10	0.5	1	7.45	0.00	12.78	1	7.34	0.00	7.34
			$\Pi^1 = -53.17$				$\Pi^2 = -45.65$			

Table 7. Results for repeated games: both firms defect (top), Firm 2 defects and Firm 1 cooperates (middle), or play other firm's last move (bottom; Firm 2 starts defecting).

4 Conclusions

Recent advances in mixed-integer nonlinear programming software allow the straightforward computation of optimal solution for problems which were very difficult to solve some years ago. This allowed us to compute equilibria in a simple but very interesting market, where a lot-sizing problem is setup in a competitive context.

The analysis of a simple example puts in evidence that the solution of a lot-sizing problem may be very different of its usual, constant demand version. This also extends, in a somewhat different way, recent work on agent-based approaches for computational equilibria [9] to a recurring, dynamic environment.

When several firms play, the existence of several periods in the lot-sizing problem makes it natural to have an iterated game where each firm plays one move per period. We proposed a very simple strategy; more sophisticated ones, e.g. mixed strategies, are an interesting subject for further research in this topic. Another direction concerns the analysis of real-world problems under this background; many markets have points in common with this model (for example, energy markets, often tackled as equilibrium models [10]). A scenario which is worthy studying is the one where firms with consecutive losses drop out of the market, as happens in most real-world cases.

The illustration presented in this paper is also interesting for broadening the range of situations analysed in the study of equilibria in Economics, putting in close relation economic equilibrium and computational complexity, two subjects seldom studied together. Iterated games are often considered excellent teaching tools [11]; we hope that the one described here will also make a contribution in this regard.

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