Exercises #3
Solving Recurrences

Theoretical Background

4 methods for solving recurrences:

- **Unrolling:** unroll the recurrence to obtain an expression (ex: summation) you can work with
- **Substitution:** guess the answer and prove by induction
- **Recursion Tree:** draw a tree representing the recursion and sum all the work done in the nodes
- **Master Theorem:** If the recurrence is of the form $\mathbf{aT(n/b) + cn^k}$ *(this is one version of the theorem)*:
  (1) $T(n) = \Theta(n^k)$ if $a < b^k$
  (2) $T(n) = \Theta(n^k \log n)$ if $a = b^k$
  (3) $T(n) = \Theta(n^{\log_b a})$ if $a > b^k$

For the following exercises, assume that $T(n)$ takes constant time for sufficiently small n.

1. Solve the following recurrences by unrolling. State the answer using Θ notation.
   (a) $T(n) = T(n-2) + 1$
   (b) $T(n) = T(n-1) + n^2$
2. Show that the following conjectures are true by using the substitution method.
   (a) $T(n) = T(n-1) + 2$ is $\Theta(n)$
   (b) $T(n) = 2T(n/2) + n$ is $\Theta(n \log n)$
3. Draw a recursion tree for the following recurrences and use it to obtain asymptotic bounds as tight as possible.
   (a) $T(n) = 3T(n/2) + n$
   (b) $T(n) = T(n/2) + n^2$
4. Solve the following recurrences using the master method:
   (a) $T(n) = 2T(n/4) + 1$
   (b) $T(n) = 2T(n/4) + \sqrt{n}$
   (c) $T(n) = 2T(n/4) + n$
   (d) $T(n) = 2T(n/4) + n^2$

5. Consider the recurrence $T(n) = 8T(n/2) + n^2$

   (a) Use the substitution method to try to prove that $T(n) = O(n^2)$. The proof should fail. Can you understand why?

   (b) Use the master method to find the a tight asymptotic bound. Try to prove that bound directly. Does the math work?

   (c) Use a stronger induction hypothesis (by subtracting a lower order term) and make a correct proof of that tighter bound.

6. Give asymptotic upper and lower bounds (as tight as possible) for the following recurrences. You can use any method you want.

   (a) $T(n) = 7T(n/3) + n^2$
   (b) $T(n) = 7T(n/2) + n^2$
   (c) $T(n) = 2T(n/4) + n^2$
   (d) $T(n) = T(n-2) + n^3$
   (e) $T(n) = T(n/2) + T(n/4) + T(n/8) + n$
   (f) $T(n) = T(n-1) + \frac{1}{n}$
   (g) $T(n) = 4T(n/3) + n \log_2 n$