

# VC 13/14 – T6

## Frequency Space

Mestrado em Ciência de Computadores  
Mestrado Integrado em Engenharia de Redes e  
Sistemas Informáticos

***Miguel Tavares Coimbra***

# Outline

- Fourier Transform
- Frequency Space
- Spatial Convolution

**Acknowledgements:** Most of this course is based on the excellent courses offered by Prof. Shree Nayar at Columbia University, USA and by Prof. Srinivasa Narasimhan at CMU, USA. Please acknowledge the original source when reusing these slides for academic purposes.

# Topic: Fourier Transform

- **Fourier Transform**
- Frequency Space
- Spatial Convolution

# How to Represent Signals?

- Option 1: Taylor series represents any function using polynomials.

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \dots$$

- Polynomials are not the best - unstable and not very physically meaningful.
- Easier to talk about “signals” in terms of its “frequencies” (how fast/often signals change, etc).

# Jean Baptiste Joseph Fourier (1768-1830)

- Had a crazy idea (1807):
- **Any** periodic function can be rewritten as a weighted sum of Sines and Cosines of different frequencies.
- Don't believe it?
  - Neither did Lagrange, Laplace, Poisson and other big wigs
  - Not translated into English until 1878!
- **But it's true!**
  - called **Fourier Series**
  - Possibly the greatest tool used in Engineering

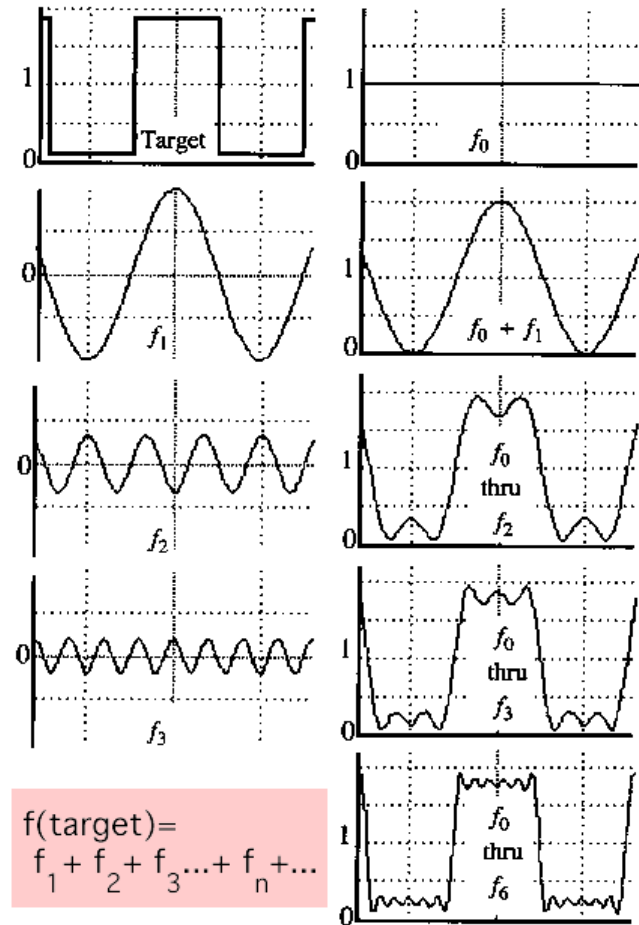


# A Sum of Sinusoids

- Our building block:

$$A \sin(\omega x + \phi)$$

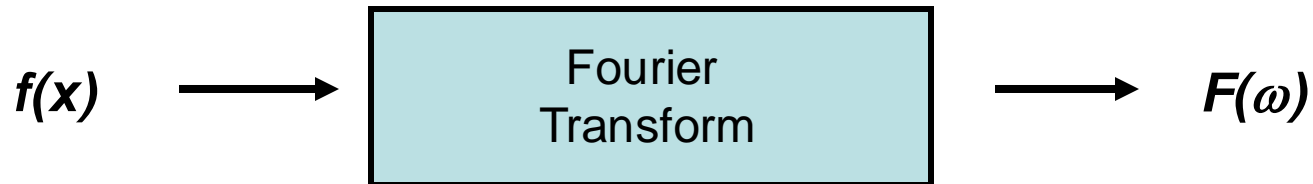
- Add enough of them to get any signal  $f(x)$  you want!
- How many degrees of freedom?
- What does each control?
- Which one encodes the coarse vs. fine structure of the signal?



$$f(\text{target}) = f_0 + f_1 + f_2 + f_3 + \dots + f_n + \dots$$

# Fourier Transform

- We want to understand the frequency  $\omega$  of our signal. So, let's reparametrize the signal by  $\omega$  instead of  $x$ :



- For every  $\omega$  from 0 to inf,  $F(\omega)$  holds the amplitude  $A$  and phase  $\phi$  of the corresponding sine
  - How can  $F$  hold both? Complex number trick!

$$F(\omega) = R(\omega) + iI(\omega)$$

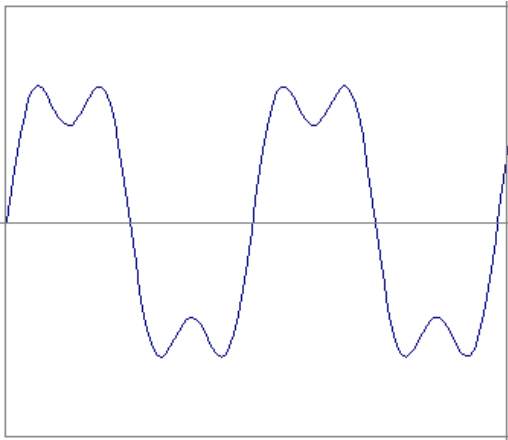
$$A = \pm \sqrt{R(\omega)^2 + I(\omega)^2}$$

$$A \sin(\omega x + \phi)$$

$$\phi = \tan^{-1} \frac{I(\omega)}{R(\omega)}$$

# Time and Frequency

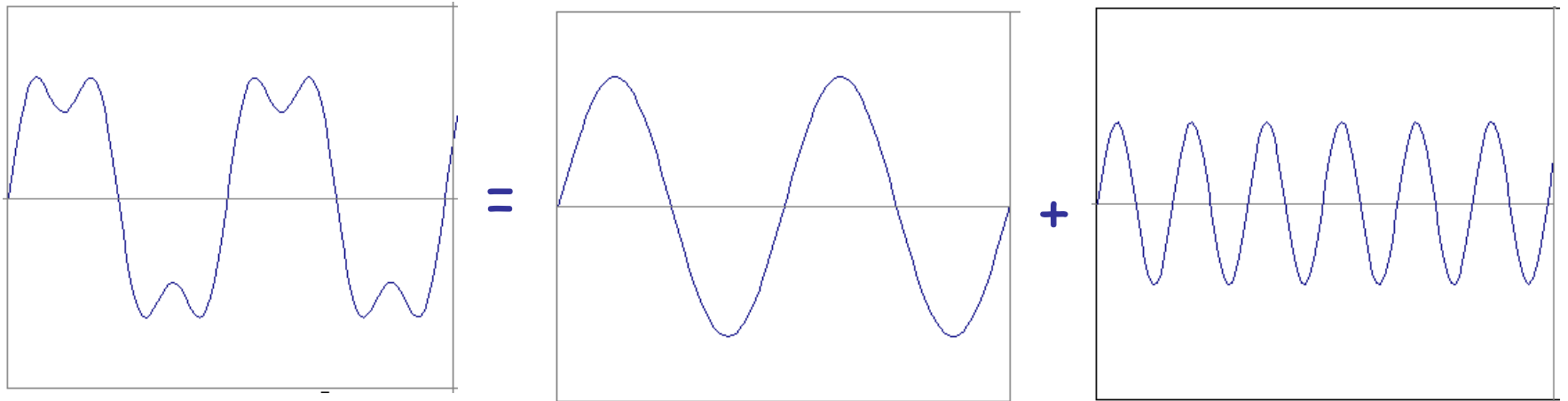
- example :  $g(t) = \sin(2\pi f t) + (1/3)\sin(2\pi(3f) t)$





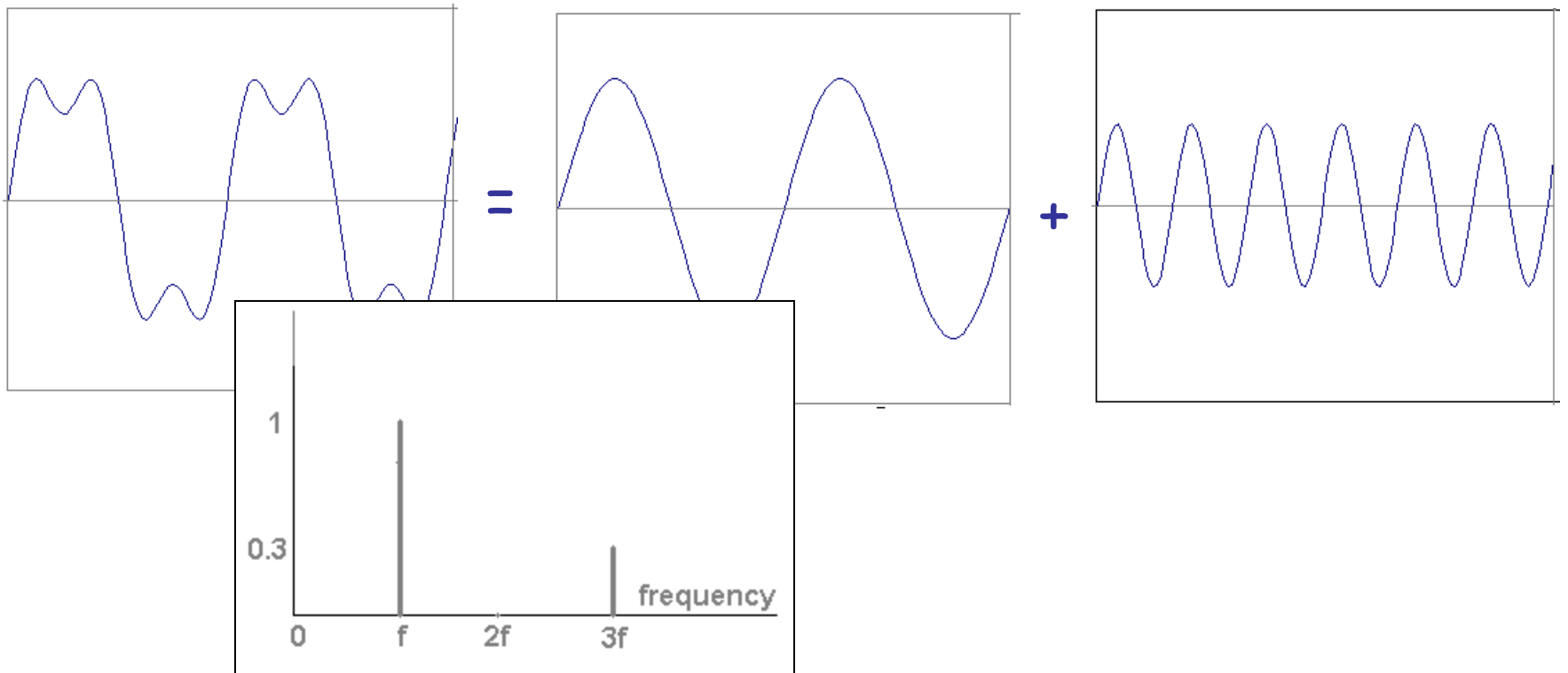
# Time and Frequency

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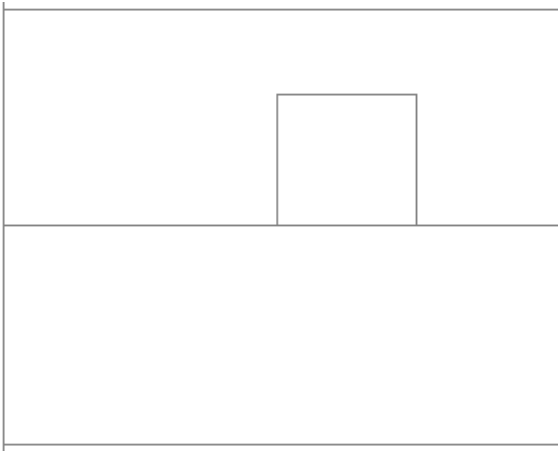
# Frequency Spectra

- example :  $g(t) = \sin(2\pi f t) + (1/3)\sin(2\pi(3f) t)$

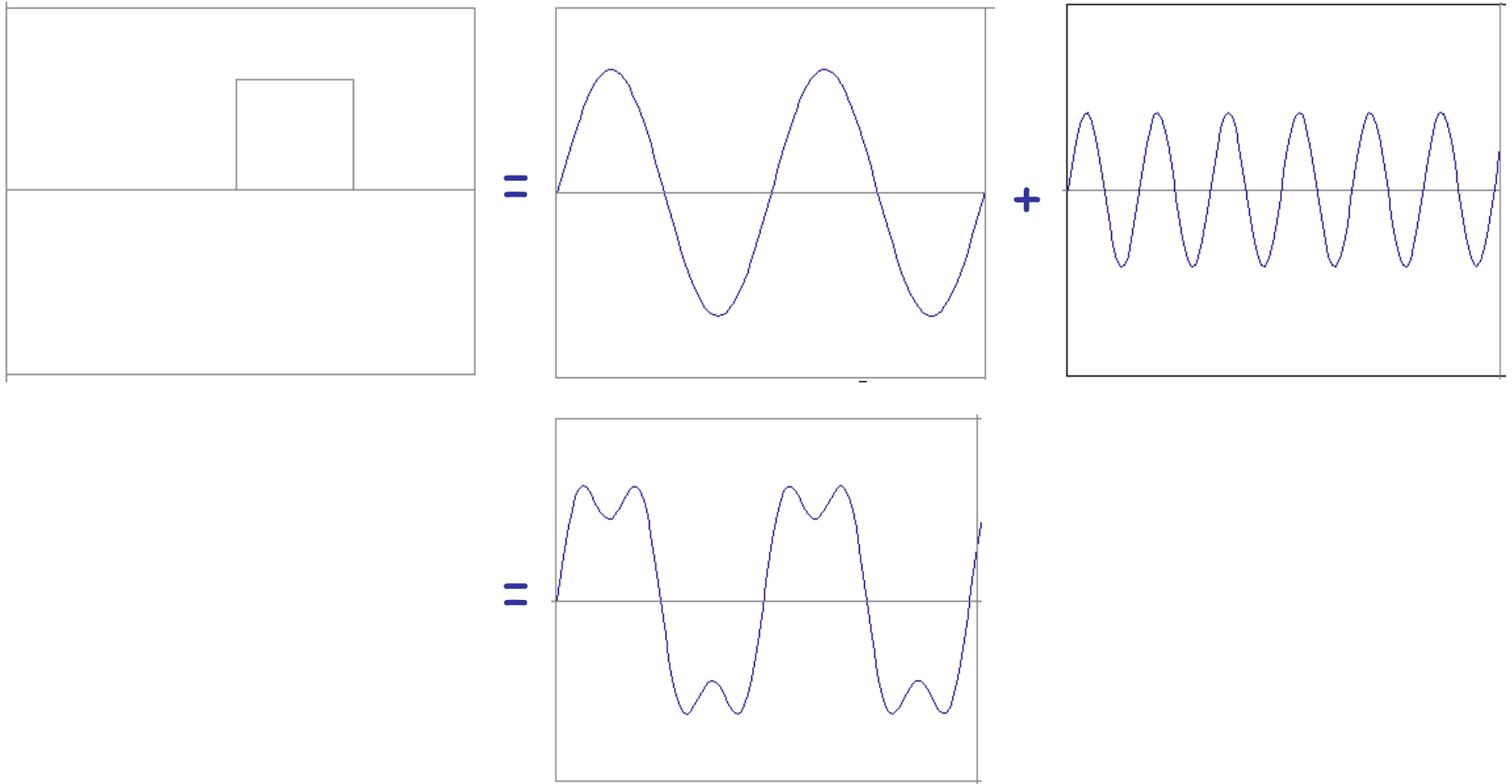


# Frequency Spectra

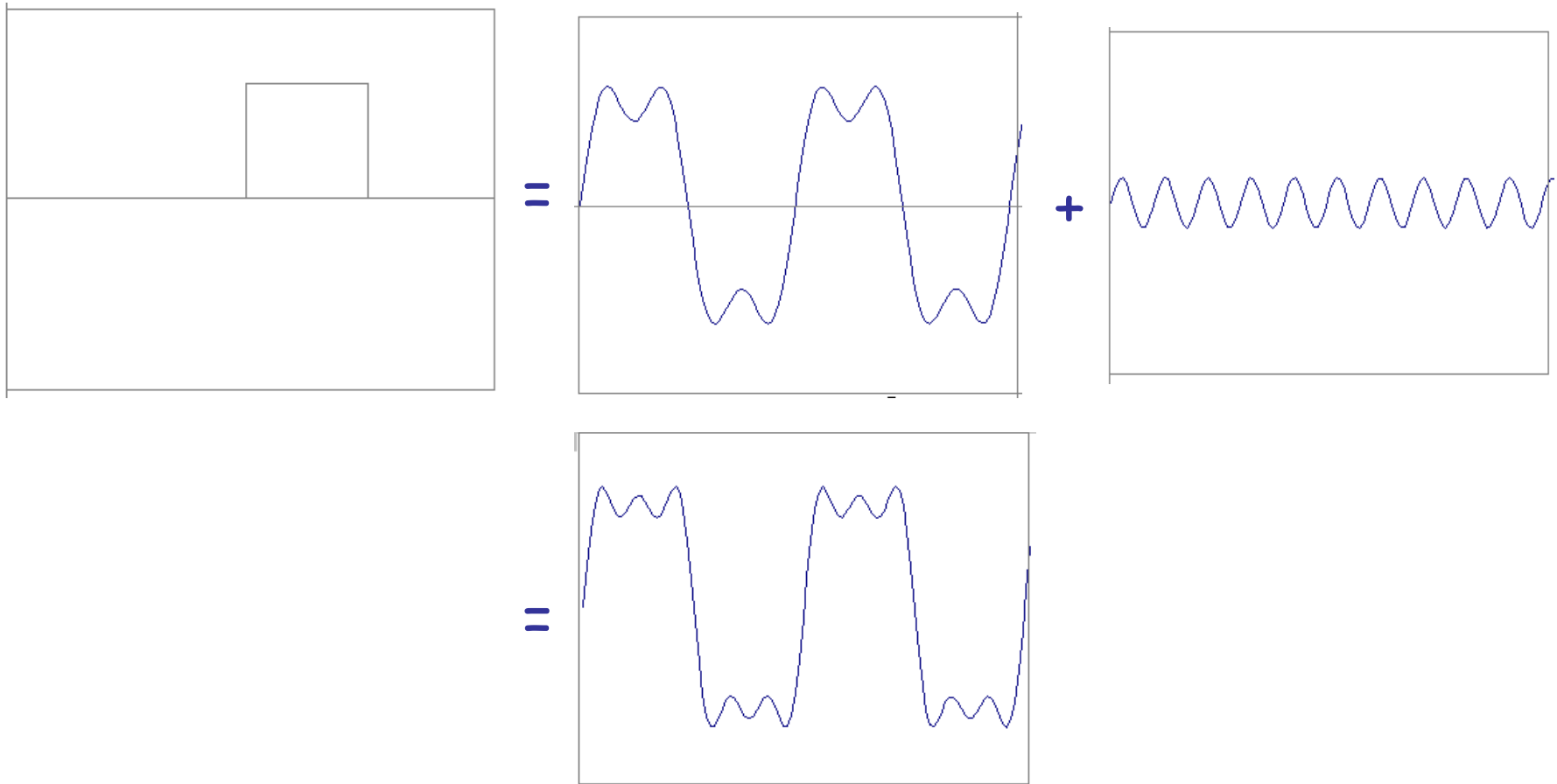
- Usually, frequency is more interesting than the phase



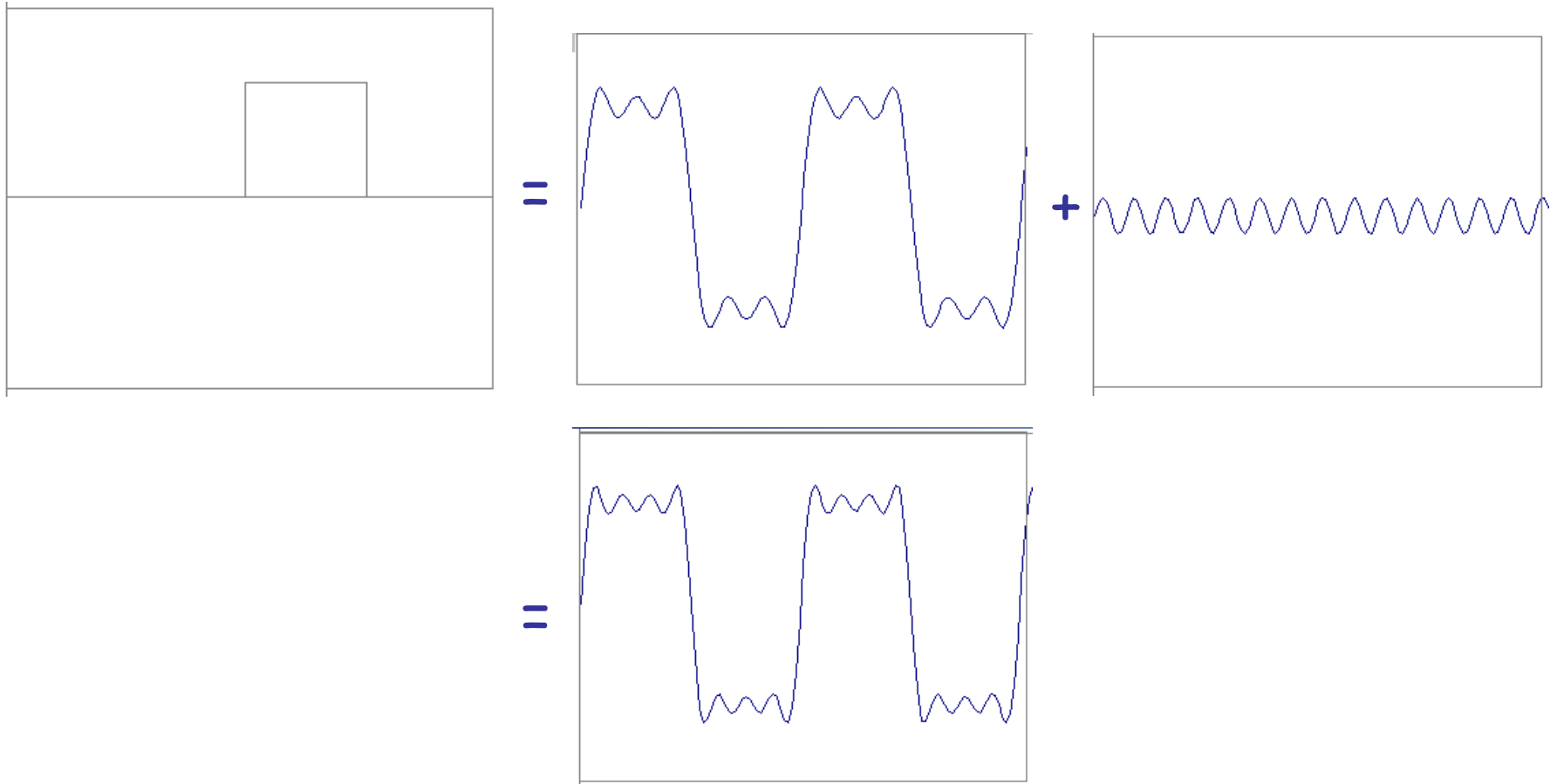
# Frequency Spectra



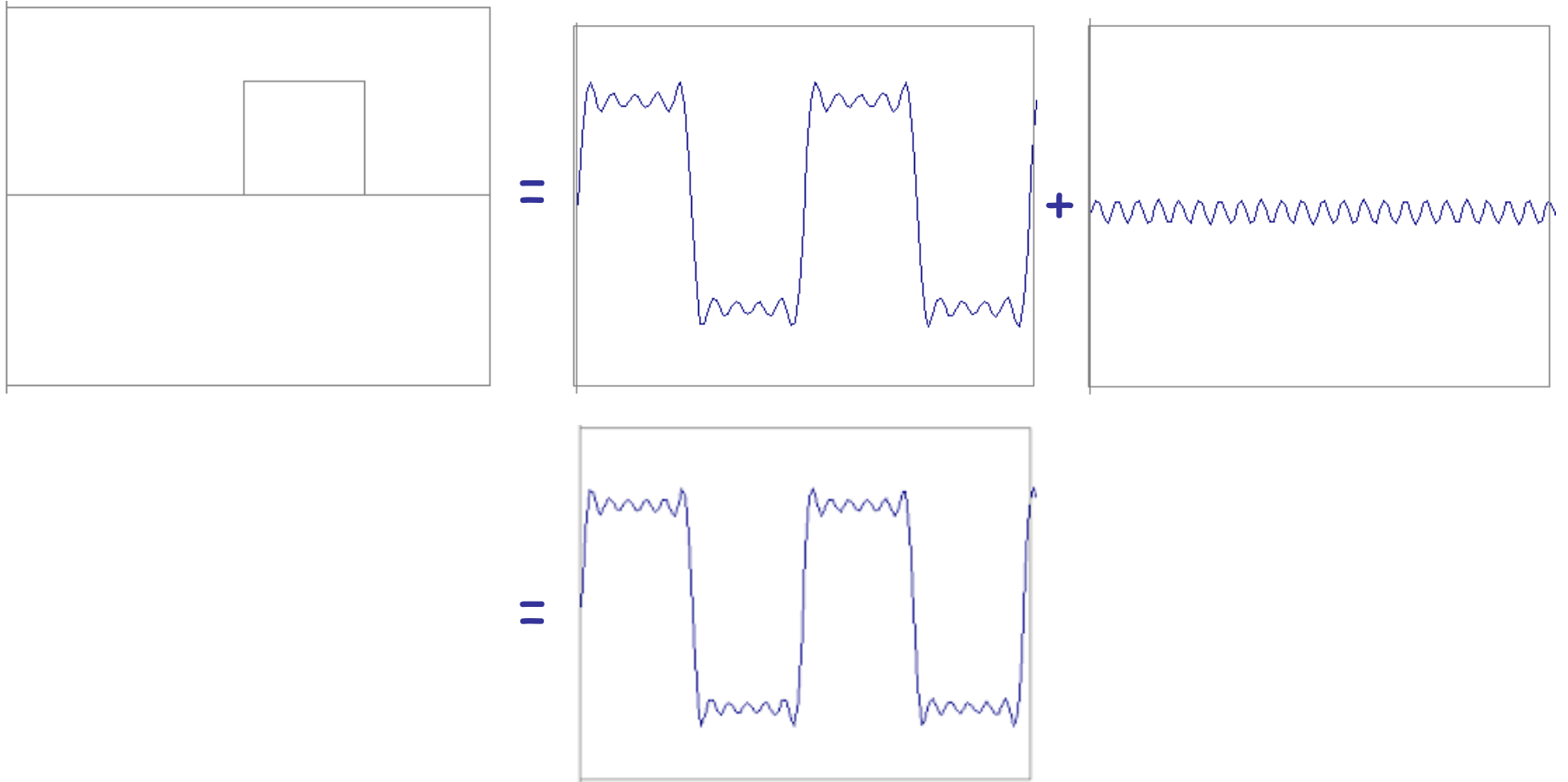
# Frequency Spectra



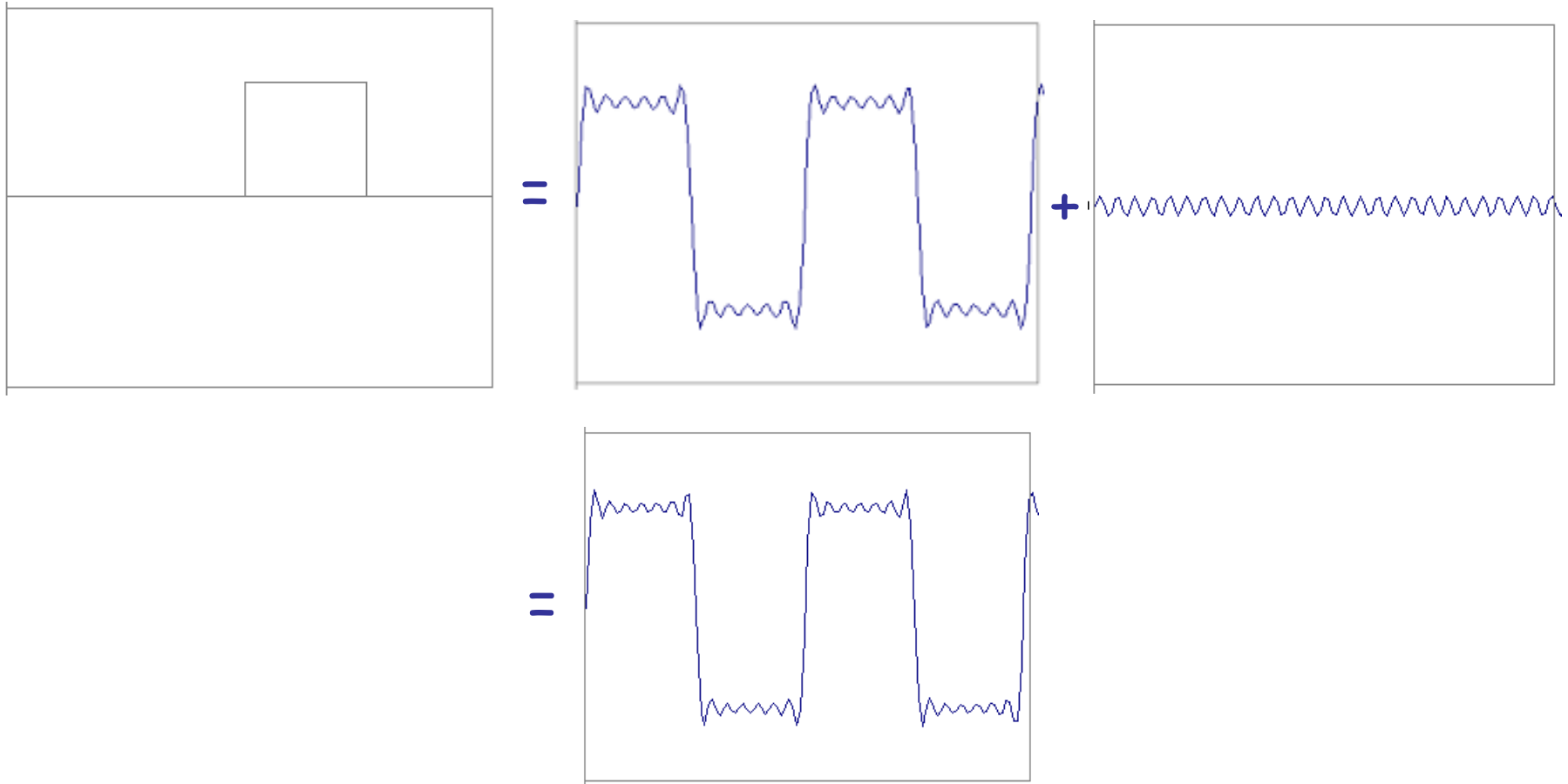
# Frequency Spectra



# Frequency Spectra

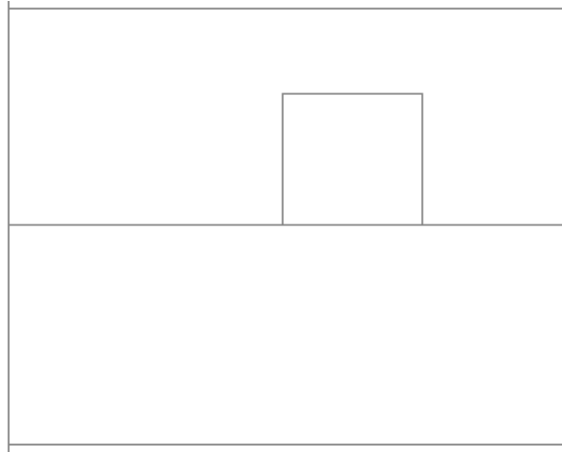


# Frequency Spectra



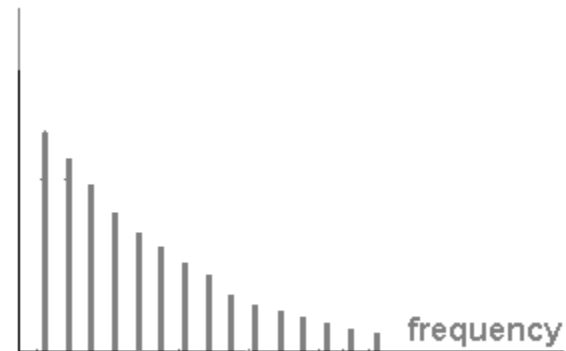


# Frequency Spectra



=

$$A \sum_{k=1}^{\infty} \frac{1}{k} \sin(2\pi kt)$$



# Fourier Transform – more formally

Represent the signal as an infinite weighted sum of an infinite number of sinusoids

$$F(u) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi ux} dx$$

Note:  $e^{ik} = \cos k + i \sin k$       $i = \sqrt{-1}$

Arbitrary function      $\longrightarrow$      Single Analytic Expression

Spatial Domain ( $x$ )      $\longrightarrow$      Frequency Domain ( $u$ )  
(Frequency Spectrum  $F(u)$ )

Inverse Fourier Transform (IFT)      $f(x) = \int_{-\infty}^{\infty} F(u) e^{i2\pi ux} dx$

# Fourier Transform

- Also, defined as:

$$F(u) = \int_{-\infty}^{\infty} f(x) e^{-iux} dx$$

Note:  $e^{ik} = \cos k + i \sin k$       $i = \sqrt{-1}$

- Inverse Fourier Transform (IFT)

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(u) e^{iux} dx$$

# Properties of Fourier Transform

|                        |                         |                |   |
|------------------------|-------------------------|----------------|---|
| <b>Linearity</b>       | $c_1 f(x) + c_2 g(x)$   |                | $c_1 F(u) + c_2 G(u)$   |
| <b>Scaling</b>         | $f(ax)$                 | Spatial Domain | $\frac{1}{ a } F\left(\frac{u}{a}\right)$<br>Frequency Domain |
| <b>Shifting</b>        | $f(x - x_0)$            |                | $e^{-i2\pi u x_0} F(u)$                                       |
| <b>Symmetry</b>        | $F(x)$                  |                | $f(-u)$   |
| <b>Conjugation</b>     | $f^*(x)$                |                | $F^*(-u)$   |
| <b>Convolution</b>     | $f(x) * g(x)$           |                | $F(u)G(u)$  |
| <b>Differentiation</b> | $\frac{d^n f(x)}{dx^n}$ |                | $(i2\pi u)^n F(u)$  |

# Topic: Frequency Space

- Fourier Transform
- **Frequency Space**
- Spatial Convolution

# How does this apply to images?

- We have defined the Fourier Transform as

$$F(u) = \int_{-\infty}^{\infty} f(x) e^{-iux} dx$$

- But images are:
  - Discrete.
  - Two-dimensional.

|   |   |   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|---|---|
| 0 | 3 | 2 | 5 | 4 | 7 | 6 | 9 | 8 |
| 3 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 2 | 1 | 0 | 3 | 2 | 5 | 4 | 7 | 6 |
| 5 | 2 | 3 | 0 | 1 | 2 | 3 | 4 | 5 |
| 4 | 3 | 2 | 1 | 0 | 3 | 2 | 5 | 4 |
| 7 | 4 | 5 | 2 | 3 | 0 | 1 | 2 | 3 |
| 6 | 5 | 4 | 3 | 2 | 1 | 0 | 3 | 2 |
| 9 | 6 | 7 | 4 | 5 | 2 | 3 | 0 | 1 |
| 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 |

What a computer sees

# 2D Discrete FT

- In a 2-variable case, the discrete FT pair is:

$$F(u, v) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) \exp[-j2\pi(ux/M + vy/N)]$$

For  $u=0, 1, 2, \dots, M-1$  and  $v=0, 1, 2, \dots, N-1$

New matrix  
with the  
same size!

AND: 
$$f(x, y) = \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u, v) \exp[j2\pi(ux/M + vy/N)]$$

For  $x=0, 1, 2, \dots, M-1$  and  $y=0, 1, 2, \dots, N-1$

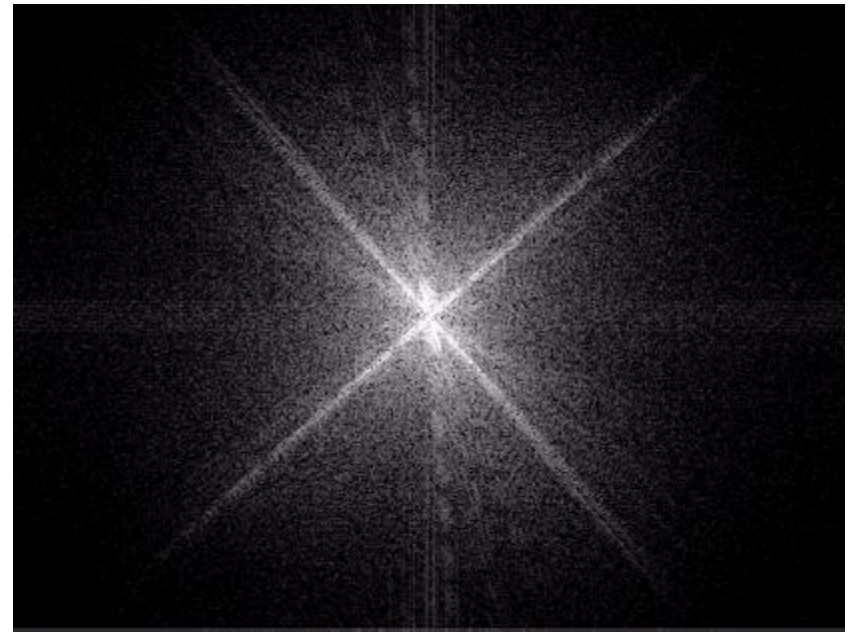
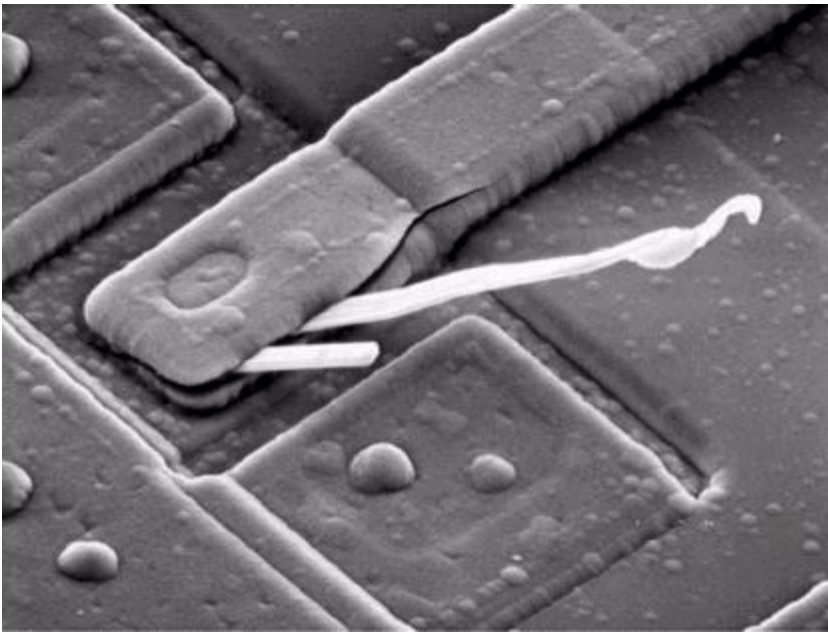
# Frequency Space

- Image Space

- $f(x,y)$
- Intuitive

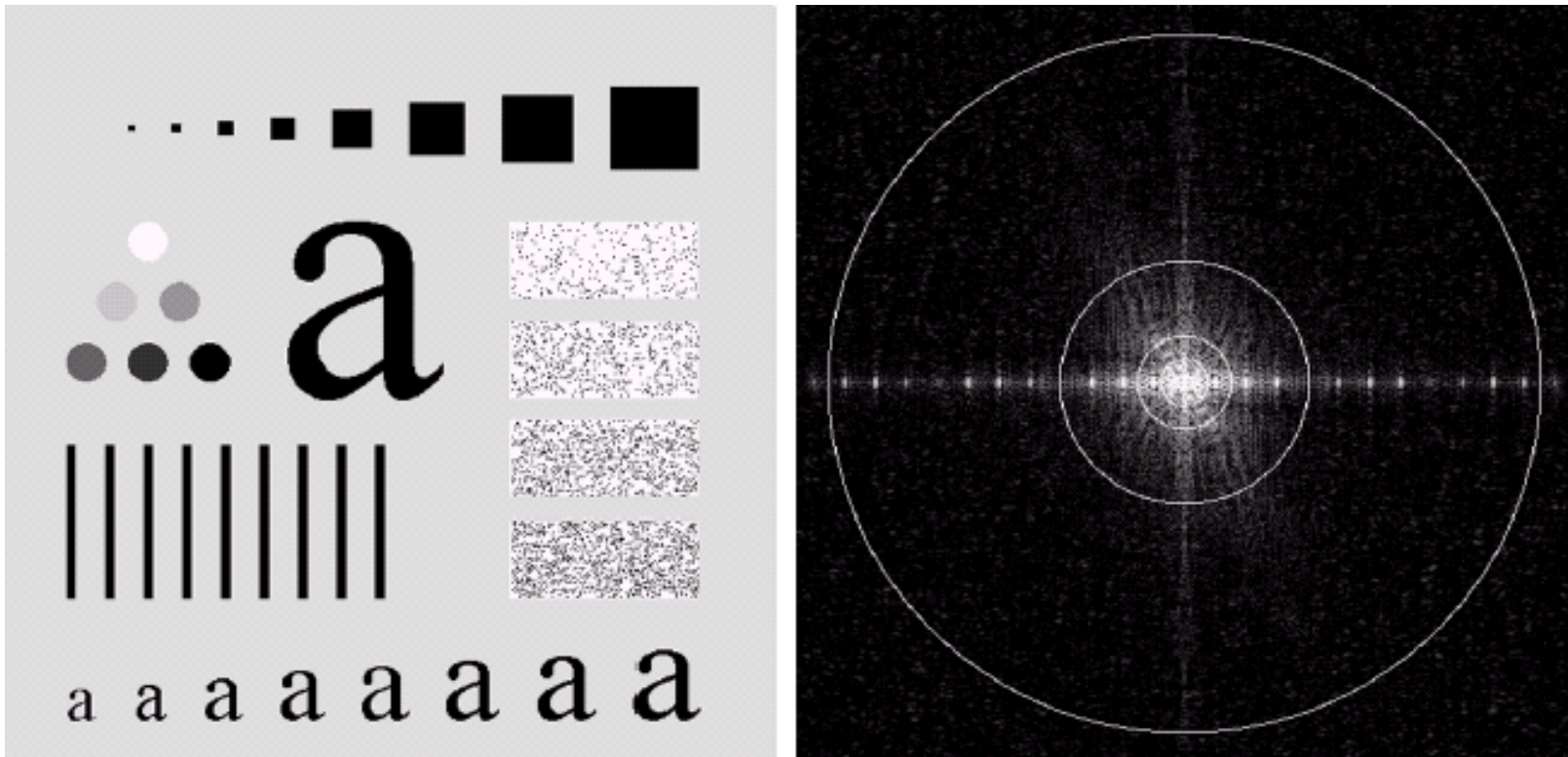
- Frequency Space

- $F(u,v)$
- What does this mean?





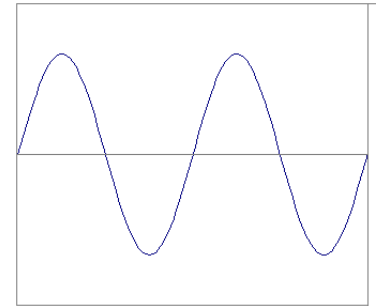
# Power distribution



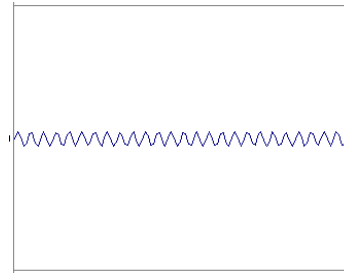
An image (500x500 pixels) and its Fourier spectrum. The super-imposed circles have radii values of 5, 15, 30, 80, and 230, which respectively enclose 92.0, 94.6, 96.4, 98.0, and 99.5% of the image power.

# Power distribution

- Most power is in low frequencies.
- Means we are using more of this:

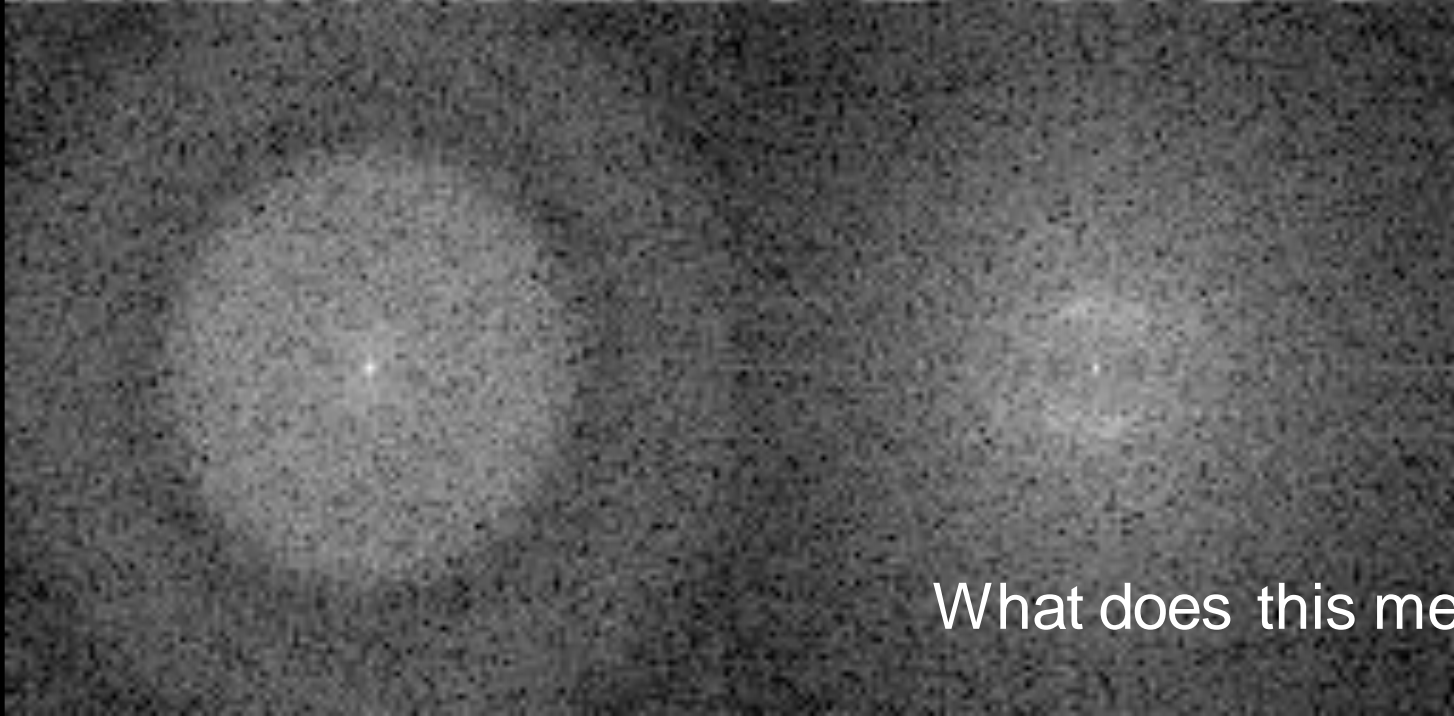
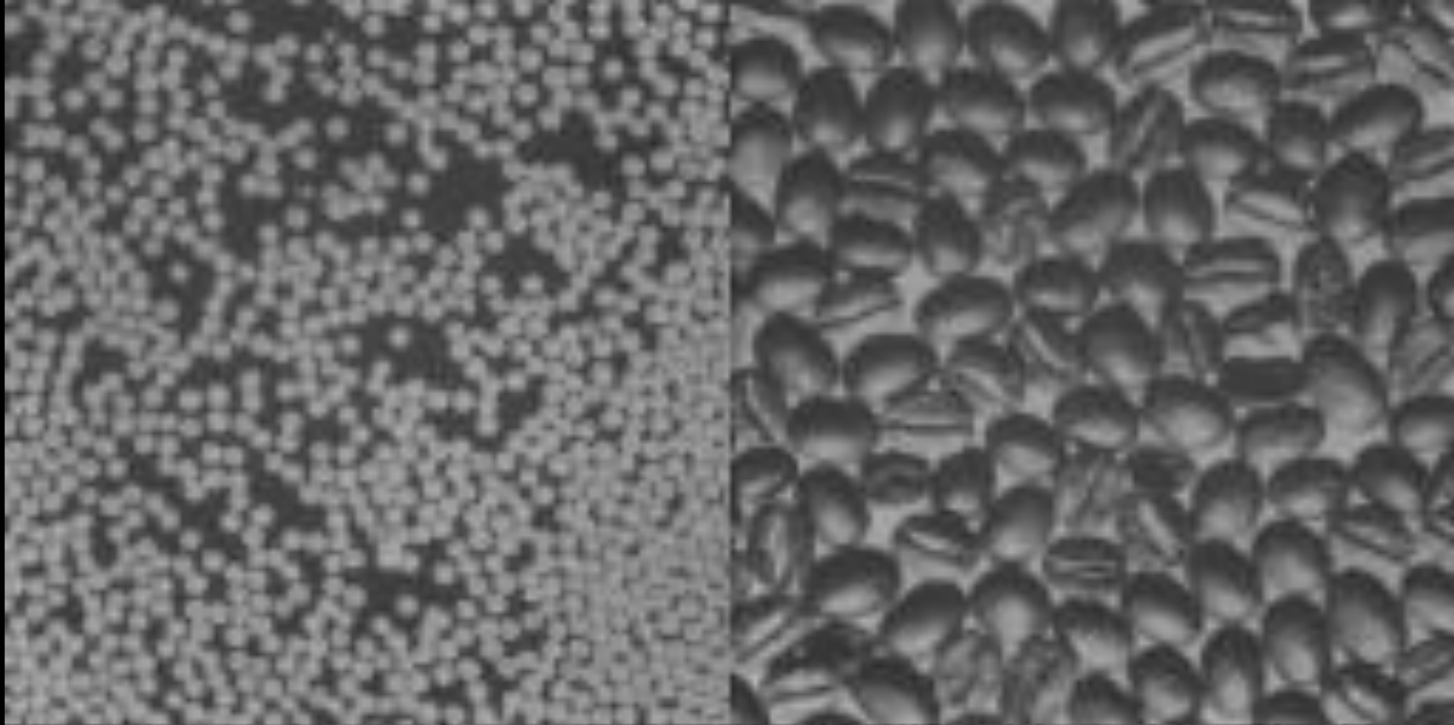


And less of this:



To represent our signal.

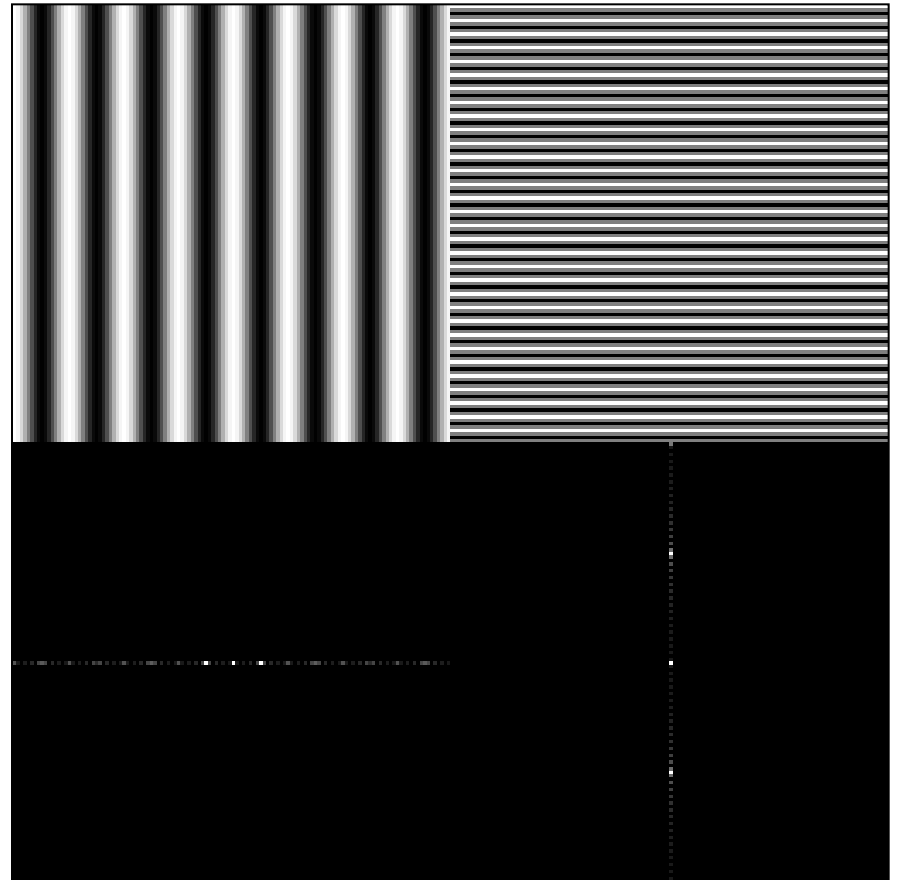
- Why?

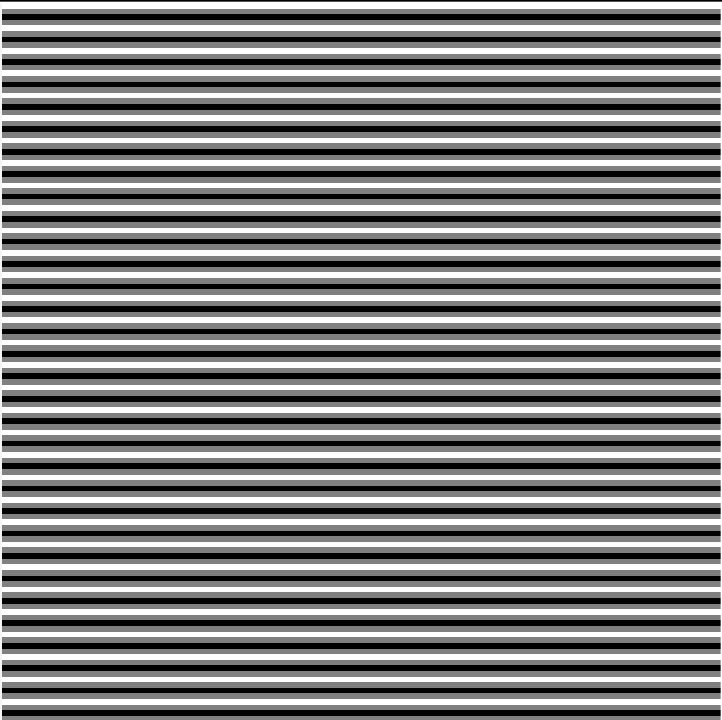
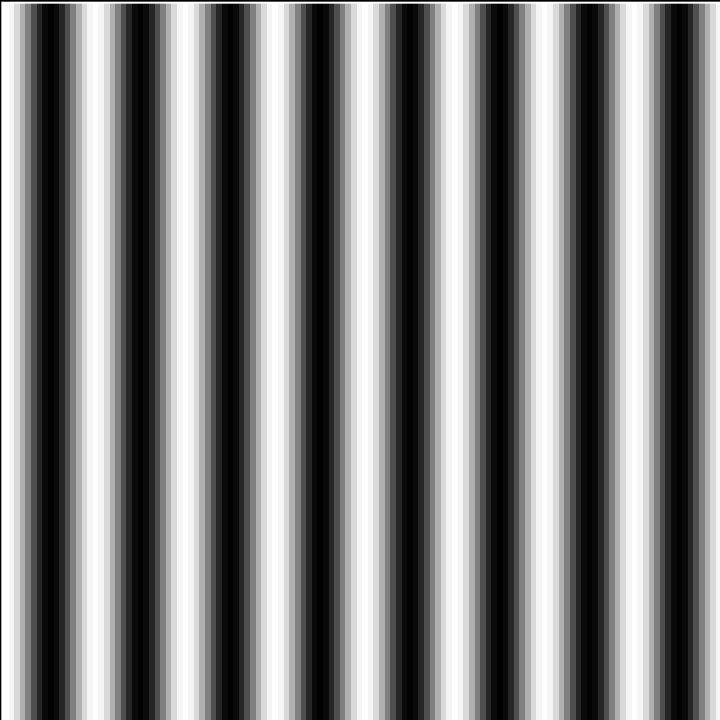


What does this mean??

# Horizontal and Vertical Frequency

- **Frequencies:**
  - Horizontal frequencies correspond to horizontal gradients.
  - Vertical frequencies correspond to vertical gradients.
- **What about diagonal lines?**







If I discard high-frequencies, I get a blurred image...  
Why?

# Why bother with FT?

- Great for filtering.
- Great for compression.
- In some situations: Much faster than operating in the spatial domain.
- Convolutions are simple multiplications in Frequency space!
- ...

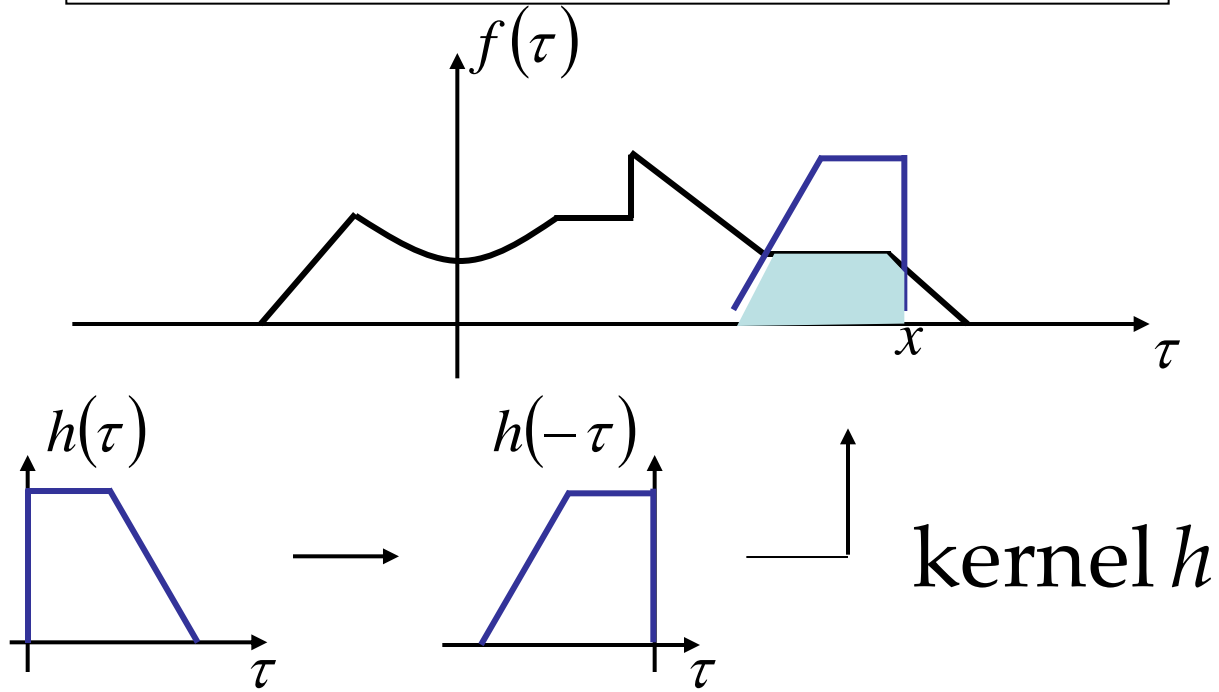
# Topic: Spatial Convolution

- Fourier Transform
- Frequency Space
- **Spatial Convolution**

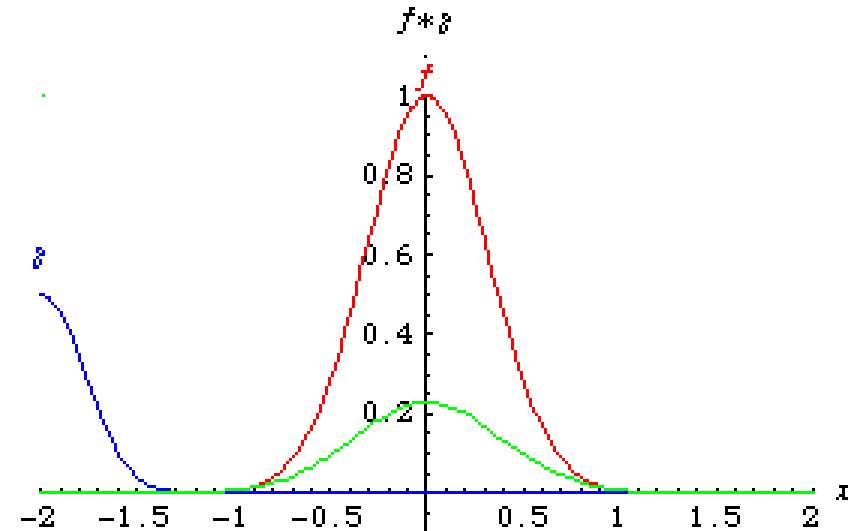
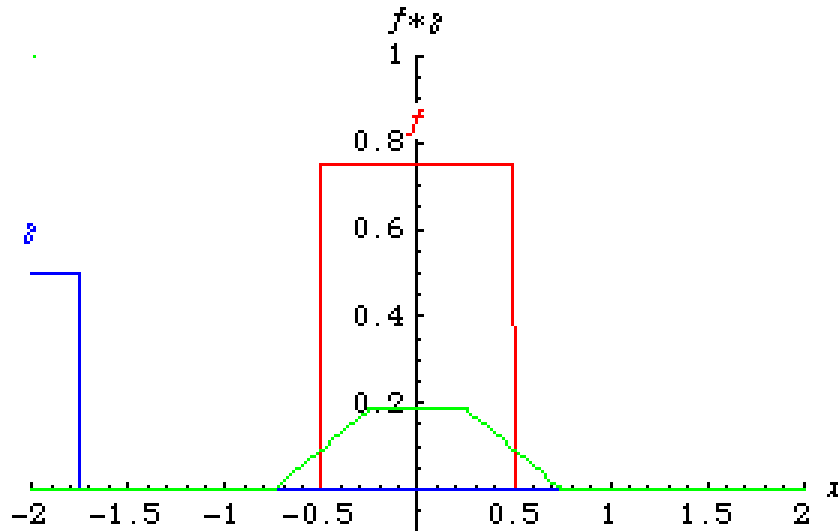


# Convolution

$$g(x) = \int_{-\infty}^{\infty} f(\tau)h(x-\tau)d\tau \quad g = f * h$$



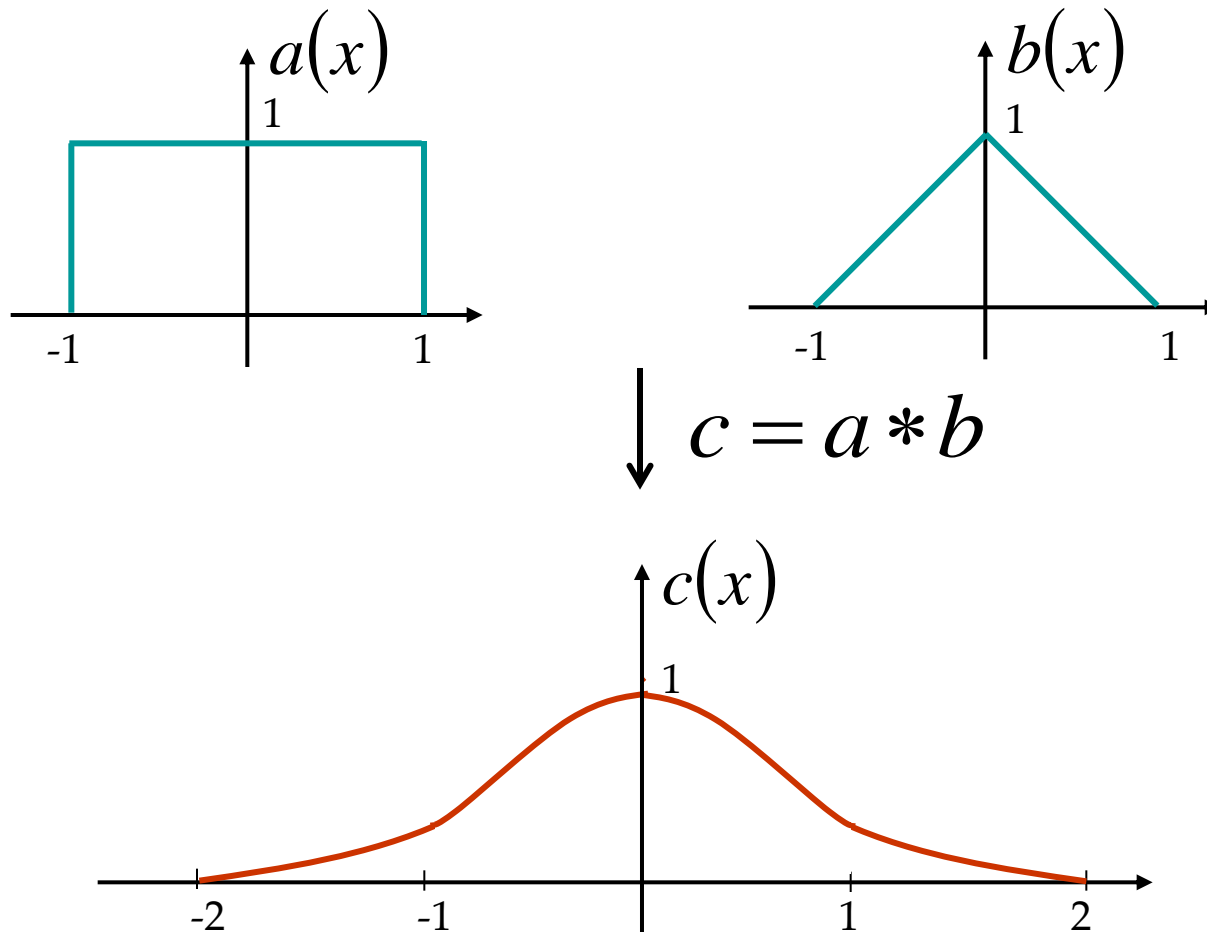
# Convolution - Example



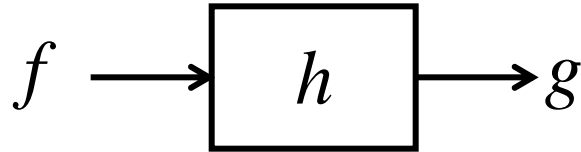
—  $f$   
—  $g$   
—  $f * g$

Eric Weinstein's Math World

# Convolution - Example



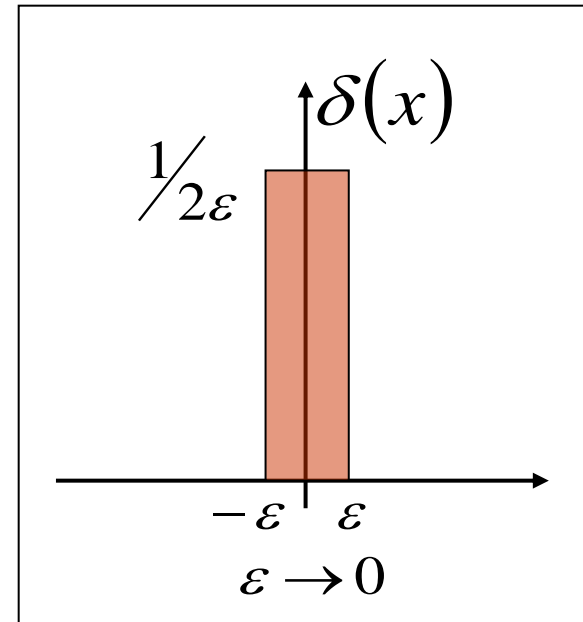
# Convolution Kernel – Impulse Response



$$g = f * h$$

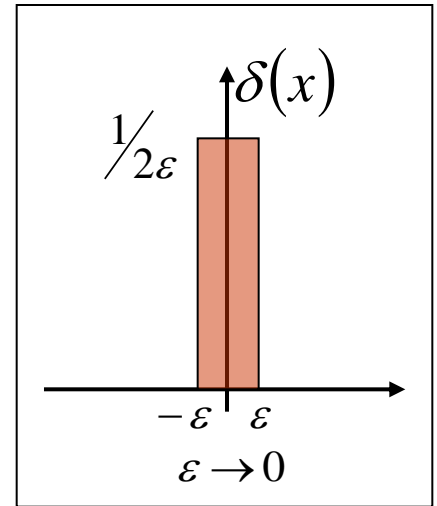
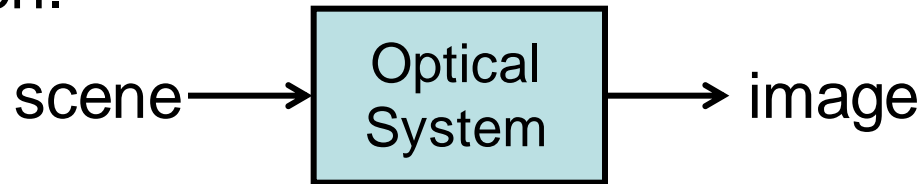
- What  $h$  will give us  $g = f$ ?

Dirac Delta Function (Unit Impulse)

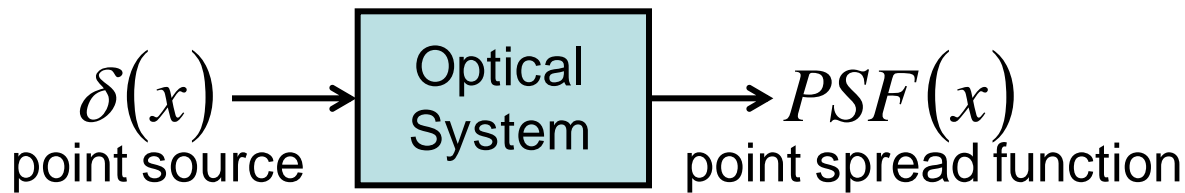


# Point Spread Function

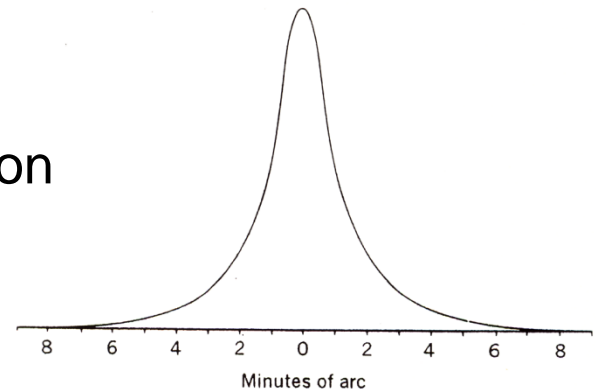
- Ideally, the optical system should be a Dirac delta function.



- However, optical systems are never ideal.



- Point spread function of Human Eyes.



# Point Spread Function



normal vision



myopia



hyperopia

# Properties of Convolution

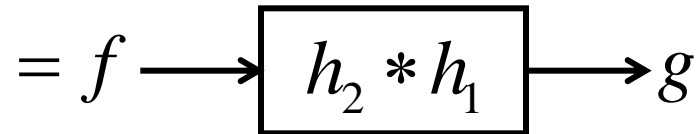
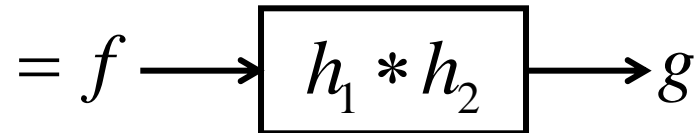
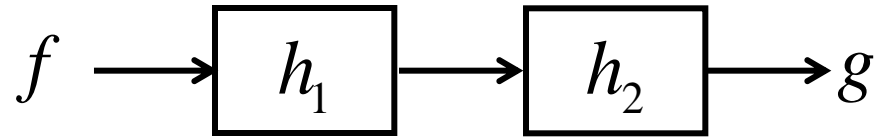
- Commutative

$$a * b = b * a$$

- Associative

$$(a * b) * c = a * (b * c)$$

- Cascade system



# Fourier Transform and Convolution

$$\begin{aligned} \text{Let } g &= f * h & \text{Then } G(u) &= \int_{-\infty}^{\infty} g(x) e^{-i2\pi u x} dx \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\tau) h(x - \tau) e^{-i2\pi u x} d\tau dx \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [f(\tau) e^{-i2\pi u \tau} d\tau] [h(x - \tau) e^{-i2\pi u (x - \tau)} dx] \\ &= \int_{-\infty}^{\infty} [f(\tau) e^{-i2\pi u \tau} d\tau] \int_{-\infty}^{\infty} [h(x') e^{-i2\pi u x'} dx'] &= F(u)H(u) \end{aligned}$$

Convolution in spatial domain

$\Leftrightarrow$  Multiplication in frequency domain





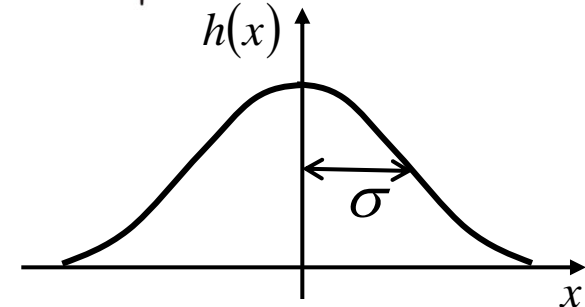
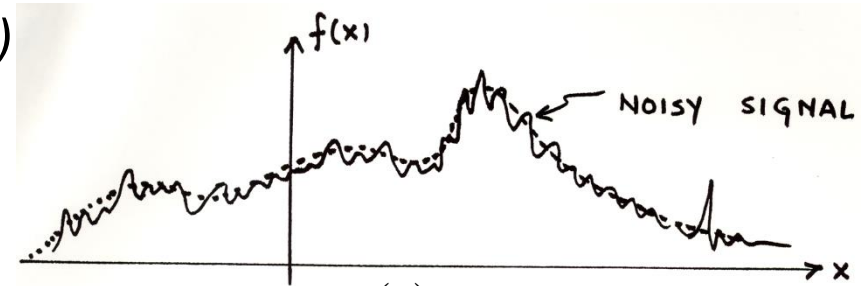
# Example use: Smoothing/Blurring

- We want a smoothed function of  $f(x)$

$$g(x) = f(x) * h(x)$$

- Let us use a Gaussian kernel

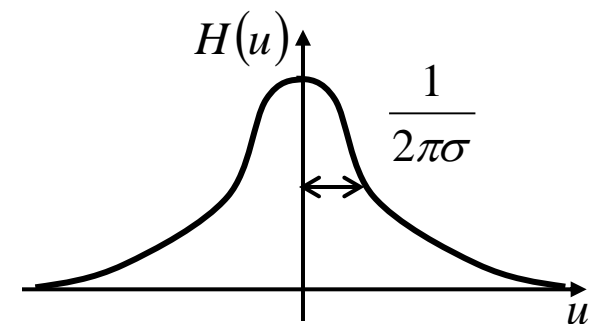
$$h(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2} \frac{x^2}{\sigma^2}\right]$$



- Then

$$H(u) = \exp\left[-\frac{1}{2} (2\pi u)^2 \sigma^2\right]$$

$$G(u) = F(u)H(u)$$



# Resources

- Russ – Chapter 6
- Gonzalez & Woods – Chapter 4