

VC 17/18 – TP6

Frequency Space

Mestrado em Ciência de Computadores
Mestrado Integrado em Engenharia de Redes e
Sistemas Informáticos

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Outline

- Fourier Transform
- Frequency Space
- Spatial Convolution

Topic: Fourier Transform

- **Fourier Transform**
- Frequency Space
- Spatial Convolution

How to Represent Signals?

- Option 1: Taylor series represents any function using polynomials.

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \dots$$

- Polynomials are not the best - unstable and not very physically meaningful.
- Easier to talk about “signals” in terms of its “frequencies” (how fast/often signals change, etc).

Jean Baptiste Joseph Fourier (1768-1830)

- Had a crazy idea (1807):
- **Any** periodic function can be rewritten as a weighted sum of **Sines** and **Cosines** of different frequencies.
- **Don't believe it?**
 - Neither did Lagrange, Laplace, Poisson and other big wigs
 - Not translated into English until 1878!
- **But it's true!**
 - called **Fourier Series**
 - Possibly the greatest tool used in Engineering

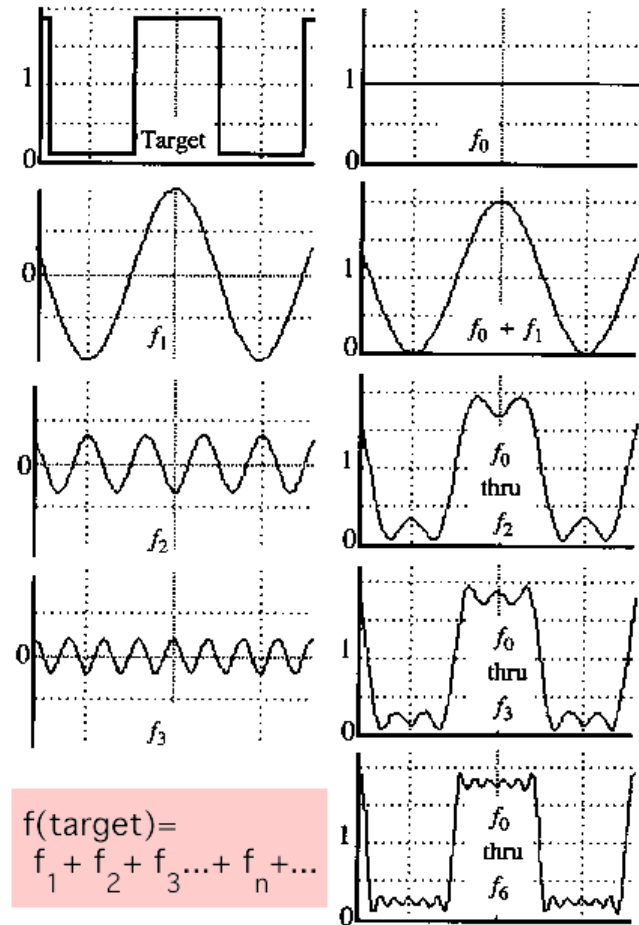


A Sum of Sinusoids

- Our building block:

$$A \sin(\omega x + \phi)$$

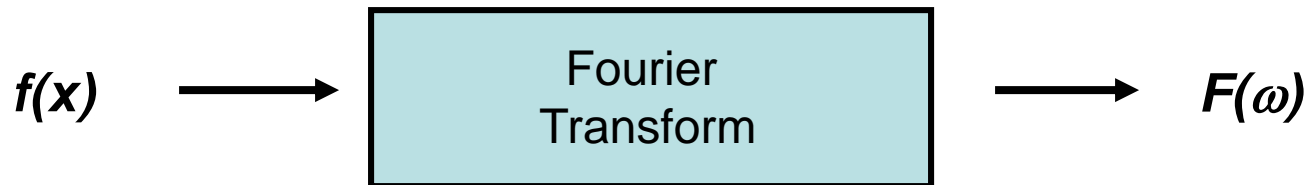
- Add enough of them to get any signal $f(x)$ you want!
- How many degrees of freedom?
- What does each control?
- Which one encodes the coarse vs. fine structure of the signal?



$$f(\text{target}) = f_0 + f_1 + f_2 + f_3 + \dots + f_n + \dots$$

Fourier Transform

- We want to understand the frequency ω of our signal. So, let's reparametrize the signal by ω instead of x :



- For every ω from 0 to inf, $F(\omega)$ holds the amplitude A and phase ϕ of the corresponding sine
 - How can F hold both? Complex number trick!

$$F(\omega) = R(\omega) + iI(\omega)$$

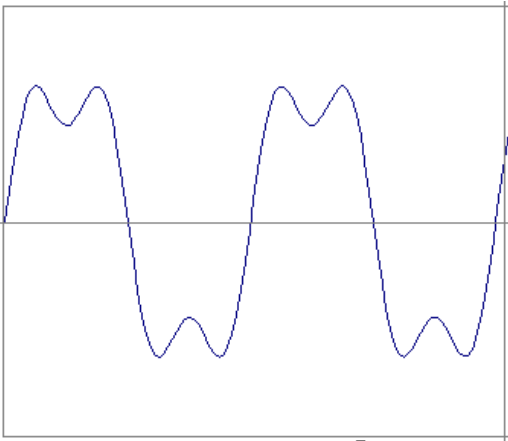
$$A = \pm \sqrt{R(\omega)^2 + I(\omega)^2}$$

$$A \sin(\omega x + \phi)$$

$$\phi = \tan^{-1} \frac{I(\omega)}{R(\omega)}$$

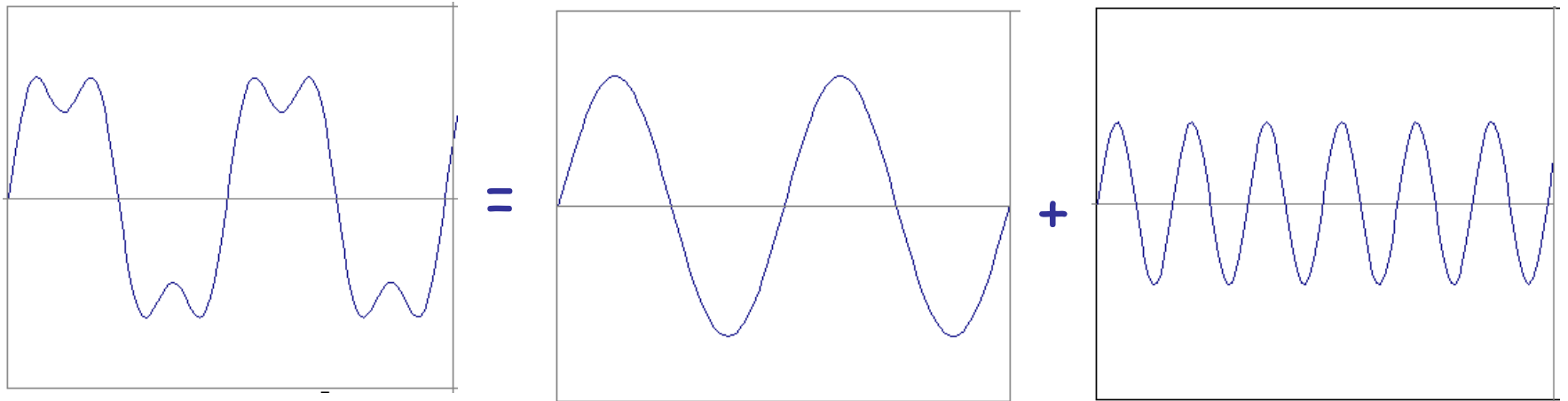
Time and Frequency

- example : $g(t) = \sin(2\pi f t) + (1/3)\sin(2\pi(3f) t)$



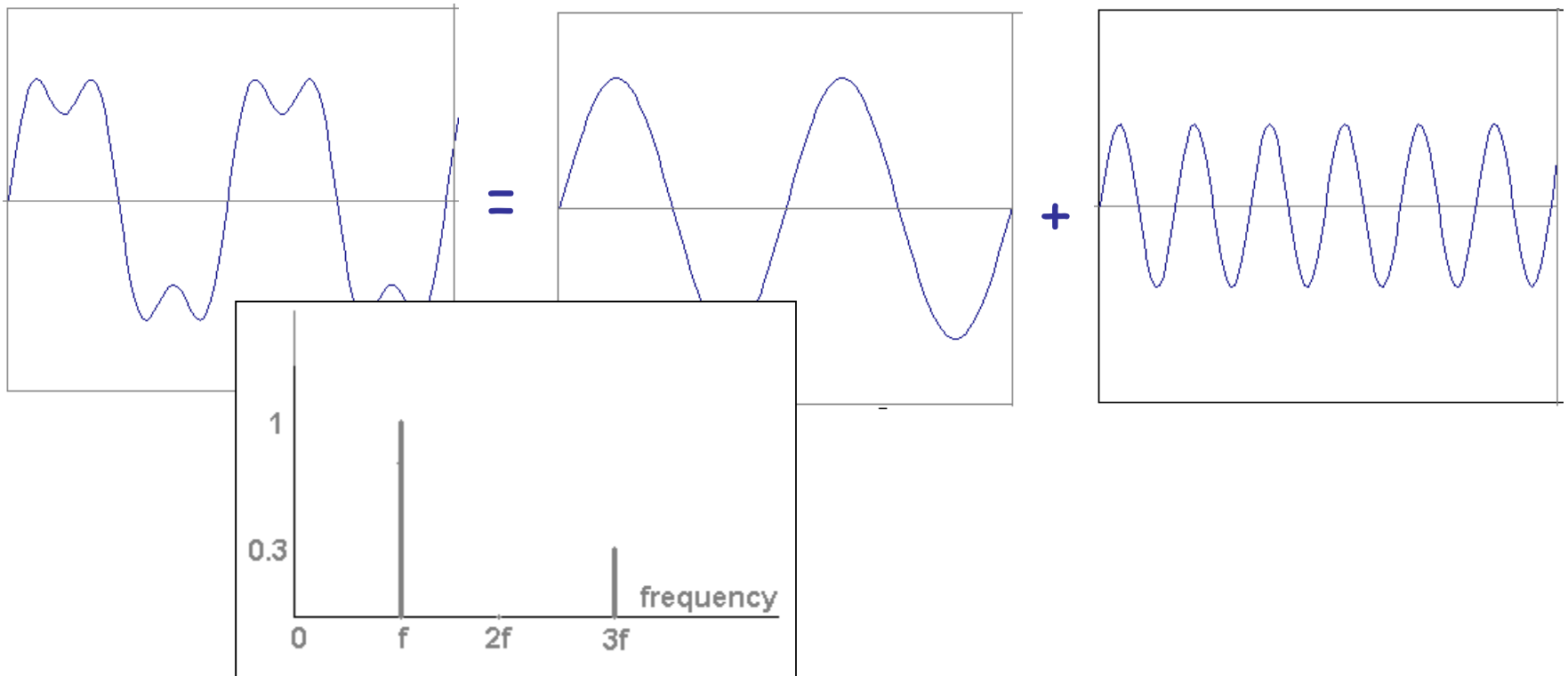
Time and Frequency

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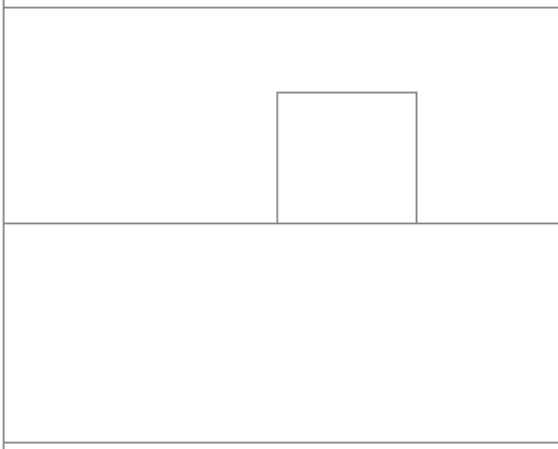
Frequency Spectra

- example : $g(t) = \sin(2\pi f t) + (1/3)\sin(2\pi(3f) t)$

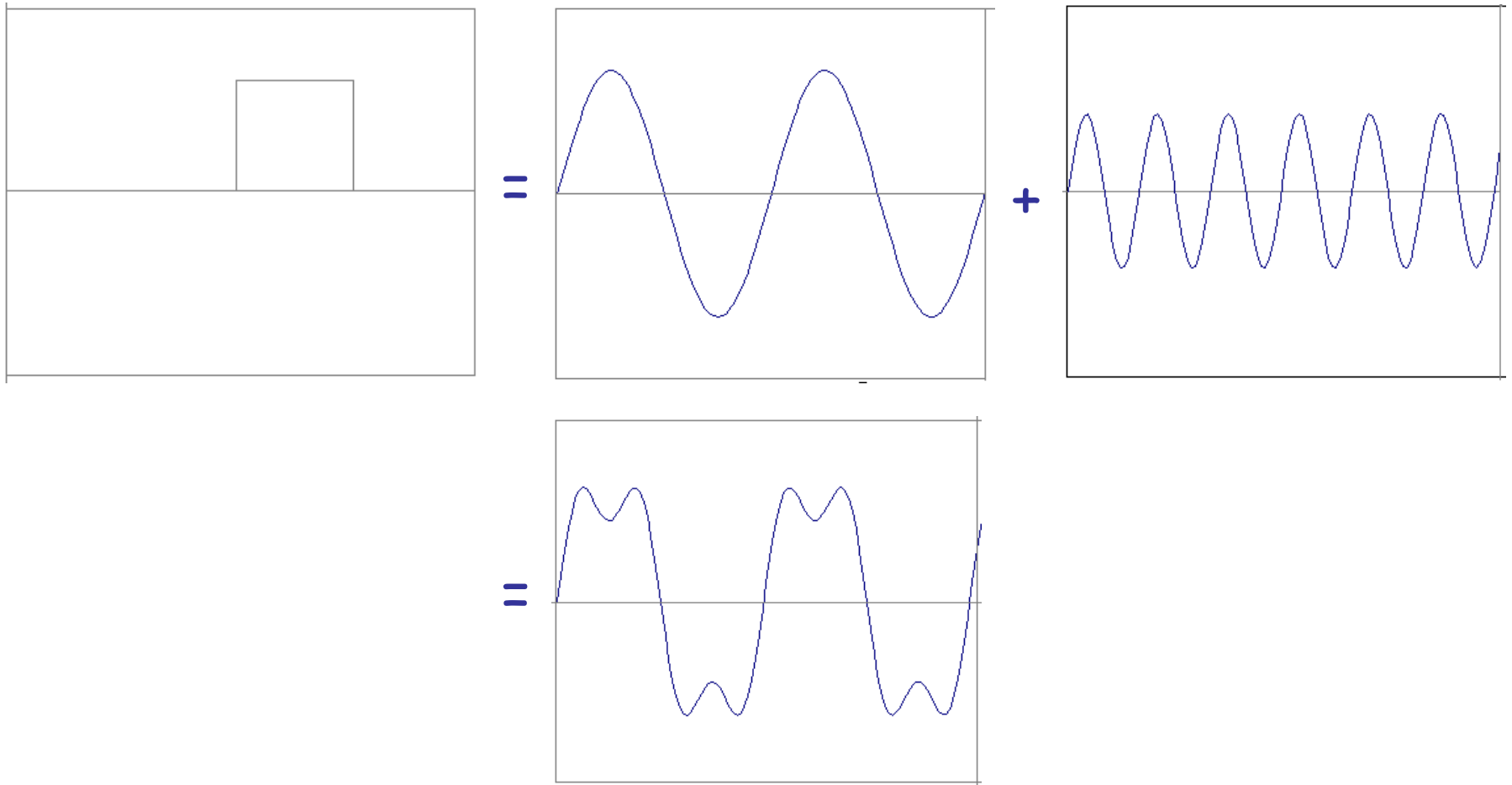


Frequency Spectra

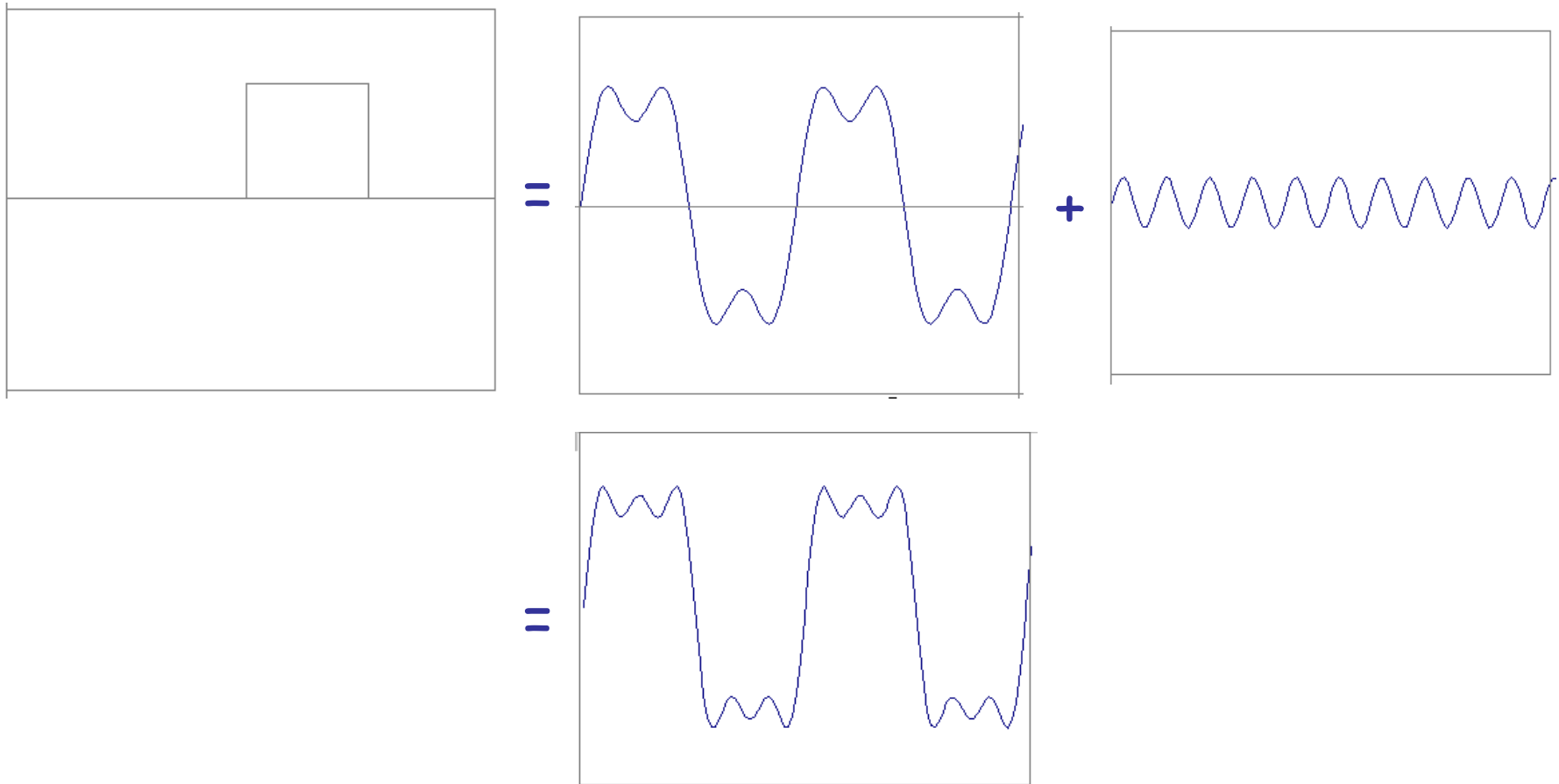
- Usually, frequency is more interesting than the phase



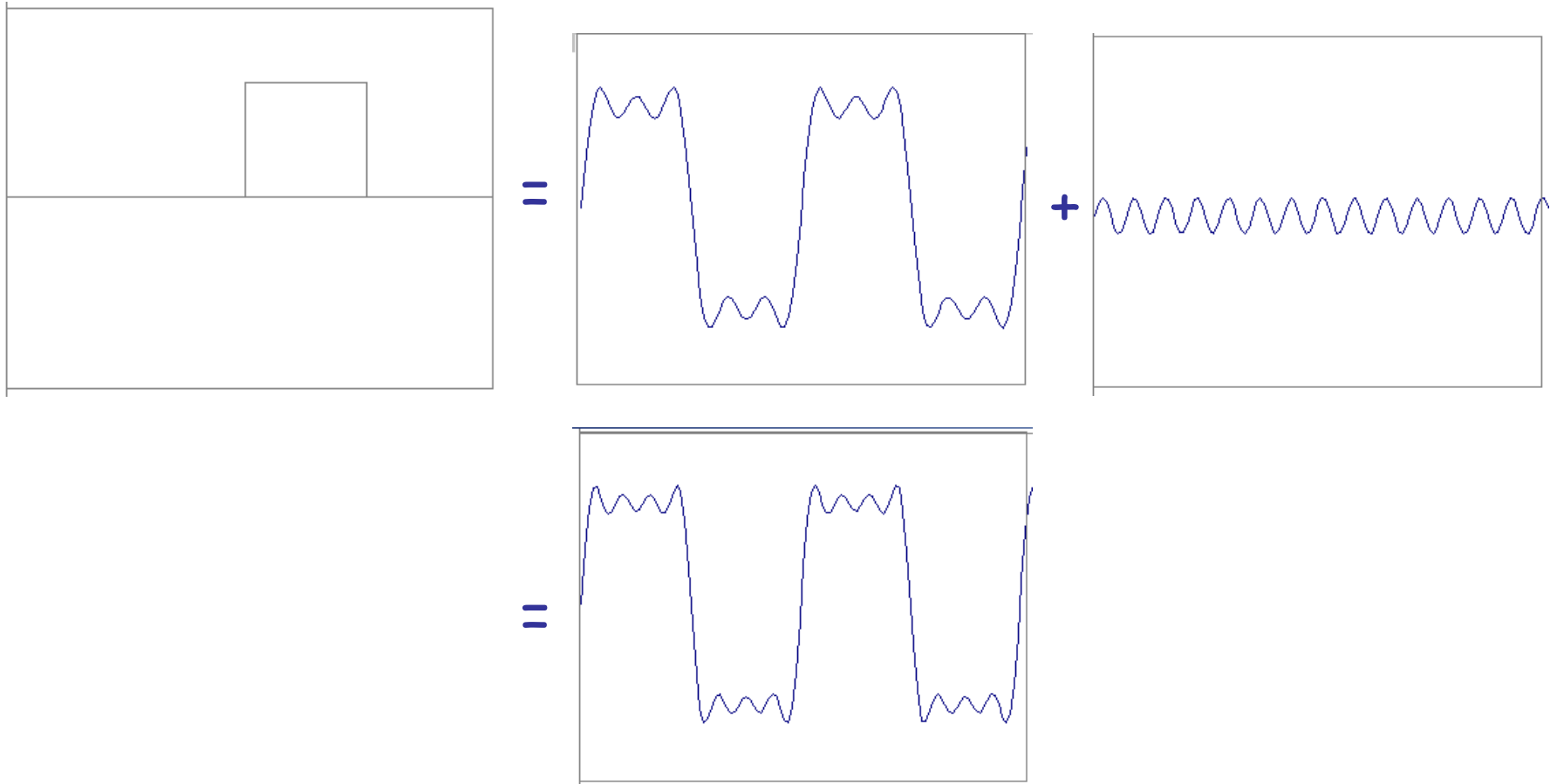
Frequency Spectra



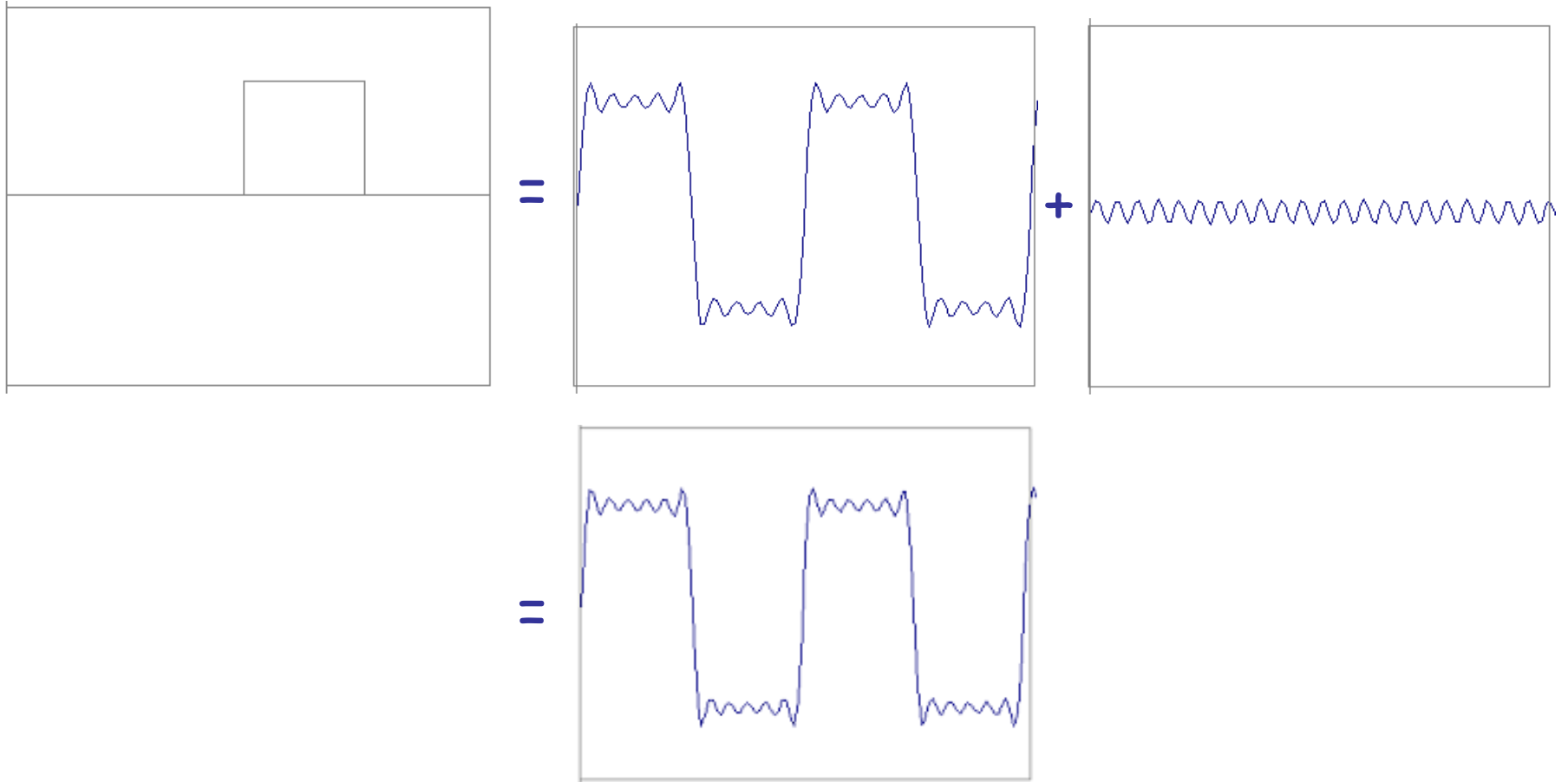
Frequency Spectra



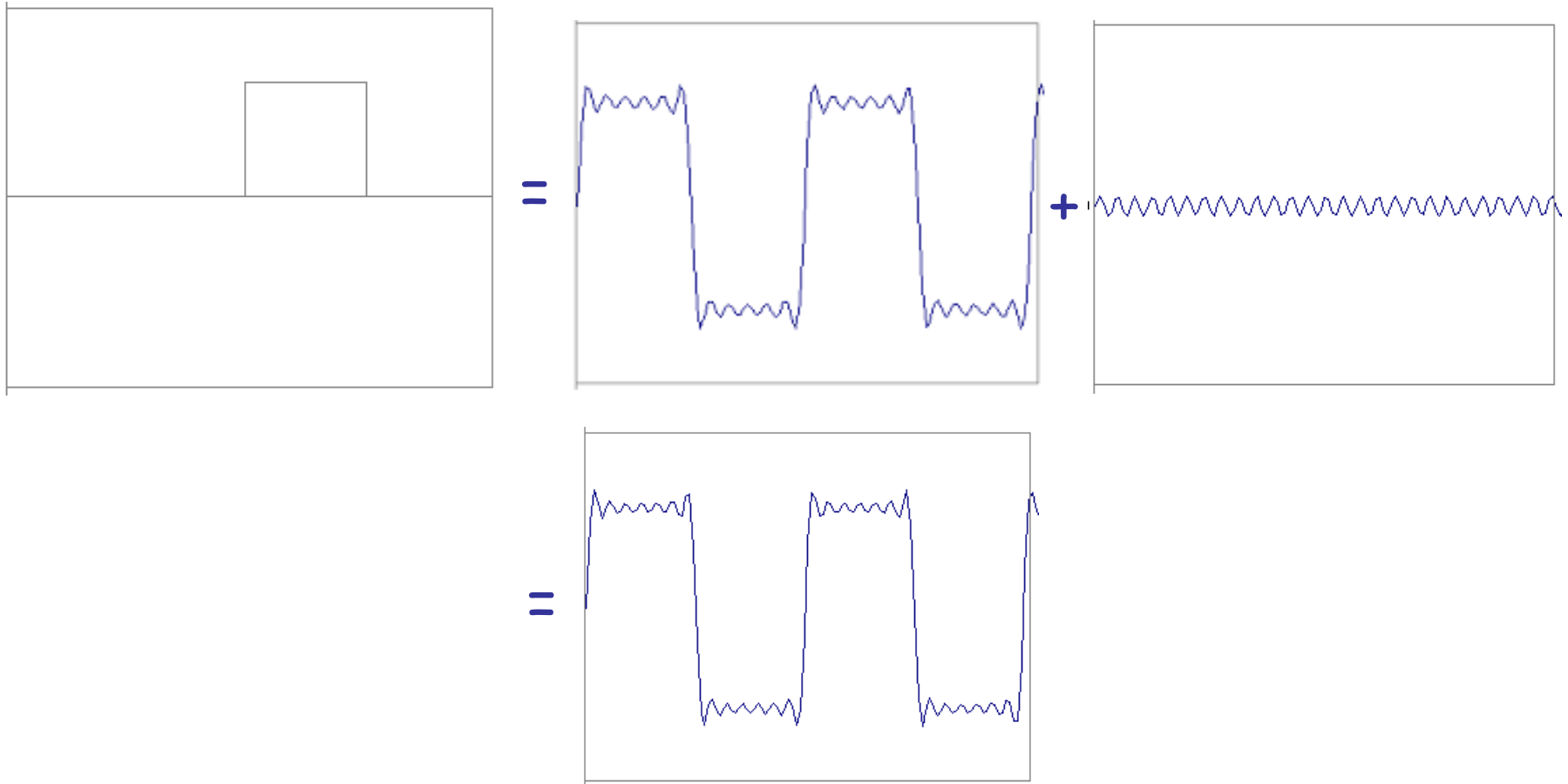
Frequency Spectra



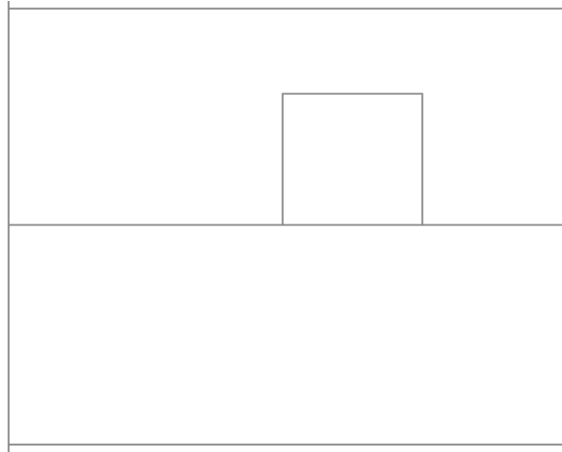
Frequency Spectra



Frequency Spectra

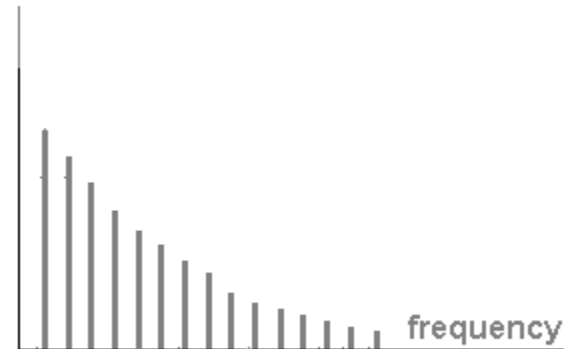


Frequency Spectra



=

$$A \sum_{k=1}^{\infty} \frac{1}{k} \sin(2\pi kt)$$



Fourier Transform – more formally

Represent the signal as an infinite weighted sum of an infinite number of sinusoids

$$F(u) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi ux} dx$$

Note: $e^{ik} = \cos k + i \sin k$ $i = \sqrt{-1}$

Arbitrary function \longrightarrow Single Analytic Expression

Spatial Domain (x) \longrightarrow Frequency Domain (u)
(Frequency Spectrum $F(u)$)

Inverse Fourier Transform (IFT) $f(x) = \int_{-\infty}^{\infty} F(u) e^{i2\pi ux} dx$

Fourier Transform

- Also, defined as:

$$F(u) = \int_{-\infty}^{\infty} f(x) e^{-iux} dx$$

Note: $e^{ik} = \cos k + i \sin k$ $i = \sqrt{-1}$

- Inverse Fourier Transform (IFT)

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(u) e^{iux} dx$$

Properties of Fourier Transform

| | | | | |
|------------------------|-------------------------|----------------|---|------------------|
| Linearity | $c_1 f(x) + c_2 g(x)$ | | $c_1 F(u) + c_2 G(u)$ | |
| Scaling | $f(ax)$ | Spatial Domain | $\frac{1}{ a } F\left(\frac{u}{a}\right)$ | Frequency Domain |
| Shifting | $f(x - x_0)$ | | $e^{-i2\pi u x_0} F(u)$ | |
| Symmetry | $F(x)$ | | $f(-u)$ | |
| Conjugation | $f^*(x)$ | | $F^*(-u)$ | |
| Convolution | $f(x) * g(x)$ | | $F(u)G(u)$ | |
| Differentiation | $\frac{d^n f(x)}{dx^n}$ | | $(i2\pi u)^n F(u)$ | |

Topic: Frequency Space

- Fourier Transform
- **Frequency Space**
- Spatial Convolution

How does this apply to images?

- We have defined the Fourier Transform as

$$F(u) = \int_{-\infty}^{\infty} f(x) e^{-iux} dx$$

- But images are:
 - Discrete.
 - Two-dimensional.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 0 | 3 | 2 | 5 | 4 | 7 | 6 | 9 | 8 |
| 3 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 2 | 1 | 0 | 3 | 2 | 5 | 4 | 7 | 6 |
| 5 | 2 | 3 | 0 | 1 | 2 | 3 | 4 | 5 |
| 4 | 3 | 2 | 1 | 0 | 3 | 2 | 5 | 4 |
| 7 | 4 | 5 | 2 | 3 | 0 | 1 | 2 | 3 |
| 6 | 5 | 4 | 3 | 2 | 1 | 0 | 3 | 2 |
| 9 | 6 | 7 | 4 | 5 | 2 | 3 | 0 | 1 |
| 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 |

What a computer sees

2D Discrete FT

- In a 2-variable case, the discrete FT pair is:

$$F(u, v) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) \exp[-j2\pi(ux/M + vy/N)]$$

For $u=0, 1, 2, \dots, M-1$ and $v=0, 1, 2, \dots, N-1$

New matrix
with the
same size!

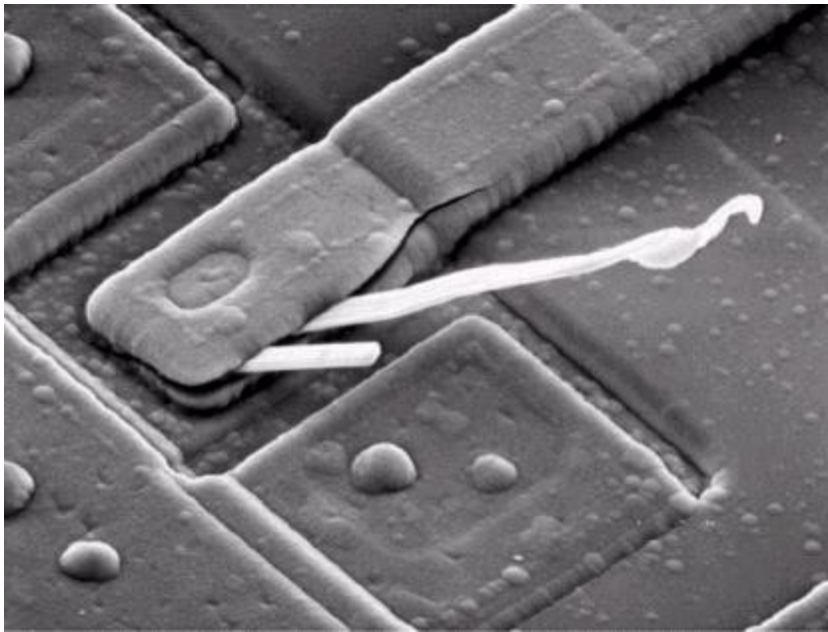
AND:
$$f(x, y) = \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u, v) \exp[j2\pi(ux/M + vy/N)]$$

For $x=0, 1, 2, \dots, M-1$ and $y=0, 1, 2, \dots, N-1$

Frequency Space

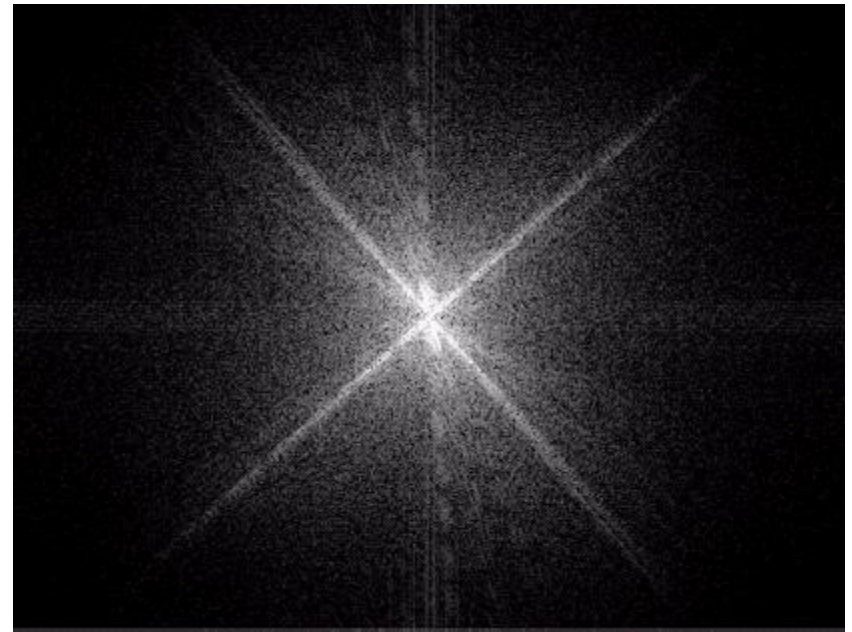
- Image Space

- $f(x,y)$
- Intuitive

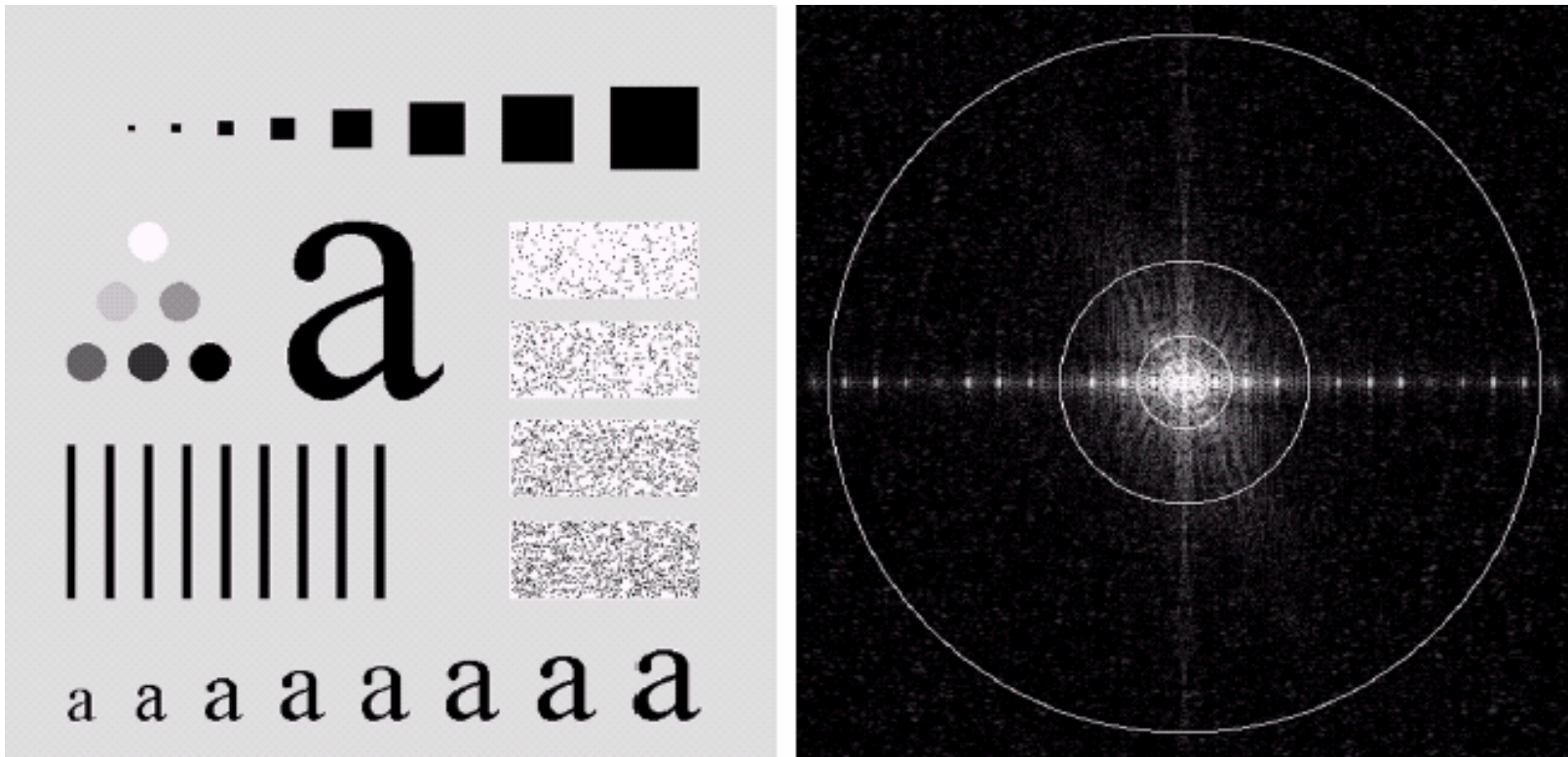


- Frequency Space

- $F(u,v)$
- What does this mean?



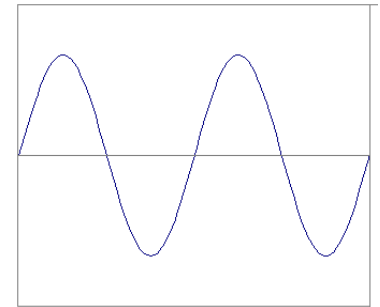
Power distribution



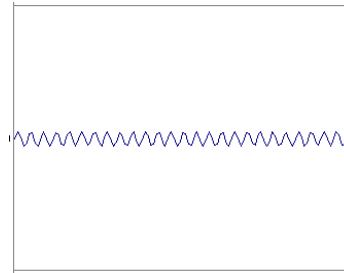
An image (500x500 pixels) and its Fourier spectrum. The super-imposed circles have radii values of 5, 15, 30, 80, and 230, which respectively enclose 92.0, 94.6, 96.4, 98.0, and 99.5% of the image power.

Power distribution

- Most power is in low frequencies.
- Means we are using more of this:

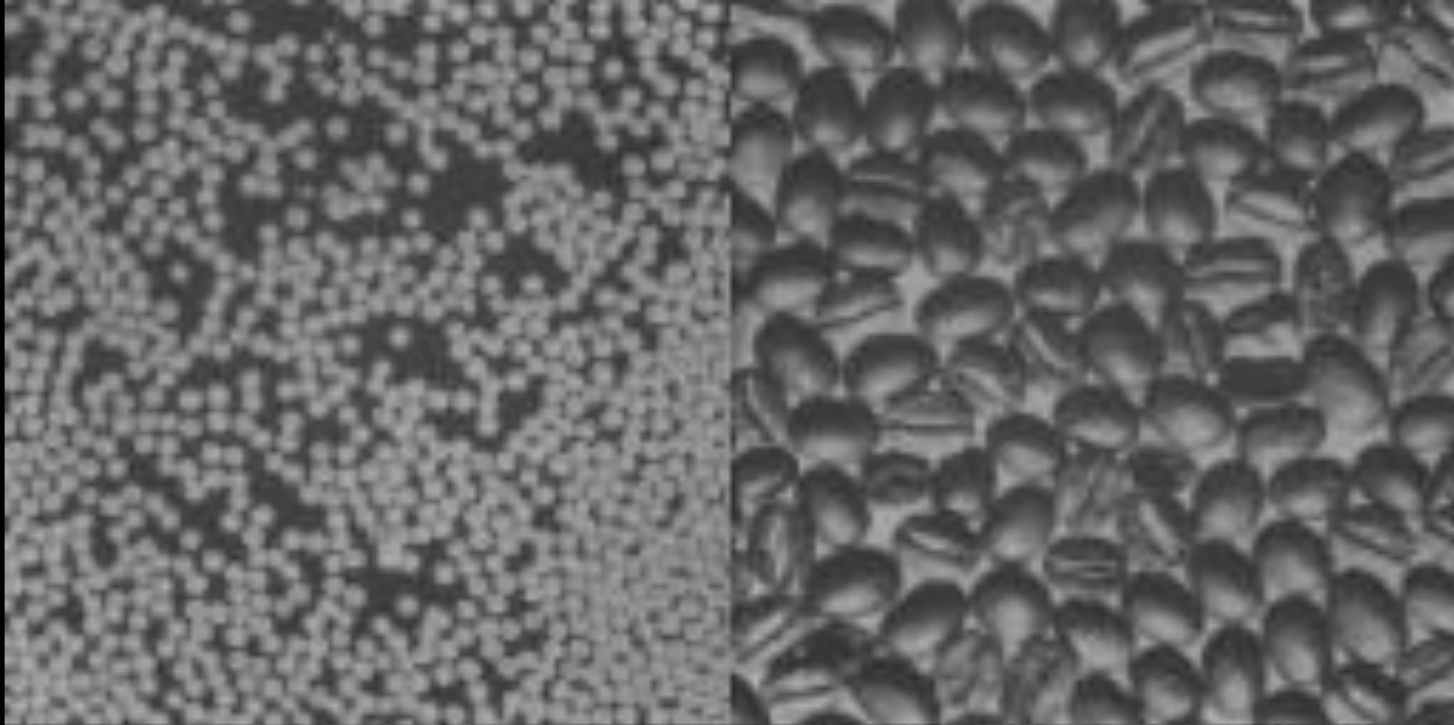


And less of this:



To represent our signal.

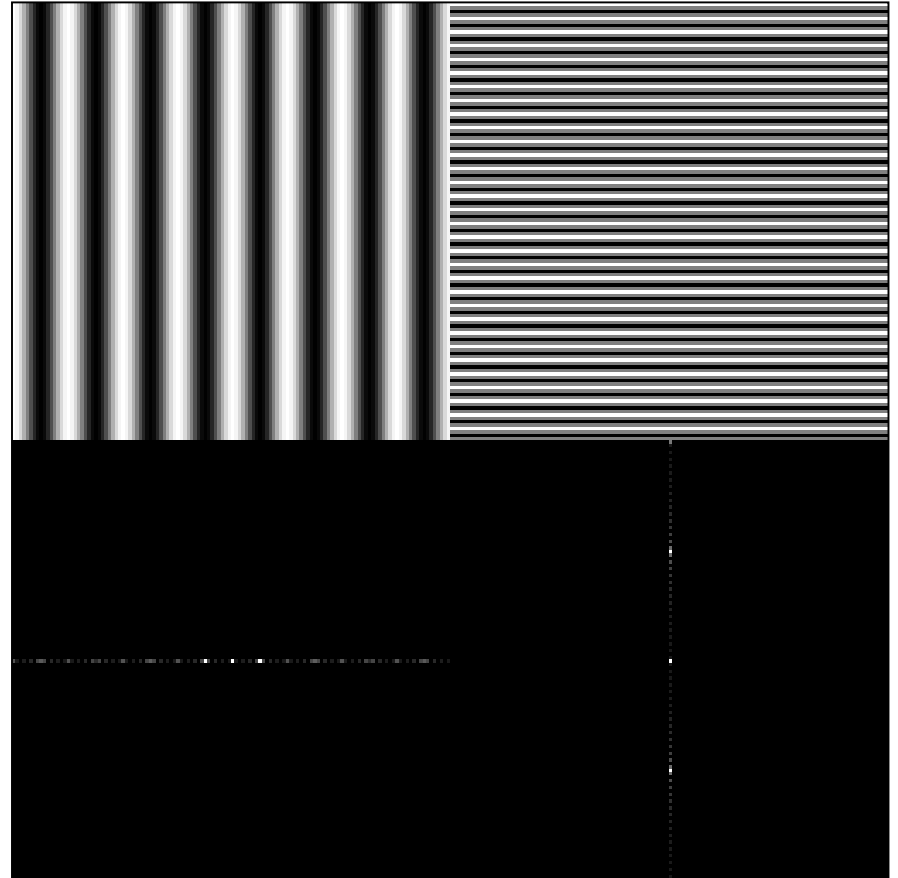
- Why?

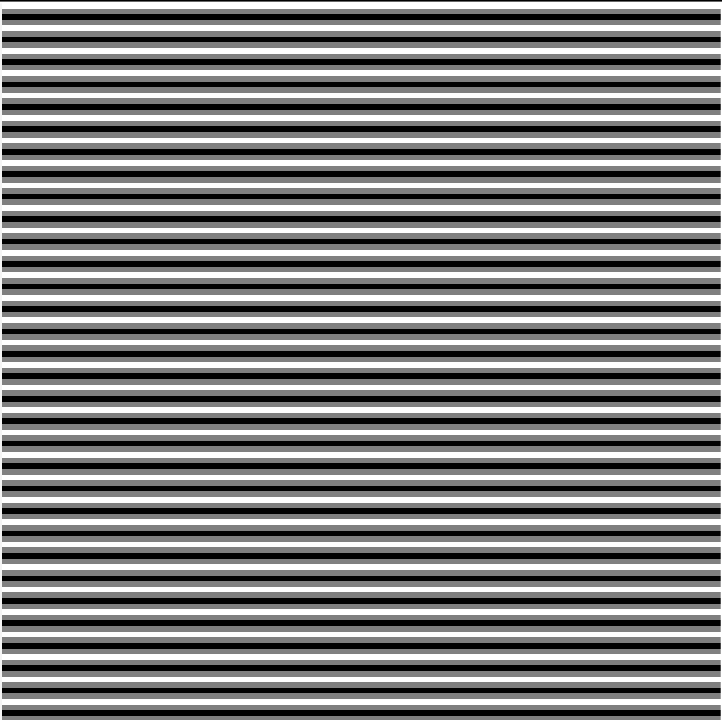
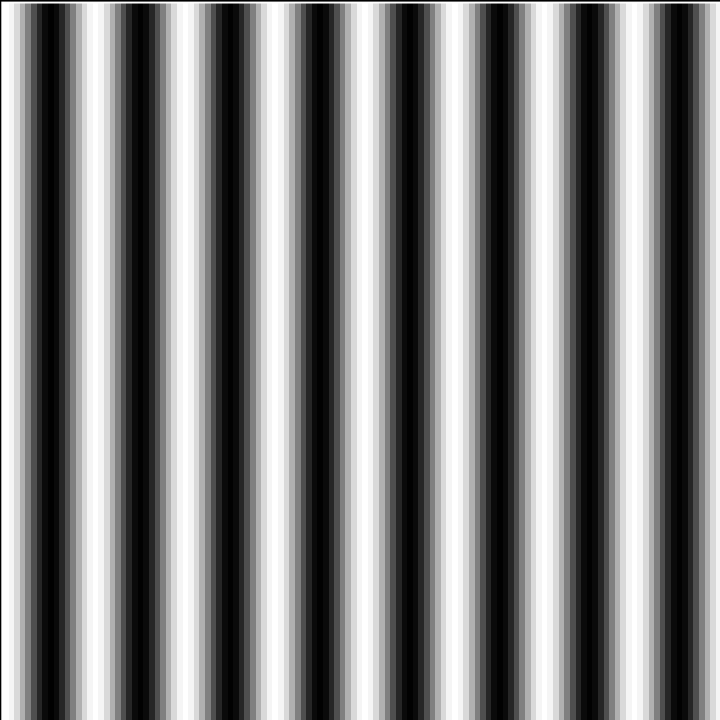


What does this mean??

Horizontal and Vertical Frequency

- **Frequencies:**
 - Horizontal frequencies correspond to horizontal gradients.
 - Vertical frequencies correspond to vertical gradients.
- **What about diagonal lines?**







If I discard high-frequencies, I get a blurred image...
Why?

Why bother with FT?

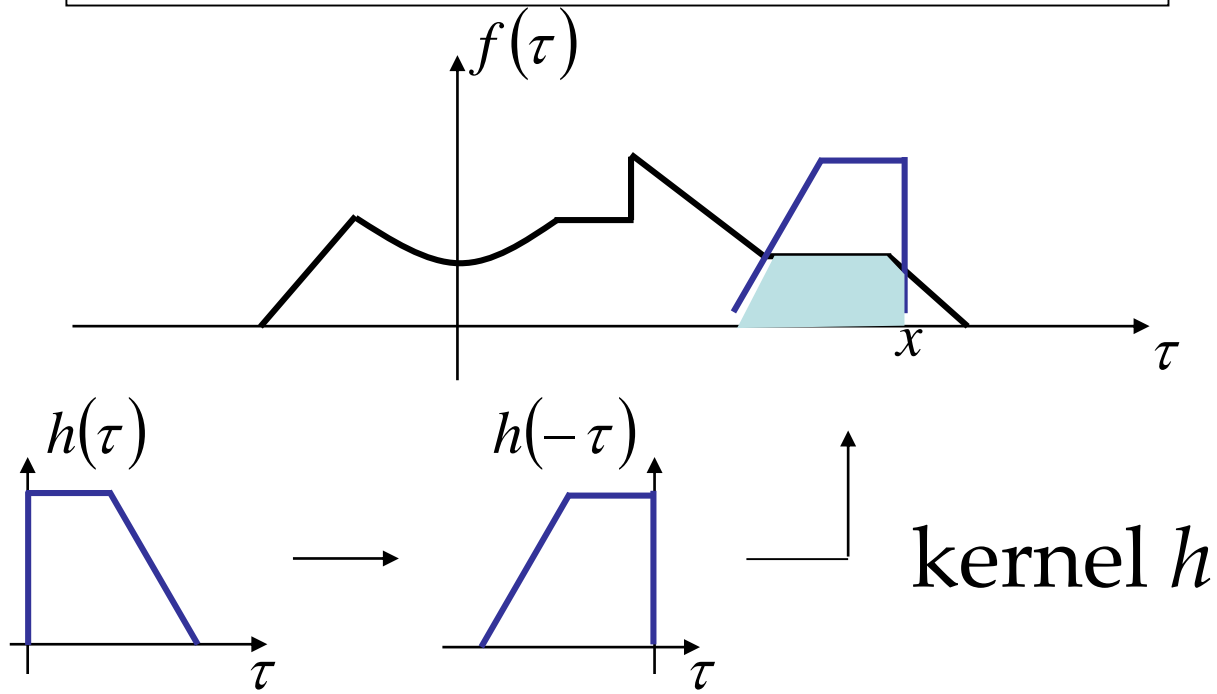
- Great for filtering.
- Great for compression.
- In some situations: Much faster than operating in the spatial domain.
- Convolutions are simple multiplications in Frequency space!
- ...

Topic: Spatial Convolution

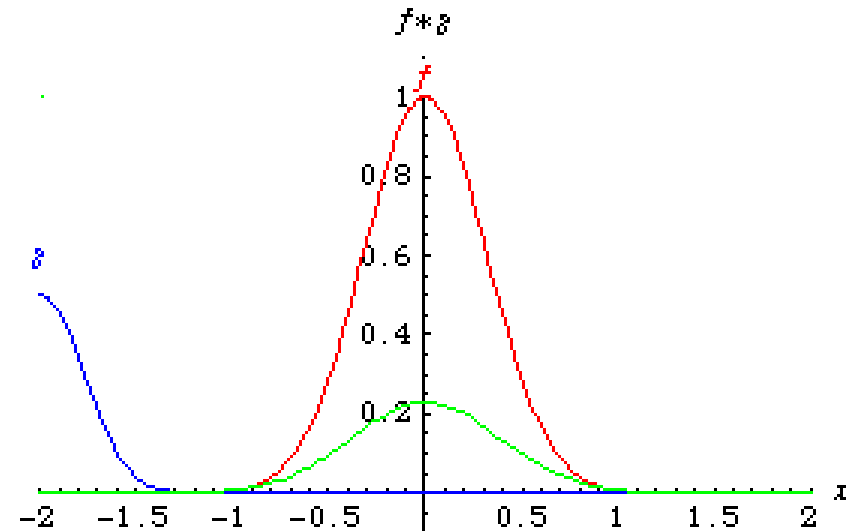
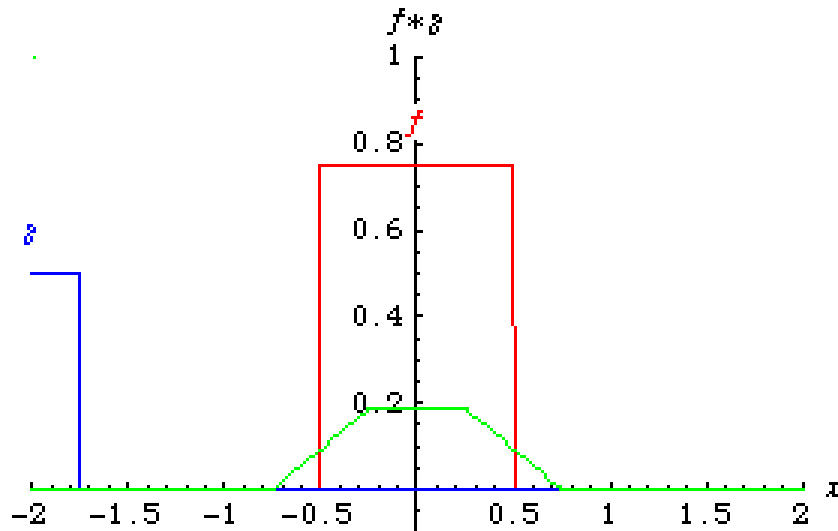
- Fourier Transform
- Frequency Space
- **Spatial Convolution**

Convolution

$$g(x) = \int_{-\infty}^{\infty} f(\tau)h(x-\tau)d\tau \quad g = f * h$$



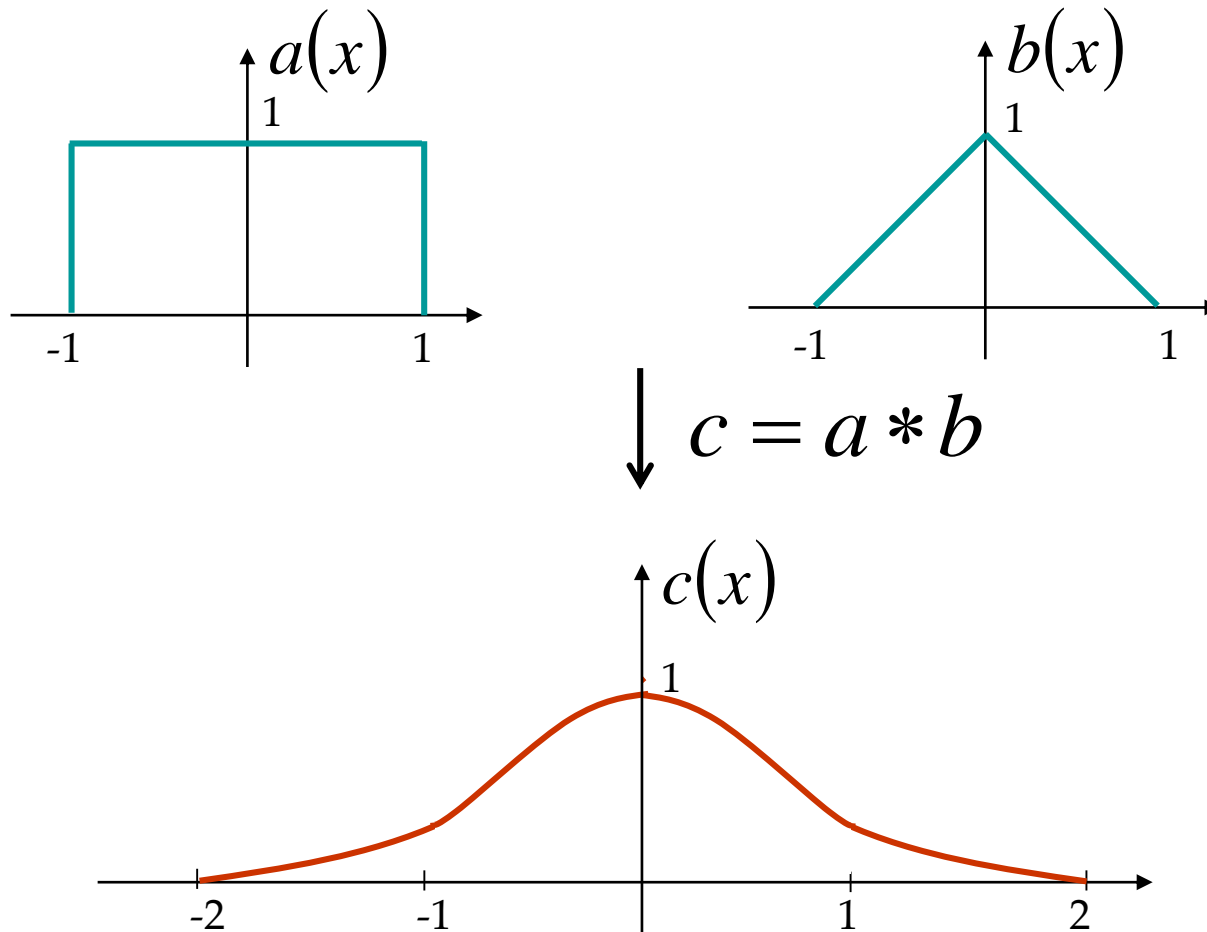
Convolution - Example



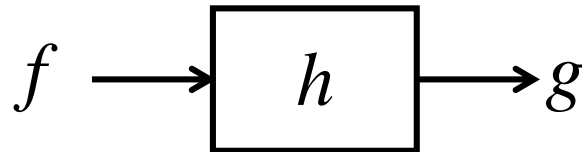
— f
— g
— $f * g$

Eric Weinstein's Math World

Convolution - Example



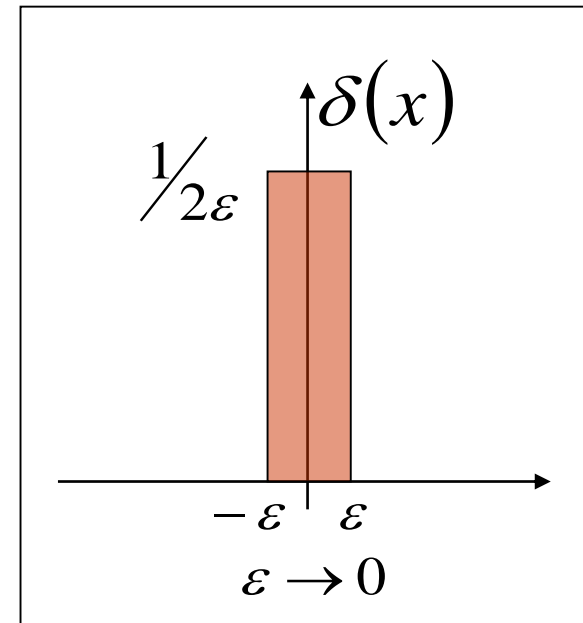
Convolution Kernel – Impulse Response



$$g = f * h$$

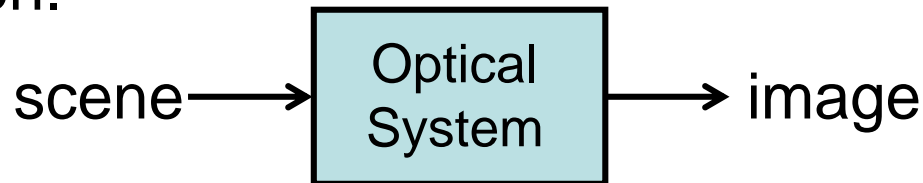
- What h will give us $g = f$?

Dirac Delta Function (Unit Impulse)

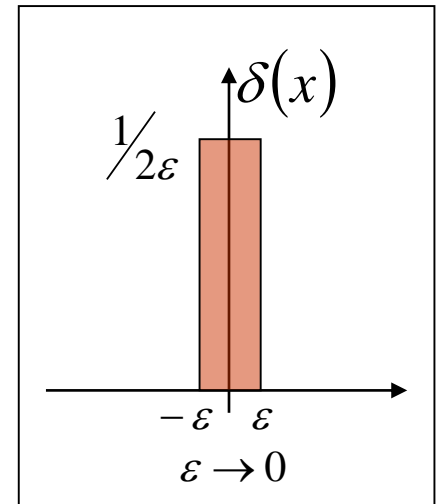
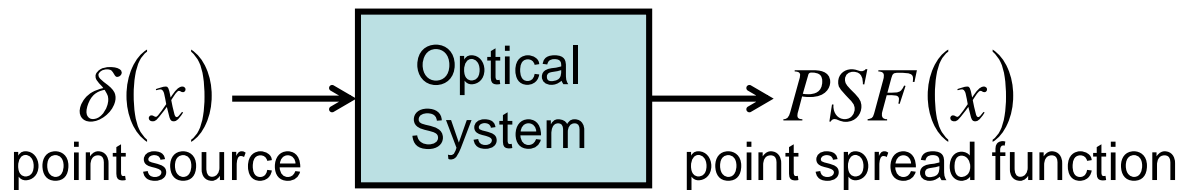


Point Spread Function

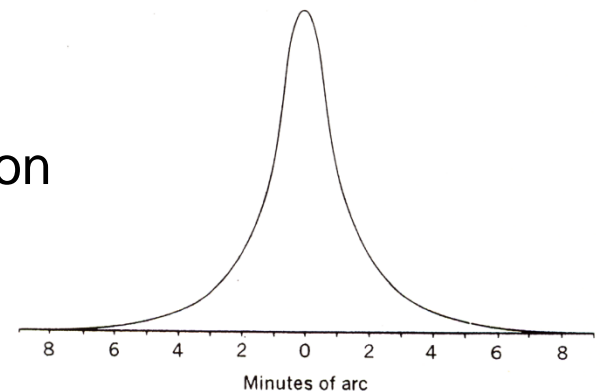
- Ideally, the optical system should be a Dirac delta function.



- However, optical systems are never ideal.



- Point spread function of Human Eyes.



Point Spread Function



normal vision



myopia



hyperopia

Properties of Convolution

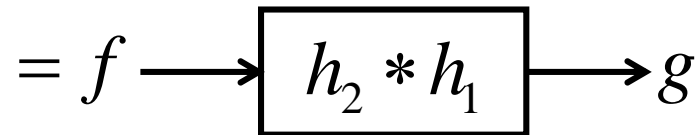
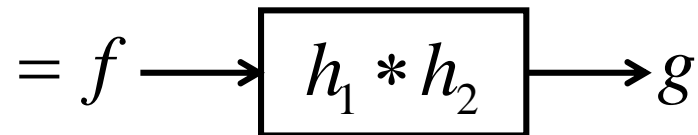
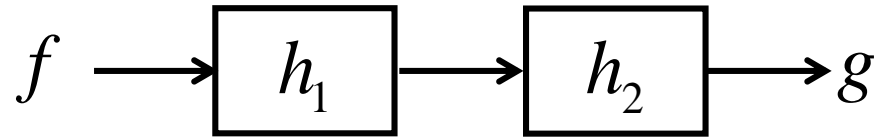
- Commutative

$$a * b = b * a$$

- Associative

$$(a * b) * c = a * (b * c)$$

- Cascade system



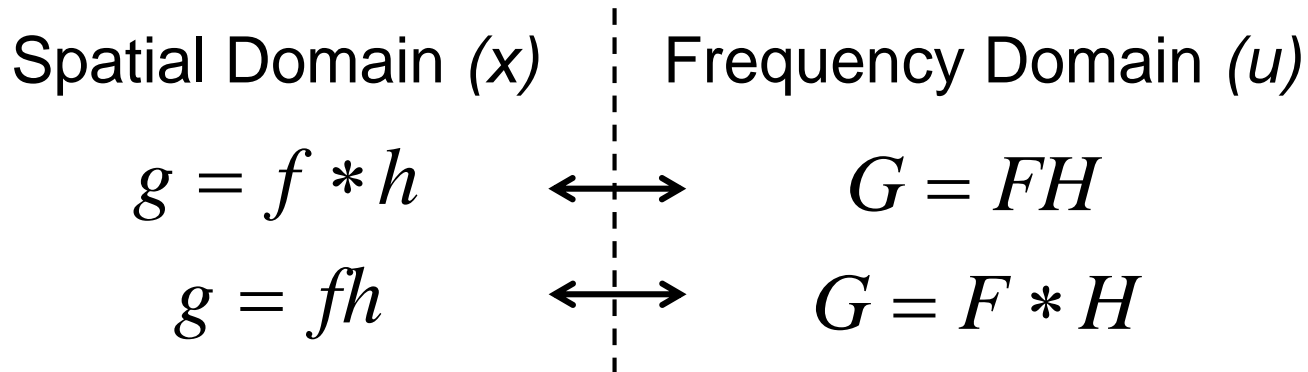
Fourier Transform and Convolution

$$\begin{aligned} \text{Let } g &= f * h & \text{Then } G(u) &= \int_{-\infty}^{\infty} g(x) e^{-i2\pi ux} dx \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\tau) h(x - \tau) e^{-i2\pi ux} d\tau dx \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [f(\tau) e^{-i2\pi u\tau} d\tau] [h(x - \tau) e^{-i2\pi u(x - \tau)} dx] \\ &= \int_{-\infty}^{\infty} [f(\tau) e^{-i2\pi u\tau} d\tau] \int_{-\infty}^{\infty} [h(x') e^{-i2\pi ux'} dx'] &= F(u)H(u) \end{aligned}$$

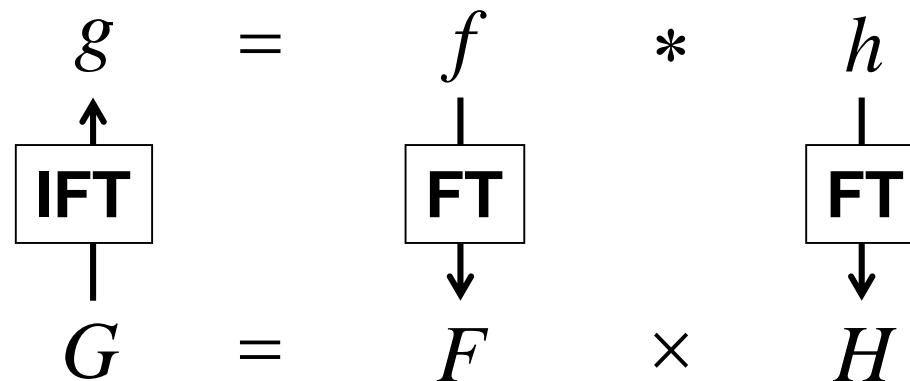
Convolution in spatial domain

\Leftrightarrow Multiplication in frequency domain

Fourier Transform and Convolution



So, we can find $g(x)$ by Fourier transform



Example use: Smoothing/Blurring

- We want a smoothed function of $f(x)$

$$g(x) = f(x) * h(x)$$

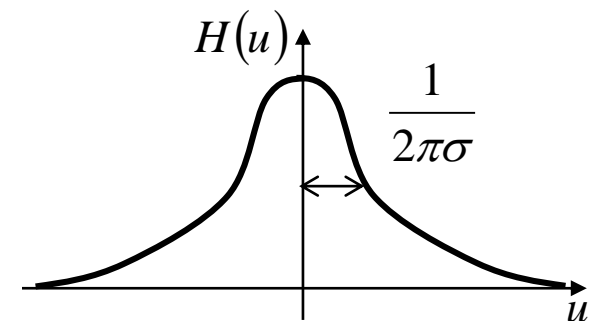
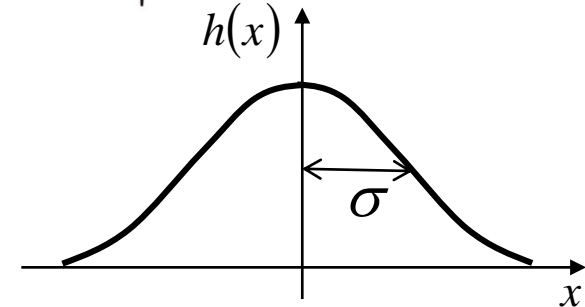
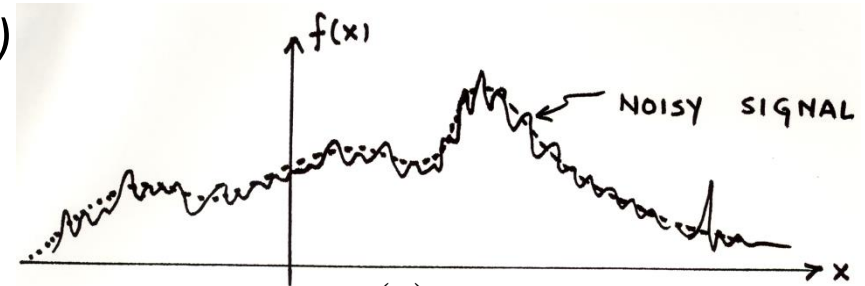
- Let us use a Gaussian kernel

$$h(x) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left[-\frac{1}{2} \frac{x^2}{\sigma^2}\right]$$

- Then

$$H(u) = \exp\left[-\frac{1}{2} (2\pi u)^2 \sigma^2\right]$$

$$G(u) = F(u)H(u)$$



Resources

- Russ – Chapter 6
- Gonzalez & Woods – Chapter 4