Sorting and variants

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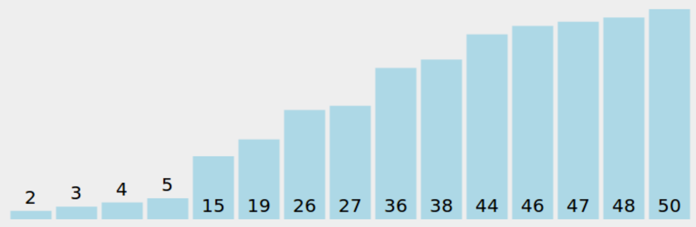


Sorting

- Sorting is an **initial step** to many other algorithms
  - Ex: finding the median
- When you don't know what to do... **sort**!
  - Ex: finding repeated elements is much easier after sorting
- **Different sorting types** might be more adequate to different scenarios
  - Ex: to less general cases, there might be $\mathcal{O}(n)$ algorithms
- It is important to know the sorting functions available on your language **libraries**
  - Ex: qsort (C), STL sort (C++), Arrays.sort (Java)

About sorting complexity

- What is the least possible complexity for a general sorting algorithm? $\Theta(\mathrm{n} \log \mathrm{n})$... but only on the **comparative model**.
  - **Comparative model:** to distinguish elements I can only use comparisons $(<, >, =, \geq, \leq)$. How many comparisons are needed?
- A sketch of the **proof** that comparative sorting is $\Omega(\mathrm{n} \log \mathrm{n})$
  - Input of size n has n! **possible permutations** (only one is the desired ordering)
  - A comparison has **two possible results** (it can distinguish between 2 different permutations)
  - Let $f(n)$ be the function that measures the **number of comparisons**
  - $f(n)$ comparisons: can **distinguish** between $2^{f(n)}$ permutations
  - We need that $2^{f(n)} \geq n!$, that is, $\mathrm{f(n)} \geq \log_2(\mathrm{n!})$
  - Using **Stirling's approximation**, we know that $\mathrm{f(n)} \geq \mathrm{n} \log_2 \mathrm{n}$

Some sorting algorithms

- **Comparative algorithms**
  - **BubbleSort** (swap elements)
  - **SelectionSort** (selected smallest/largest)
  - **InsertionSort** (insert on correct position)
  - **MergeSort** (divide in two, sort halves, merge sorted parts)
  - **HeapSort** (create heap with all elements, remove one by one)
  - **QuickSort** (divide according to a pivot and sort recursively)

- **Non Comparative Algorithms**
  - **CountingSort** (count number of elements of each type)
  - **RadixSort** (sort according to "digits")

Non Comparative Algorithms

- To simplify, let's assume that the **elements to sort are numbers**
- Idea can be **generalized** to other data types
- Suppose we have n elements to sort, stored on an array v with indexes from 0 to $n - 1$

CountingSort

- **Key idea:** count the amount of numbers of each type

CountingSort

$count[max\_size]$ $\leftarrow$ `frequencies array`
For $i = 0$ **to** $n-1$ **do**
    $count[v[i]]++$ `(one more v[i] element)`
$i = 0$
For $j = min\_size$ **to** max_size **do**
    While $count[j] > 0$ **do**
        $v[i] = j$ `(put element on array)`
        $count[j]--$ `(one less element of that size)`
        $i++$ `(increments first free position on the array)`

You can check an animation at VisuAlgo

- Let k be the largest number
- This algorithm will take $O(n+k)$

RadixSort

- **Key idea:** sort digit by digit

A possible RadixSort (starting on the least significant digit)

$bucket[10] \leftarrow$ `array of lists of numbers (one per digit)`
For $pos = 1$ **to** max_number_digits **do**
    For $i = 0$ **to** $n-1$ **do** (for each number)
        Put $v[i]$in $bucket[digit\_position\_pos(v[i])]$
    For $i = 0$ **to** 9 **do** (for each possible digit)
        While $size(bucket[i]) > 0$ **do**
            Take first number of $bucket[i]$ and add it to $v[]$

You can check an animation at VisuAlgo

- Let k be the largest quantity of digits in a single number
- This algorithm will take $O(k \times n)$

Some sorting algorithms

There are many more!

Exchange sorts	Bubble sort · Cocktail sort · Odd–even sort · Comb sort · Gnome sort · Quicksort · Stooge sort · Bogosort
Selection sorts	Selection sort · Heapsort · Smoothsort · Cartesian tree sort · Tournament sort · Cycle sort
Insertion sorts	Insertion sort · Shellsort · Splaysort · Tree sort · Library sort · Patience sorting
Merge sorts	Merge sort · Cascade merge sort · Oscillating merge sort · Polyphase merge sort · Strand sort
Distribution sorts	American flag sort · Bead sort · Bucket sort · Burstsort · Counting sort · Pigeonhole sort · Proxmap sort · Radix sort · Flashsort
Concurrent sorts	Bitonic sorter · Batcher odd–even mergesort · Pairwise sorting network
Hybrid sorts	Block sort · Timsort · Introsort · Spreadsort · JSort
Other	Topological sorting · Pancake sorting · Spaghetti sort

(source of picture: http://en.wikipedia.org/wiki/Sorting_algorithm)

Overview

- There are **many** sorting algorithms
- The **"best"** algorithm depends on the use case
- It is possible to **combine** several algorithms (hybrid approaches)
  - Ex: RadixSort might have as internal step another algorithm, as long as it is a **stable sort** (keep initial order in case of a tie)
- In practice, on **real implementations**, this is what is done (to combine):
  (Note: the exact implementation depends on compiler and version)
  - **Java:** uses **Timsort** (MergeSort + InsertionSort)
  - **C++ STL**: uses **IntroSort** (QuickSort + HeapSort) + InsertionSort

Example use cases of sorting

Repetitions

Problem: finding **repeated** elements

Input

```
 9 21 27 38 34 53 19 38 43
51  1  9 10 39 50  6 26 44
 5 32 16 20 50 22 41 30 39
 3 32 30 31 40 50 56 13 19
46 32 56 26 20 57 32 27 31
17 32 54 61 34 22 14 54  9
34 30 38 10 30  5 37 61 44
```

Input

```
 1| 3| 5  5| 6| 9  9  9|10
10|13|14|16|17|19 19|20 20|
21|22 22|26 26|27 27|30 30
30 30|31 31|32 32 32 32 32|
34 34 34|37|38 38 38|39 39|
40|41|43|44 44|46|50 50 50|
51|53|54 54|56 56|57|61 61
```

Equal elements are together when sorted!

Example use cases of sorting

Others

Problem: find the **frequency** of elements
(equal elements are in consecutive positions after being sorted)

Problem: find **closest** pair of points
(sort and see differences between consecutive numbers)

Problem: find the k-**th** number
(sort and seek position k)

Problem: sort o **top**-k
(sort and seek first k numbers)

Problem: set **union**
(sort and "merge" - like in mergesort)

Problem: ser **intersection**
(sort and traverse - similar to mergesort)

Example use cases of sorting

Anagrams

Problem: Finding anagrams
(words/sets of words that use the same letters)

Exemples:

- **amor**, **ramo**, **mora** and **Roma** [amor]
- **Ricardo**, **criador** and **corrida** [acdiorr]
- **algorithm** and **logarithm** [aghilmort]
- **Tom Marvolo Riddle** and **I am Lord Voldemort** [addeillmmooorrtv]
- **Clint Eastwood** and **Old West action** [acdeilnoosttw]

Problem: Searching for elements in sorted arrays

Binary search - $\Theta(\log n)$

Binary search

A definition

Binary search on a sorted array (bsearch)

Input:

- an array v[] of n sorted number in increasing order
- a key to look for

Output:

- **Position** of *key* in array *v*[] (if it exists)
- **-1** (if it is not found)

Example:

v =

2	5	6	8	9	12

bsearch(v, 2) = 0
bsearch(v, 4) = -1
bsearch(v, 8) = 3
bsearch(v, 14) = -1

Binary search on a sorted array

```
bsearch(v, low, high, key)
    While (low ≤ high ) do
        middle = low + (high − low)/2
        If (key == v[middle])        return(middle)
        Else If (key < v[middle])    high = middle − 1
        Else                         low = middle + 1
    return(-1)
```

v =

2	5	6	8	9	12

bsearch(v, 0, 5, 8)

$low = 0, high = 5, middle = 2$

Since $8 > v[2]$: $low = 3, high = 5, middle = 4$

Since $8 < v[4]$: $low = 3, high = 3, middle = 3$

Since $8 = v[3]$: **return(3)**

Binary Search

A generalization

We can generalize **binary search** to cases where we have something like:

no	no	no	no	no	yes	yes	yes	yes	yes	yes

We want to find the **first yes** (or in some cases the **last no**)

Example:

- Searching for the least number bigger or equal than a certain *key* (**lower_bound** of C++)

2	5	6	8	9	12
no	no	no	yes	yes	yes

lower_bound(7) $\rightarrow$ condition: $v[i] >= 7$

[the smallest number bigger than 7 in this array is 8]

Binary Search

A generalization

Binary for smallest k such that $condition(k)$ is "yes"

```
bsearch(low, high, condition)
While (low < high ) do
    middle = low + (high − low)/2
    If (condition(middle) == yes)) high = middle
    Else                           low = middle + 1
If (condition(low) == no) return(-1)
return(low)
```

v =

2	5	6	8	9	12
no	no	no	yes	yes	yes

bsearch(0, 5, ≥ 7)

$low = 0$, $high = 5$, $middle = 2$

Since $v[2] \geq 7$ is não: $low = 3$, $high = 5$, $middle = 4$

Since $v[4] \geq 7$ is yes: $low = 3$, $high = 4$, $middle = 3$

Since $v[3] \geq 7$ is yes: $low = 3$, $high = 3$ (exits while)

Since $v[3] \geq 7$ is yes: **return(3)**

Balanced partition problem

Input: a sequence $\langle a_1, \ldots, a_n \rangle$ of n positive integers e an integer k
Output: a way of partitioning the sequence into k contiguous subsequences, minimizing the sum of the biggest partition

Example:
7 9 3 8 2 2 9 4 3 4 7 9 9 $\quad k = 4$ (4 partitions)

7 9 3|8 2 2|9 4 3|4 7 9 9 $\rightarrow$ 19 + 12 + 16 + 29
7 9 3 8|2 2 9|4 3 4 7|9 9 $\rightarrow$ 27 + 13 + 18 + 18
7 9|3 8 2 2|9 4 3 4|7 9 9 $\rightarrow$ 16 + 15 + 20 + 25
...

Which one is the best (with the smallest maximum)?

Binary Search

A different example - Balanced Partition

- Exhaustive search would have to test all possible partitions! (can you estimate how many are they?)
- This problem could also be solved with dynamic programming, but that is for another class
- Here we will discuss how to solve it with... **binary search!**

Let's think on a "similar" problem: **It is possible to create a partition where the sum of the largest partition is $\leq X$?**

"Greedy" idea: keep extending the partition while the sum is $< X$!

Examples:

Let $X = 21$ and $k = 4$
7 9 3|8 2 2 9 4 3 4 7 9 9
7 9 3|8 2 2 9|4 3 4 7 9 9
7 9 3|8 2 2 9|4 3 4 7|9 9
7 9 3|8 2 2 9|4 3 4 7|9 9 - OK!

Seja $X = 20$ and $k = 4$
7 9 3|8 2 2 9 4 3 4 7 9 9
7 9 3|8 2 2|9 4 3 4 7 9 9
7 9 3|8 2 2|9 4 3 4|7 9 9
7 9 3|8 2 2|9 4 3 4|7 9|9 - Wrong! We would need more than 4 partitions

It is possible to create a partition where the sum of the largest partition is $\leq X$?

If we think about the X for which the answer is yes, we have a search space where:

no	no	no...	no	no	yes	yes	yes	...	yes	yes

We can apply **binary search on X**!

- Let s be the sum of all numbers
- X will be at least 1 (or in alternative the largest a_i)
- X will be at most s
- Verify answer for a certain X: $\Theta(n)$
- Binary search on X: $\Theta(\log s)$
- Global time: $\Theta(n \log s)$

Example: 7 9 3 8 2 2 9 4 3 4 7 9 9 $\qquad k = 4$ (4 partitions)

low = 1, high = 76, middle = 38 $\rightarrow$ possible(38)? Yes
low = 1, high = 38, middle = 19 $\rightarrow$ possible(19)? No
low = 20, high = 38, middle = 29 $\rightarrow$ possible(29)? Yes
low = 20, high = 29, middle = 24 $\rightarrow$ possible(24)? Yes
low = 20, high = 24, middle = 22 $\rightarrow$ possible(22)? Yes
low = 20, high = 22, middle = 21 $\rightarrow$ possible(21)? Yes
low = 20, high = 21, middle = 20 $\rightarrow$ possible(20)? No
low = 21, high = 21

Exits the cycle and verifies that **possible(21)** is true, and 21 is the answer!

7 9 3|8 2 2 9|4 3 4 7|9 9 $\rightarrow$ 19 + 21 + 18 + 18

2nd Example: 7 9 3 8 2 2 9 4 3 4 7 9 9 $k = 3$ (3 partitions)

low $=$ 1, high $=$ 76, middle $=$ 38 $\rightarrow$ possible(38)?Sim
low $=$ 1, high $=$ 38, middle $=$ 19 $\rightarrow$ possible(19)? Yes
low $=$ 20, high $=$ 38, middle $=$ 29 $\rightarrow$ possible(29)? Yes
low $=$ 20, high $=$ 29, middle $=$ 24 $\rightarrow$ possible(24)? No
low $=$ 25, high $=$ 29, middle $=$ 27 $\rightarrow$ possible(27)? Yes
low $=$ 25, high $=$ 27, middle $=$ 26 $\rightarrow$ possible(26)? No
low $=$ 27, high $=$ 27

Exits the cycle and verifies that **possible(27)** is true, and 27 is the answer!

7 9 3 8|2 2 9 4 3 4|7 9 9 $\rightarrow$ 27 $+$ 24 $+$ 25

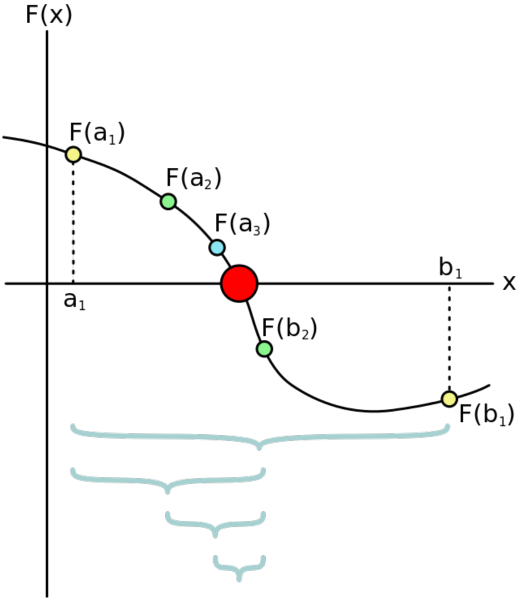
Bisection Method

A similar idea do binary search can be used to find the **root of a function**

- Let $f(n)$ be a **continuous** function defined on an interval $[\mathrm{a}, \mathrm{b}]$ and where $f(a)$ and $f(b)$ have **opposite signs**
- $f(n)$ must have **at least one root** on the interval $[a, b]$

- Starting in $[a, b]$, look at **middle point** c and according to $f(c)$ **reduce the interval** to $[a, c]$ or $[c, b]$

Bisection Method



(image: Wikipedia)

Bisection Method

Example: $f(x) = x^3 - x - 2$

(1) Find *a* and *b* with opposite signals:

$f(1) = 1^3 - 1 - 2 = -2$ $\qquad$ $f(2) = 2^3 - 2 - 2 = 4$

(2) Make successive divisions:

#	a	b	c	f(c)
1	1.0	2.0	1.5	-0.125
2	1.5	2.0	1.75	1.6093750
3	1.5	1.75	1.625	0.6660156
4	1.5	1.625	1.5625	0.2521973
5	1.5	1.5625	1.5312500	0.0591125
6	1.5	1.5312500	1.5156250	-0.0340538
7	1.5156250	1.5312500	1.5234375	0.0122504
8	1.5156250	1.5234375	1.5195313	-0.0109712
9	1.5195313	1.5234375	1.5214844	0.0006222
10	1.5195313	1.5214844	1.5205078	-0.0051789
11	1.5205078	1.5214844	1.5209961	-0.0022794
12	1.5209961	1.5214844	1.5212402	-0.0008289
13	1.5212402	1.5214844	1.5213623	-0.0001034

Método da Bisseção

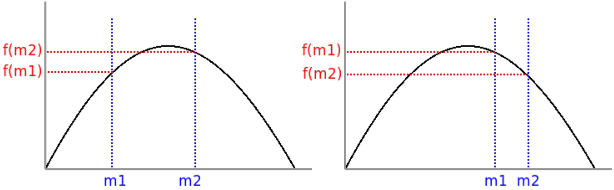
- Stop when you have the **required precision**
  or
- Stop when you reach your **desired number of iterations**

- There are other methods that **converge more rapidly**
  - Newton's method
  - Secant method

Ternary Search

Another similar idea can be used to find the **maximum** (or minimum) of an **unimodal** function (*that is, with a "single peak"*)

- Let $f(n)$ be a **unimodal** function defined on an interval $[\mathrm{a}, \mathrm{b}]$
- Take any two points m_1 and m_2 such that $a < m_1 < m_2 < b$. Then:
  - $f(m_1) < f(m_2)$ then max cannot be in $[a, m_1]$. Continue in $[m_1, b]$
  - $f(m_1) > f(m_2)$ then max cannot be in $[m_2, b]$. Continue in $[a, m_2]$
  - $f(m_1) = f(m_2)$ then max should be in $[m_1, m_2]$.



- We can choose m_1 and $m2$ to be $1/3$ and $2/3$ of $[a, b]$
- With each iteration we will eliminate at least $1/3$ of the search space!
  Runtime: $T(n) = T(2n/3) + \Theta(1) = \Theta(\log n)$

Binary Search

- Binary search is very **useful** and **flexible**
- It can be used on a **vast number of applications**
- There are many other **variations** on it (besides the ones we already described)
  - Interpolated (binary) search
    (instead of going into the middle, estimate position)
  - Exponential (binary) search
    (Start by fixing interval in $low = 2^a$ and $high = 2^{a+1}$)
  - ...