Equilibria on the Day-Ahead Electricity Market

Margarida Carvalho
INESC Porto, Portugal
Faculdade de Ciências, Universidade do Porto, Portugal
margarida.carvalho@dcc.fc.up.pt

João Pedro Pedroso
INESC Porto, Portugal
Faculdade de Ciências, Universidade do Porto, Portugal
jpp@fc.up.pt

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FC FACULDADE DE CIÊNCIAS
UNIVERSIDADE DO PORTO

Departamento de Ciência de Computadores

Faculdade de Ciências da Universidade do Porto
Rua do Campo Alegre, 1021/1055,
4169-007 PORTO,
PORTUGAL

Tel: 220 402 900 Fax: 220 402 950
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Equilibria on the Day-Ahead Electricity Market

Margarida Carvalho[a,b], João Pedro Pedroso[a,b]

[a]*Faculdade de Ciências, Universidade do Porto,*
Rua do Campo Alegre, 4169-007 Porto, Portugal
[b]*INESC TEC, Rua Dr. Roberto Frias 378, 4200-465 Porto, Portugal*

Abstract

In the energy sector, there has been a transition from monopolistic to oligopolistic situations (pool markets); each time more companies' optimization revenues depend on the strategies of their competitors. The market rules vary from country to country. In this work, we model the Iberian Day-Ahead Duopoly Market and find exactly which are the outcomes (Nash equilibria) of this auction using game theory.

Keywords: Duopoly, Nash Equilibria, Day-Ahead Market

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1. Introduction

Over the last years the way electricity is produced and delivered has changed considerably. Market mechanisms were implemented in several countries and electricity markets are no longer vertically integrated. Nowadays, in many countries the electricity markets are based on a pool auction to purchase and sell power. Producing companies offer electricity in a market and the buyers submit acquisition proposals.

An electricity pool market is characterized by a single price (market clearing price) for electricity paid to all the proposals accepted in the market. The way in which the market clearing price is determined varies from one country to another: the last accepted offer, the first rejected offer, multiple unit Vickey (see Anderson and Xu (2004), Son et al. (2004) and Zimmerman et al. (1999)). The Iberian market uses the last accepted offer mechanism, which seems to provide a competitive price and an appropriate incentive for investment and new entry. Indeed, most pool markets use the last accepted block rule (see Son et al. (2004)).

To analyze the companies' behavior in the electricity market, game theory has been used as a generalization of decision theory (see Singh (1999)). The concept of Nash equilibrium (NE) for a game is used as a solution. The NE leads to the strategies that maximize the companies' profit (see for example Hasan and Galiana (2010), Hobbs et al. (2000) and Son

Email addresses: `margarida.carvalho@dcc.fc.up.pt` (Margarida Carvalho), `jpp@fc.up.pt` (João Pedro Pedroso)

and Baldick (2004)). This means that in a NE nobody has advantage in moving unilaterally from it.

As stressed in Baldick (2006), for a model to be tractable it must abstract away from at least some of the detail. Solvable electricity market models do not use, for example, transition constraints. First it is necessary to understand the effect changes have on market rules or structures. In Anderson and Xu (2004), the Australian power market is considered in detail, as we will do here for the Iberian case. There are some crucial differences between the formulation in Anderson and Xu (2004) and ours, such as the pool auction structure and the demand shape. In this context, the authors of Son et al. (2004) analyze the market equilibria with the first rejected mechanism and the pay-as-bid pricing. Here, the amount supplied by each generator is a discrete value; this also differs from our model. In Lee and Baldick (2003) an electricity market with three companies is formulated, and the space of strategies is discretized in order to find a NE.

Recently, in an attempt to predict market prices and market outcomes, more complex models have been used. However, many times that does not allow the use of analytical studies. Thus, techniques from evolutionary programming (see Barforoushi et al. (2010) and Son and Baldick (2004)) and mathematical programming (see Hobbs et al. (2000) and Pereira et al. (2005)) have been used in these new models.

In our Iberian market duopoly model, we do not consider network constraints. We will provide a detailed application of non cooperative game theory in our formulation of the electricity market. To the best of our knowledge, such theoretical treatment has not been considered before. In our formulation, demand elasticity will be a parameter as this is a realistic approach that only has been considered in the simulation of markets, but not in a theoretical way. Apart of being the first detailed approach of the Iberian market, it points out the existence of NE in pure strategies in all the instances of this game, showing how the NE are conditioned by capacity, production costs and elasticity parameters. Some potential properties about the oligopoly case are also highlighted.

In Section 2, the Iberian duopoly market model will be presented and the concept of NE will be formalized. In Section 3, some remarks will be made which allow us to find the NE in a constructive way. Section 4 concludes this detailed NE classification.

2. Iberian Duopoly Market Model

In this section we will begin by explaining our model and fixing notation. Then, game theory will be introduced with the aim of giving us tools to analyze this market. We set up a model for a game with two producing firms, which will represent the players, labeled as Firm 1 and Firm 2. It is assumed that each Firm i owns a generating unit with marginal cost c_i and capacity $E_i > 0$. Both firms submit simultaneously a bid to the market, using a pair $(q_i, p_i) \in S_i = [0, E_i] \times [0, b]$ for Firm i, where q_i is the proposal quantity, p_i is the bid price and S_i is the space of strategies. Firm i's payoff $\Pi_i(q_1, p_1, q_2, p_2)$ is a function that depends on the strategic choices of the rival firm and its own.

The demand is a segment $P = mQ + b$ characterized by the real constants $m < 0$ and $b > 0$. It is assumed that demand, the firms' marginal costs and capacities are known

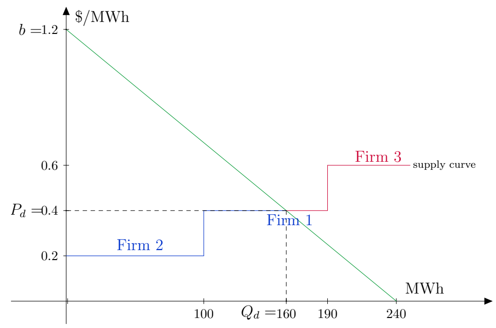

Figure 2.1: Economic Dispatch

by all agents. These assumptions can be justified by the knowledge of information on the technology available for each firm, fuel costs, and precise demand forecasts. Without loss of generality $c_1 < c_2 < b$.

Once, the market operator has the firms' proposals and the demand, it finds the intersection between the demand representation and the supply curve, which gives the market clearing price P_d and quantity Q_d. For example, suppose that $m = \frac{-1}{200}$, $b = \frac{6}{5}$ and Firm 1 bid $(90, 0.4)$, Firm 2 bid $(100, 0.2)$ and Firm 3 bid $(60, 0.6)$. The Market Operator organizes the proposals by ascending order of prices which gives the supply curve, see Figure 2.1. Then, P_d and Q_d are found. The accepted bids are the ones in the left side of Q_d.

Therefore the revenue of Firm i is given by:

$$\Pi_i = (P_d - c_i)\, g_i$$

where $g_i \leq q_i$ is the accepted quantity in the economic dispatch.

Note that with this structure (linear demand and linear production cost) the firms' profit is concave and thefore the optimum strategies are an extreme point or a stationary point.

As a tie breaking rule, we proportionally divide the quantities proposed by the firms declaring the same price, in case the total quantity is not fully required. If the demand segment intersects the supply curve in a discontinuity, all the proposals with prices below the intersection are fully accepted and the market clearing price is given by the last accepted proposal.

Now we are able to define a NE in pure strategies.

Definition 1. *In a game with n players a point $s^{NE} = \left(s_1^{NE}, s_2^{NE}, \ldots, s_n^{NE}\right)$, where s_i^{NE} specifies a strategy over the set of strategies of player i, is a Nash Equilibrium if* $\forall i \in \{1, 2, \ldots, n\}$*:*

$$\Pi_i\left(s^{NE}\right) \geq \Pi_i\left(s_i, s_{-i}^{NE}\right) \quad \forall s_i \in S_i$$

where Π_i is the utility of player i, S_i is his space of strategies and s_{-i}^{NE} is s^{NE} except s_i^{NE}.

Our task is to classify all the NE in this model when players choose their actions deterministically. This is a hard task because the classical approach of reaction functions is inadequate, since the payoff functions are non-continuous.

During this work, besides the NE, it was observed that ϵ-equilibria are also likely to be market outcomes.

Definition 2. *In a game with n players a point $s^{NE_\epsilon} = \left(s_1^{NE_\epsilon}, s_2^{NE_\epsilon}, \ldots, s_n^{NE_\epsilon}\right)$, where $s_i^{NE_\epsilon}$ specifies a strategy over the set of strategies of player i, is an ϵ-equilibrium if $\forall i \in \{1, 2, \ldots, n\}$ and for a real non-negative parameter ϵ:*

$$\Pi_i\left(s^{NE_\epsilon}\right) \geq \Pi_i\left(s_i, s_{-i}^{NE_\epsilon}\right) - \epsilon \qquad \forall s_i \in S_i.$$

In our duopoly case, ϵ is infinitesimal and just one of the firms has an ϵ advantage in changing its strategy. Thus, we also found ϵ-equilibria.

3. Nash Equilibria Classification

The goal of this section is to describe Nash equilibria that may arise in the duopoly case, characterizing Nash equilibria in terms of the parameters: c_1, c_2, E_1, E_2, m and b. For that purpose, we first eliminate some cases where there cannot be NE. Recall the assumption $c_1 < c_2 < b$.

Lemma 3. *In the duopoly market model no Nash equilibrium in pure strategies has a tie in the proposals' prices, which means that $p_1 = p_2$.*

Proof. In the tied case both firms bid $p_1 = p_2 = P_d$ which implies

$$\Pi_1^{tied} = (P_d - c_1)\left(q_1 \frac{Q_d}{q_1 + q_2}\right) \quad \texttt{and} \quad \Pi_2^{tied} = (P_d - c_2)\left(q_2 \frac{Q_d}{q_1 + q_2}\right)$$

with $q_1 + q_2 > Q_d$. (Note that if $q_1 + q_2 = Q_d$, the firms' proposals are totally accept.)

If $P_d < c_1$, then $\Pi_1^{tied} < 0$ an thus, Firm 1 has incentive to change its strategy to $p_1 = c_1$, since its payoff will increase to zero.

If $P_d = c_1 < c_2$, then $\Pi_2^{tied} < 0$, thus Firm 2 has incentive to change its behavior as Firm 1 did in the previous case.

If $P_d > c_1$, Firm 1 has stimulus to choose $p_1^{new} < P_d = p_2$ and

$$q_1^{new} = \begin{cases} E_1 & \text{if } E_1 < \frac{P_d - b}{m} \\ \frac{P_d - b}{m} - \varepsilon & \textit{otherwise} \end{cases}$$

with $\varepsilon > 0$ infinitesimal. The reason is that with this new choice Firm 1 has an accepted quantity $g_1 = q_1^{new} > q_1 \frac{Q_d}{q_1 + q_2} = q_1 \frac{\frac{P_d - b}{m}}{q_1 + q_2}$ and the market clearing price is maintained, up to an infinitesimal quantity. Therefore, revenue of Firm 1 increased.

Since there is always at least one firm that benefits from changing its behavior unilaterally, this cannot be an equilibrium.

Note that this lemma can be easily generalized for the oligopoly case, as long as the marginal costs are different for all firms.

We have just eliminated from the possible set of NE the cases with a tie in prices. Another particular situation occurs when the demand intersects the supply curve in a discontinuity. This possibility can also be discarded from the potential NE.

Lemma 4. *In the duopoly market model, a Nash equilibrium in pure strategies always intersects the supply curve.*

PROOF. Let us prove the lemma by contradiction. In the duopoly market model, suppose that there is an equilibrium such that the demand curve intersects the supply curve in a discontinuity. It suffices to note that the firm with the last proposal being accepted takes advantage increasing the price of this proposal unilaterally up to the intersection point, because the market clearing price increases and the market clearing quantity is maintained. Therefore, this leads to a contradiction, since it was assumed that there was an equilibrium.

Potential equilibria will have: Firm 1 monopolizing the market with $P_d = p_1 < p_2 = c_2$, Firm 1 deciding $P_d = p_1 > p_2$ or Firm 2 deciding $P_d = p_2 > p_1$. Firm 2 never monopolizes the market since it is the less competitive company ($c_2 > c_1$).

The proposition below summarizes the interesting strategies in an equilibrium, that is, the potential equilibria. In the following sections we will evaluate under which conditions they are an equilibrium.

Proposition 5. *In the duopoly market model the equilibria have*

1. *Firm 1 deciding the market clearing price and both firms producing; this means $P_d = p_1 > p_2$, in which case Firm 2 plays the largest possible quantity, as long as p_1 remains greater than c_2, and Firm 1 may play:*
   (a) *the duopoly optimum $\left(q_1 \geq \frac{c_1 - b - q_2 m}{2m}, p_1 = \frac{c_1 + b + q_2 m}{2}\right)$ – stationary point;*
   (b) *the duopoly optimum $(q_1 = E_1, p_1 = (E_1 + q_2)\,m + b)$ – extreme point;*
2. *Firm 1 monopolizing; which means $P_d = p_1 < p_2 = c_2$ and in this case Firm 1 may play:*
   (a) *the monopoly optimum $\left(q_1 \geq \frac{c_1 - b}{2m}, p_1 = \frac{c_1 + b}{2}\right)$ – stationary point;*
   (b) *the monopoly optimum $(q_1 = E_1, p_1 = E_1 m + b)$ – extreme point;*
   (c) *a price close to Firm 2's marginal cost $\left(q_1 \geq \frac{c_2 - \varepsilon - b,}{m}, p_1 = c_2 - \varepsilon\right)$ (or equivalently $\left(q_1 = \frac{c_2 - b}{m} - \varepsilon, p_1 < c_2\right)$) for $\varepsilon > 0$ arbitrary small; ;*
3. *Firm 2 deciding the market clearing price and both firms producing; this means $P_d = p_2 > p_1$, in which case Firm 1 plays the largest possible quantity, as long as $P_d = p_2$, and Firm 2 may play:*
   (a) *the duopoly optimum $\left(q_2 \geq \frac{c_2 - b - q_1 m}{2m}, p_2 = \frac{c_2 + b + q_1 m}{2}\right)$ – stationary point;*
   (b) *the duopoly optimum $(q_2 = E_2, p_2 = (E_2 + q_1)\,m + b)$ – extreme point.*

Proof. Let us start with the simplest case:

(2) Consider an equilibrium with $P_d = p_1 < p_2 = c_2$. The optimal strategy in the monopoly case is:

$$\text{maximize}\, \Pi_1(q_1, p_1) = \left(\frac{p_1 - b}{m}\right)(p_1 - c_1)$$

$$\text{subject to } p_1 \in [0, b]$$

A stationary point is at

$$\frac{\partial \Pi_1}{\partial p_1} = 0 \Leftrightarrow p_1^* = \frac{c_1 + b}{2}$$

with $\frac{\partial^2 \Pi_1}{\partial p_1^2} < 0$, $q_1^* \geq \frac{p_1 - b}{m} = \frac{c_1 - b}{2m} \geq E_1$ and $\frac{c_1 + c_2}{2} < c_2$. When $E_1 < \frac{c_1 - b}{2m}$ the monopoly optimum is an extreme point: $q_1 = E_1$ and $p_1 = E_1 m + b$. Therefore, Firm 1's best strategy is one of the just derived if $c_2 > E_1 m + b > \frac{c_1 + b}{2}$. Otherwise, for Firm 1 to monopolize, it has to bid a price below c_2. In this case, the strategy with the highest payoff is $\left(q_1 \geq \frac{c_2 - \varepsilon - b}{m}, p_1 = c_2 - \varepsilon\right)$, for ε arbitrarily small. To make this bid Firm 1's capacity must satisfy $E_1 \geq \frac{c_2 - \varepsilon - b}{m}$. Equivalently, Firm 1 could bid $\left(q_1 \geq \frac{c_2 - b}{m} - \varepsilon, p_1 < c_2\right)$, allowing Firm 2 to decide $P_d = c_2$, but Firm 2's participation in the market would be as small as ε.

(1) Considerer an equilibrium with $P_d = p_1 > p_2$. In this case, Firm 2 is playing a quantity bid as high as possible, as long as $p_1 > c_2$. Now, we just have to see which is the best strategy for Firm 1 when $P_d = p_1 > p_2$ (which implies $q_2 < \frac{p_1 - b}{m}$):

$$\text{maximize } \Pi_1(q_1, p_1, q_2, p_2) = \left(\frac{p_1 - b}{m} - q_2\right)(p_1 - c_1)$$

$$\text{subject to } p_1 \in [0, b]$$

A stationary point is at

$$\frac{\partial \Pi_1}{\partial p_1} = 0 \Leftrightarrow p_1^* = \frac{c_1 + b + q_2 m}{2}$$

which implies $q_1 \geq \frac{p_1^* - b}{m} - q_2 = \frac{c_1 - b - q_2 m}{2m}$ or if $E_1 < \frac{c_1 - b - q_2 m}{2m} \Rightarrow p_1^* = (E_1 + q_2)\, m + b$ (extreme point). The bid quantity for Firm 2 makes sense since once $P_d = p_1^*$ is fixed Firm 2's profit increases with q_2.

(3) Completely analogous to the above case.

In Figure 3.1, the above proposition is summarized.

Using this proposition, we will compute the conditions in which the above equilibria exist; this means that neither of the firms will have advantage in unilaterally moving from the chosen strategies. Note that $g_1 = \frac{c_1 - b}{2m}$ and $p_1 = \frac{c_1 + b}{2}$ is the monopoly optimum. Therefore, we will use these two values to start our equilibria classification. Figure 3.2 represents the initial division of the space of parameters which will allow us to start a classifying the equilibria that may occur in this market.

Before the computation of equilibria, we prove the following theorem.

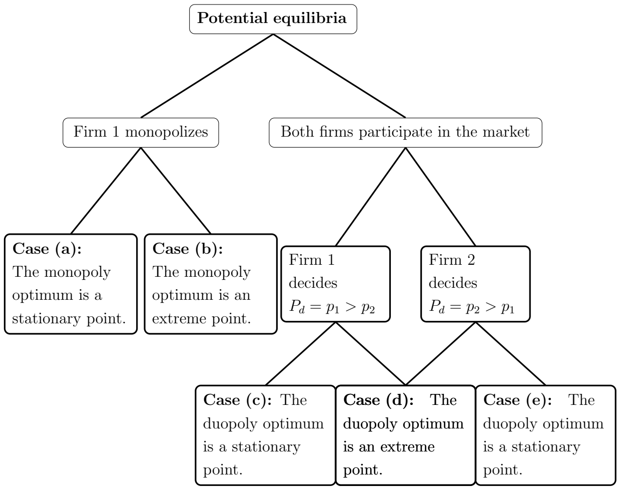

Figure 3.1: Potential equilibria.

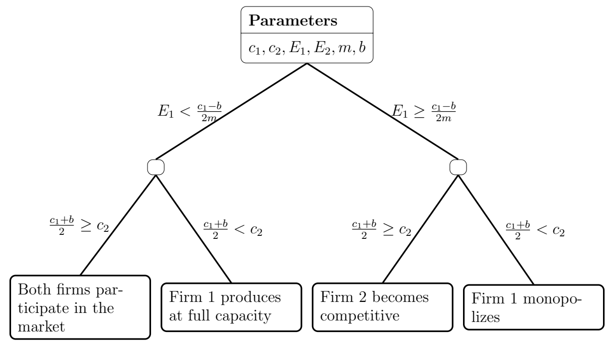

Figure 3.2: Decision tree.

Theorem 6. *There is always an equilibrium in the Iberian duopoly market model. The equilibrium is a Nash equilibrium or an ϵ-equilibrium with ϵ infinitesimal.*

PROOF. Suppose that there is an algorithm A which given the proposals' firms (q_1, p_1, q_2, p_2), is able to output the strictly best reaction for each of them. Therefore, $A_1(q_1, p_1, q_2, p_2) = (q_1', p_1')$ and $A_2(q_1, p_1, q_2, p_2) = (q_2', p_2')$ are the bids that maximize the profit for Firm 1 and Firm 2, respectively, given (q_1, p_1, q_2, p_2).

Our goal is to prove that if we iteratively apply algorithm A for some initial proposals, it will find a fixed point of A. This is $A_1(q_1, p_1, q_2, p_2) = (q_1, p_1)$ and $A_2(q_1, p_1, q_2, p_2) = (q_2, p_2)$, which is an equilibrium of the game.

Let the initial bids be $(q_1, p_1, q_2, p_2) = (E_1, p_1', E_2, c_2)$, where p_1' is the best bid price for Firm 1 such that it is lower than c_2. Remember, as mentioned in Proposition 5, that in an equilibrium the firms bid their entire production capacity.

Applying $A_1(E_1, p_1', E_2, c_2)$, we can obtain:

1. $A_1(E_1, p_1', E_2, c_2) = (E_1, p_1')$, meaning that Firm 1 is already making its best proposal according to Firm 2's strategy. Now, we apply $A_2(E_1, p_1', E_2, c_2)$ to see if Firm 2 has advantage in increasing its price bid.
   (a) If $A_2(E_1, p_1', E_2, c_2) = (E_2, c_2)$, then (E_1, p_1', E_2, c_2) is a fixed point of A and thus it is an equilibrium. Case 2 of Proposition 5 describes this situation.
   (b) If $A_2(E_1, p_1', E_2, c_2) = (E_2, p_2^*)$, where p_2^* is equal to the one described in case 3 of Proposition 5, then (E_1, p_1', E_2, p_2^*) is an equilibrium. Firm 1 does not have advantage in changing from p_1' to the price of case 1 in Proposition 5, since if it has, Firm 1 had done that in the previous step.
2. $A_1(E_1, p_1', E_2, c_2) = (E_1, p_1^*)$, where p_1^* is equal to the one of case 1 in Proposition 5. Firm 1 had advantage in increasing its price bid to p_1^*. Computing $A_2(E_1, p_1^*, E_2, c_2)$ we obtain:

(a) $A_2(E_1, p_1^*, E_2, c_2) = (E_2, c_2)$. Firm 2 keeps its bid which implies that (E_1, p_1^*, E_2, c_2) is an equilibrium.

(b) $A_2(E_1, p_1^*, E_2, c_2) = (E_2, p_2^*)$, where p_2^* is equal to the one of case 3 in Proposition 5. Since A, and in particular A_2, only changes to bids that strictly increase profit, we can conclude for this case that $p_1^* < p_2^*$. By Proposition 5, p_1^*, is the best strategy to Firm 1 when it decides on Pd and Firm 2 is biding a price lower than p_1^*. Therefore, $A_1(E_1, p_1^*, E_2, p_2^*) = (E_1, p_1^*)$ which means that (E_1, p_1^*, E_2, p_2^*) is an equilibrium.

In AppendixI the existence of equilibria is proven, but using a merge of the cases presented in the following sections and algebraic arguments.

3.1. Firm 1 monopolizes: $E_1 \geq \frac{c_1-b}{2m}$ and $\frac{c_1+b}{2} < c_2$

Firm 1 monopolizes the market, which means that its capacity is high enough, and its marginal cost low enough, to keep Firm 2 out of the market. This is the case (a) of Figure 3.1. In this case, the Nash equilibrium is given by:

Strategies 7.

$$\textit{Firm 1: } s_1^{NE} = \left(q_1 \in \left[\frac{c_1 - b}{2m}, E_1\right], \frac{c_1 + b}{2}\right) \tag{3.1}$$

$$\textit{Firm 2: } s_2^{NE} = (q_2 \in [0, E_2], p_2 \in [c_2, b]) \tag{3.2}$$

or

$$\textit{Firm 2: } s_2^{NE} = (0, p_2 \in [0, b]) \tag{3.3}$$

With these bids, the market clears with price $P_d = \frac{c_1+b}{2} < c_2$ and quantity $Q_d = \frac{c_1-b}{2m}$. Firm 1 is at the monopoly's optimum, and Firm 2 has no influence on the market.

3.2. Firm 1 produces at full capacity: $E_1 < \frac{c_1-b}{2m}$ and $\frac{c_1+b}{2} < c_2$

We are going to have cases (b), (e) and (d) of Figure 3.1 as equilibria.

Let us start by considering that $E_1 m + b < c_2$, meaning that, Firm 2 cannot enter the market with positive profit. Note that $\frac{c_2-b}{m} < E_1 < \frac{c_1-b}{2m}$. In this case, Firm 1 will monopolize the electricity market, although its capacity is lower than the optimum $\frac{c_1-b}{2m}$. The Nash equilibrium is given by:

Strategies 8.

$$\textit{Firm 1: } s_1^{NE} = (E_1, E_1 m + b) \tag{3.4}$$

Firm 2: see Equations 3.2 or 3.3

As before, Firm 1 does not have an incentive to change its strategy, as this will mean that $(q_1, p_1) = (E_1, E_1 m + b)$ is not an optimum.

Let us now consider $E_1 m + b \geq c_2$. Firm 1 cannot monopolize the market and, in this case, the marginal cost of Firm 2 is lower than the monopoly price $E_1 m + b$, i.e., $E_1 m + b \geq c_2$. Moreover, in an equilibrium for this case, Firm 1 never decides the price, that is., $P_d = p_2 > p_1$. Otherwise, by Proposition 5, if $P_d = p_1 > p_2$, Firm 1 would be bidding the duopoly optimum price $P_d = p_1^* = \frac{c_1+b+q_2 m}{2}$ requiring:

$$p_1^* = \frac{c_1 + b + q_2 m}{2} > c_2$$

but this would imply a negative quantity for Firm 2, which is absurd:

$$q_2 < \frac{2c_2 - c_1 - b}{m} < 0.$$

The inequality $\frac{2c_2-c_1-b}{m} < 0$ is equivalent to $c_2 > \frac{c_1+b}{2}$, which holds by assumption.

Therefore, we discarded the possibility of Firm 1 deciding $P_d = p_1 > p_2$ (case (c) of Figure 3.1). The cases (d) and (e) in Figure 3.1 remain as potential NE, which we will discuss below.

Suppose that Firm 1's bidding price is $p_1 < c_2 \leq p_2 = P_d$. The best reaction for Firm 2 is the duopoly optimum in Proposition 5: $p_2^* = \frac{c_2+b+q_1 m}{2}$ and the corresponding quantity is $q_2^* \geq \frac{c_2-b-q_1 m}{2m}$. Indeed $p_2^* \geq c_2$:

$$p_2^* = \frac{c_2 + b + q_1 m}{2} \geq \frac{c_2 + b + E_1 m}{2} \geq c_2 \Leftrightarrow E_1 m + b \geq c_2$$

therefore, the bid price p_2^* makes sense, since $p_2^* > c_2$. Furthermore, Firm 1 bids $q_1^* = E_1$ (and $p_1^* < c_2$), otherwise it would have advantage in increasing its quantity and this would not be an equilibrium.

Firm 2 does not have incentive to change its move $\left(q_2^* \geq \frac{c_2-b-E_1 m}{2m}, p_2^* = \frac{c_2+b+E_1 m}{2}\right)$, as this is the optimum (stationary point) when $p_1 < c_2$. In order to have an equilibrium, neither of the firms can benefit from unilaterally changing their behavior.

When Firm 2 decides $P_d = p_2 > p_1$, this firm will not benefit from changing its behavior, because that would mean decreasing the price p_2, $p_1 > p_2$, and this does not increase Firm 2's profit, since p_1 is lower than c_2.

May Firm 1 be encouraged to reconsider its strategy? In other words, would Firm 1 be interested in increasing price p_1? The answer is no, because as we have seen at the beginning: if Firm 1 picks the price $P_d = p_1 > p_2$ it will be $p_1^* = \frac{c_1+b+q_2^* m}{2}$ but $p_1^* < c_2 \leq p_2^*$. Therefore, now we just have to distinguish the duopoly optimum as a stationary point and the duopoly optimum as an extreme point:

- In the case $E_2 \geq \frac{c_2-b-E_1 m}{2m}$, Firm 2's optimum is a stationary point. As we assume $E_1 m + b > c_2$, Firm 1 is not monopolizing , and the Nash Equilibrium is given by

Strategies 9.

$$\textit{Firm 1: } s_1^{NE} = (E_1, p_1 \in [0, c_2[) \tag{3.5}$$

$$\textit{Firm 2: } s_2^{NE} = \left(q_2 \in \left[\frac{c_2 - b - E_1 m}{2m}, E_2\right], \frac{c_2 + b + E_1 m}{2}\right) \tag{3.6}$$

For $E_1 m + b = c_2$ and $\varepsilon, \varepsilon' > 0$ arbitrary small, the equilibrium is given by

Strategies 10.

$$\textit{Firm 1: } s_1^{NE} = (E_1 - \varepsilon, p_1 \in [0, c_2[) \tag{3.7}$$

$$\textit{Firm 2: } s_2^{NE} = (q_2 \in [\varepsilon', E_2], c_2) \tag{3.8}$$

- In the case $E_2 < \frac{c_2 - b - E_1 m}{2m}$ both firms bid at full capacity. Here, Firm 2 does not have enough capacity to produce the quantity $\frac{c_2 - b - E_1 m}{2m}$ required, so $q_2^* = E_2$ and $p_2^* = (q_1 + E_2)\, m + b$. The best reaction quantity for Firm 1 is $q_1^* = E_1$ as before. Firm 2 is playing $(q_2 = E_2, p_2 = (E_1 + E_2)\, m + b)$ which, by construction, is the best reaction when Firm 1 plays $(q_1, p_1) = (E_1, p_1 < c_2)$. These strategies are a Nash equilibrium.

Strategies 11.

$$\textit{Firm 1: } s_1^{NE} = (E_1, p_1 \in [0, (E_1 + E_2)\, m + b]) \tag{3.9}$$

$$\textit{Firm 2: } s_2^{NE} = (E_2, (E_1 + E_2)\, m + b) \tag{3.10}$$

Clearly, we can invert the prices of each firm, reaching the Nash equilibrium:

Strategies 12.

$$\textit{Firm 1: } s_1^{NE} = (E_1, (E_1 + E_2)\, m + b) \tag{3.11}$$

$$\textit{Firm 2: } s_2^{NE} = (E_2, p_2 \in [0, (E_1 + E_2)\, m + b]) \tag{3.12}$$

The conclusions of this section are summarized in the decision tree of Figure 3.3.

3.3. Firm 2 becomes competitive: $E_1 \geq \frac{c_1 - b}{2m}$ and $\frac{c_1 + b}{2} \geq c_2$

In this case, we will have cases (c), (b) and (e) of Figure 3.1 as equilibria.

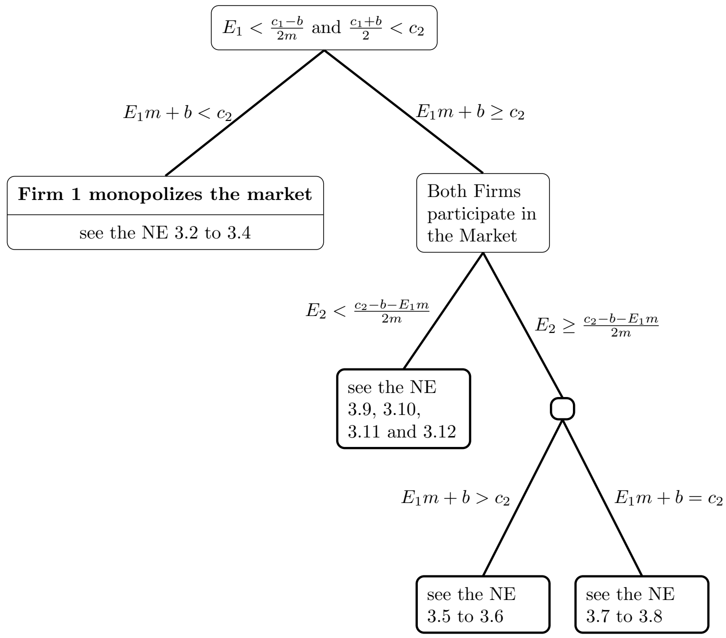

Figure 3.3: Equilibria for Section 3.2.

3.3.1. Firm 1 decides $P_d = p_1 > p_2$

We are interested in finding under which conditions case (c) in Figure 3.1 is a NE. Clearly $p_1 > c_2$ and therefore, we assume $p_2^* = c_2$. Under the results of Proposition 5, Firm 1 must be bidding the stationary point $p_1^* = \frac{c_1+b+q_2 m}{2}$ and $q_1^* \geq \frac{c_1-b-q_2 m}{2m}$. This requires $p_1^* = \frac{c_1+b+q_2 m}{2} \geq c_2$ and thus, we must have $q_2 \leq \frac{2c_2-c_1-b}{m}$. Similarly, the quantity produced by Firm 1 ($\frac{c_1-b-q_2 m}{2m}$) must be positive. This leads to the inequality $q_2 \leq \frac{c_1-b}{m}$, which is weaker than the former one, since $b > c_2 > c_1$. The best strategy for Firm 2 is:

$$q_2^* = \begin{cases} \frac{2c_2-c_1-b}{m} - \varepsilon & \text{if } E_2 \geq \frac{2c_2-c_1-b}{m} \\ E_2 & \textit{otherwise.} \end{cases} \tag{3.13}$$

Note that the quantity q_2^* mentioned above is positive as long as $c_2 \leq \frac{c_1+b}{2}$, which holds. Therefore, Firm 2 is playing the largest possible quantity, as stated in Proposition 5.

For the sake of simplicity, let us first consider the case of Firm 1 playing $q_1^* = \frac{c_1-b-q_2^* m}{2m}$, rather than $q_1^* > \frac{c_1-b-q_2^* m}{2m}$. Hence, $q_1^* \leq \frac{c_1-b}{2m}$ which by assumption is lower than E_1 and therefore, it makes sense to bid this quantity at price p_1^*.

For the profile of strategies $s_1 = (q_1^*, p_1^*)$ and $s_2 = (q_2^*, c_2)$ to be an equilibrium, it is required that neither firm changes its strategy.

If Firm 2 has an incentive to choose another strategy it would be $\tilde{p}_2 > p_1^*$. The question now is whether there is a strategy $\tilde{s}_2 = (\tilde{q}_2, \tilde{p}_2)$ such that $\Pi_2(\tilde{q}_2, \tilde{p}_2, q_1^*, p_1^*) > \Pi_2(q_2^*, p_2^*, q_1^*, p_1^*)$. When Firm 2 increases the price, the best strategy is to choose $\tilde{p}_2 =$

$\frac{c_2+b+q_1^* m}{2} = \frac{b+c_1+2c_2-q_2^* m}{4}$. In order to have $\tilde{p_2} > p_1^*$, the following inequality must hold

$$q_2^* > \frac{2c_2 - c_1 - b}{3m}. \tag{3.14}$$

Inequality 3.14 depends on our instance of the problem. Notice that $\frac{2c_2-c_1-b}{m} > \frac{2c_2-c_1-b}{3m}$, so Inequality 3.14 holds whenever $E_2 > \frac{2c_2-c_1-b}{3m}$. Our goal is to see under which conditions Firm 2 does not change the price to $\tilde{p_2}$.

In this context, if $E_2 \leq \frac{2c_2-c_1-b}{3m}$ then $q_2^* = E_2$ and Firm 2 does not have advantage in picking a strategy different from (E_2, c_2).

Suppose $E_2 > \frac{2c_2-c_1-b}{3m}$; then $\tilde{p_2} = \frac{b+c_1+2c_2-q_2^* m}{4} > p_1^* = \frac{c_1+b+q_2^* m}{2}$. Can this be the case that

$$\Pi_2\left(q_2^*, c_2, q_1^*, p_1^*\right) = \left(\frac{c_1 + b + q_2^* m}{2} - c_2\right) q_2^*$$

is larger than

$$\Pi_2\left(\frac{\tilde{p_2} - b}{m} - q_1^*, \tilde{p_2}, q_1^*, p_1^*\right) =$$

$$= \left(\frac{b + c_1 + 2c_2 - q_2^* m}{4} - c_2\right)\left(\frac{b + c_1 + 2c_2 - q_2^* m - 4b}{4m} - \frac{c_1 - b - q_2^* m}{2m}\right)?$$

The answer is no. This proof falls naturally, see in AppendixA. Consequently, we are only interested in the case where $q_2^* = E_2 \leq \frac{2c_2-c_1-b}{3m}$.

Now, we have to study under which conditions Firm 1 does not have advantage in picking $\tilde{p_1} < p_2 = c_2$.

If $E_1 < \frac{c_2-b}{m}$ then Firm 1 bids $\tilde{p_1} < c_2$ and $\tilde{q_1} = E_1$. Otherwise, if $E_1 \geq \frac{c_2-b}{m}$, Firm 1 bids $\tilde{p_1} < c_2$ and $\tilde{q_1} = \frac{c_2-b}{m} - \varepsilon$ or $\tilde{p_1} = c_2 - \varepsilon$ and $\tilde{q_1} \geq \frac{c_2-\varepsilon-b}{m}$. Obviously in the second case ($E_1 \geq \frac{c_2-b}{m}$) the Firm 1's profit is higher, so we use it to compare with its profit of our possible NE:

$$\Pi_1\left(q_1^*, p_1^*, E_2, c_2\right) = \left(\frac{c_1 + b + E_2 m}{2} - c_1\right)\left(\frac{c_1 - b - E_2 m}{2m}\right) = \frac{-\left(b + E_2 m - c_1\right)^2}{4m}.$$

So

$$\Pi_1\left(q_1^*, p_1^*, E_2, c_2\right) \geq \lim_{\varepsilon \to 0} \Pi_1\left(\tilde{q_1}, \tilde{p_1}, E_2, c_2\right)$$

when

$$E_2 \in \left[0, \min\left(\frac{c_1 - b + 2\sqrt{(c_1 - c_2)(c_2 - b)}}{m}, \frac{2c_2 - c_1 - b}{3m}\right)\right] \tag{3.15}$$

see AppendixB.

Thus, when Equation 3.15 holds and $E_1 \geq \frac{c_2-b}{m}$ we have the equilibrium:

Strategies 13.

$$\textit{Firm 1: } s_1^{NE} = \left(q_1 \in \left[\frac{c_1 - b - E_2 m}{2m}, E_1\right], \frac{c_1 + b + E_2 m}{2}\right) \tag{3.16}$$

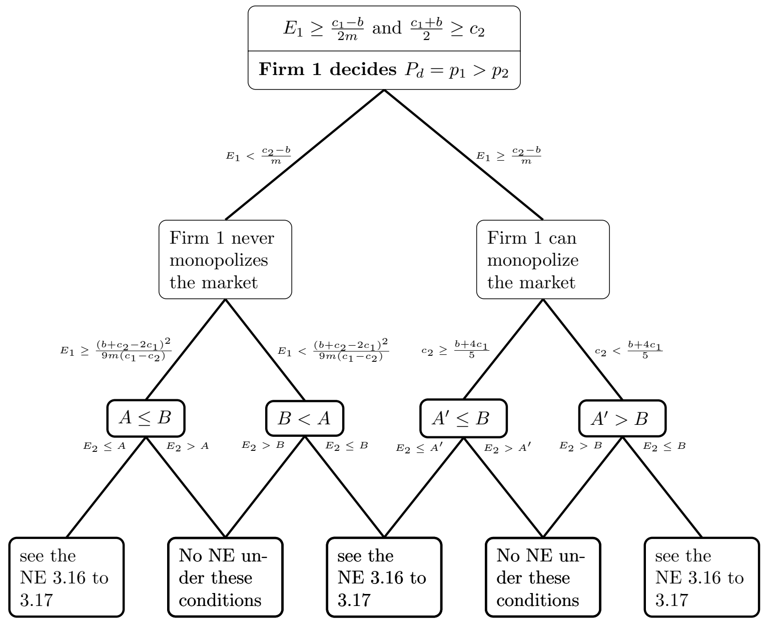

Figure 3.4: Firm 1 decides $P_d = p_1 > p_2$.

$$\textit{Firm 2: } s_2^{NE} = (E_2, p_2 \in [0, c_2]) \tag{3.17}$$

Note that we have relaxed Firm 2's biding price. This is possible because the profits do not depend on it.

When $E_1 < \frac{c_2-b}{m}$ and Equation 3.18 holds and we also have the above equilibrium (see Equations 3.16 and 3.17).

$$E_2 \in \left[0, \min\left(\frac{c_1 - b + 2\sqrt{E_1 m (c_1 - c_2)}}{m}, \frac{2c_2 - c_1 - b}{3m}\right)\right] \tag{3.18}$$

Let $A = \frac{c_1-b+2\sqrt{E_1 m(c_1-c_2)}}{m}$, $A' = \frac{c_1-b+2\sqrt{(c_1-c_2)(c_2-b)}}{m}$ and $B = \frac{2c_2-c_1-b}{3m}$. We built the decision tree in Figure 3.4. In order to verify that all the decisions in the tree make sense, that is, that all the regions in its leafs are non-empty, let us observe the following:

- $\frac{c_1-b}{2m} \leq \frac{(b+c_2-2c_1)^2}{9m(c_1-c_2)}$ when $c_2 \leq \frac{c_1+b}{2}$;
- $\frac{(b+c_2-2c_1)^2}{9m(c_1-c_2)} < \frac{c_2-b}{m}$ when $c_2 \in \left]\frac{4c_1+b}{5}, \frac{c_1+b}{2}\right[$.

This supports the fact that the decision tree makes sense or, in other words, that decisions do not lead us to empty spaces.

Remember, that we used $q_1^* = \frac{c_1-b-q_2^* m}{2m}$. In this case, the capacity of Firm 2 had to be lower than $\frac{2c_2-c_1-b}{3m}$, otherwise, Firm 2 would change its strategy with benefit.

It could be possible to find more general conditions for the Nash equilibrium with Firm 1 deciding $P_d = p_1 > p_2$.

We did not try the strategy $q_1^* > \frac{c_1-b-q_2^*m}{2m}$. Note that with bids $(s_1, s_2) = (q_1^*, p_1^*, q_2^*, p_2) = \left(q_1^* > \frac{c_1-b-q_2^*m}{2m}, \frac{c_1+b+q_2^*m}{2}, q_2^*, c_2\right)$, Firm 1 only produces the quantity $g_1 = \frac{c_1-b-q_2^*m}{2m}$. However Firm 1 may play a quantity q_1 larger than g_1, in order to reduce Firm 2's incentive in changing the price p_2. So, our goal is to find a lower bound for q_1^* when $E_2 \geq \frac{2c_2-c_1-b}{3m}$.

Let us see how larger q_1^* has to be. If Firm 2 increases the price c_2 to $\tilde{p}_2 = \frac{c_2+b+q_1m}{2}$ this implies:

$$\frac{c_2+b+q_1^*m}{2} > p_1^* = \frac{c_1+b+q_2^*m}{2} \Leftrightarrow q_1^* < \frac{c_1-c_2}{m} + q_2^* < \frac{c_2-b}{m}$$

and the new quantity dispatched should be positive:

$$\frac{\frac{c_2+b+q_1^*m}{2} - b}{m} - q_1^* > 0 \Leftrightarrow q_1^* < \frac{c_2-b}{m}$$

therefore $q_1^* < \frac{c_1-c_2}{m} + q_2^*$ is the strongest condition until now.

We still have to add the condition that leads Firm 2 to a higher profit

$$\Pi_2\left(q_1^*, p_1^*, \frac{\tilde{p}_2-b}{m} - q_1^*, \tilde{p}_2\right) = \left(\frac{c_2+b+q_1^*m}{2} - c_2\right)\left(\frac{c_2-b-q_1^*m}{2m}\right)$$

is larger than

$$\Pi_2\left(q_1^*, p_1^*, q_2^*, c_2\right) = \frac{q_2^*}{2}\left(c_1+b+q_2^*m-2c_2\right)$$

when $q_1^* < \frac{c_2-b+\sqrt{K_2}}{m}$, see AppendixC. If $q_1^* \geq \frac{c_2-b+\sqrt{K_2}}{m}$ Firm 2 does not have advantage in changing its strategy.

Now, we merely need the conditions for Firm 1 to keep the strategy (q_1^*, p_1^*). Proceeding as before:

1. Let $E_1 \geq \frac{c_2-b}{m}$. Firm 1 does not change its strategy $(\left(q_1^* > \frac{c_2-b+\sqrt{K_2}}{m}, p_1^*\right))$ if

$$q_2^* \in \left[0, \frac{c_1-b+2\sqrt{(c_2-b)(c_1-c_2)}}{m}\right] \cup \left[\frac{c_1-b-2\sqrt{(c_2-b)(c_1-c_2)}}{m}, \infty\right[.$$

   Note that

$$\frac{c_1-b-2\sqrt{(c_2-b)(c_1-c_2)}}{m} > \frac{2c_2-c_1-b}{m}$$
$$\Leftrightarrow \underbrace{-2\sqrt{(c_2-b)(c_1-c_2)}}_{<0} < \underbrace{2(c_2-c_1)}_{>0}$$

   and

$$\frac{c_1-b+2\sqrt{(c_2-b)(c_1-c_2)}}{m} < \frac{2c_2-c_1-b}{m}$$

$$\Leftrightarrow -2c_2^2 + c_2(3c_1 + b) - c_1 b - c_1^2 > 0$$

$$\Leftrightarrow c_2 \in \left]c_1, \frac{c_1 + b}{2}\right[$$

thus

$$q_2^* = E_2 \leq \frac{c_1 - b + 2\sqrt{(c_2 - b)(c_1 - c_2)}}{m}.$$

The Nash equilibrium is given by:

Strategies 14.

$$\textit{Firm 1: } s_1^{NE} = \left(q_1, \frac{c_1 + b + E_2 m}{2}\right) \textit{ with } q_1 \in \left[\frac{c_2 - b + \sqrt{K_2}}{m}, E_1\right] \tag{3.19}$$

Firm 2: see Equation 3.17

2. Let $E_1 < \frac{c_2 - b}{m}$. Firm 1 does not change its strategy if

$$q_2^* \in \left[0, \frac{c_1 - b + 2\sqrt{E_1 m (c_1 - c_2)}}{m}\right] \cup \left[\frac{c_1 - b - 2\sqrt{E_1 m (c_1 - c_2)}}{m}, \infty\right[.$$

   Note that:

$$\frac{c_1 - b - 2\sqrt{E_1 m (c_1 - c_2)}}{m} > \frac{2c_2 - c_1 - b}{m}$$

   and

$$\frac{c_1 - b + 2\sqrt{E_1 m (c_1 - c_2)}}{m} < \frac{2c_2 - c_1 - b}{m}$$

$$\Leftrightarrow c_2 \in [c_1, c_1 - E_1 m]$$

   and

$$c_1 - E_1 m > c_1 - \frac{c_1 - b}{2m} m = \frac{c_1 + b}{2} \Rightarrow \left[c_1, \frac{c_1 + b}{2}\right] \subset [c_1, c_1 - E_1 m]$$

   thus for

$$q_2^* = E_2 < \frac{c_1 - b + 2\sqrt{E_1 m (c_1 - c_2)}}{m}$$

   we have the Nash equilibrium of 3.19 and 3.17.

The decision tree is in Figure 3.5.

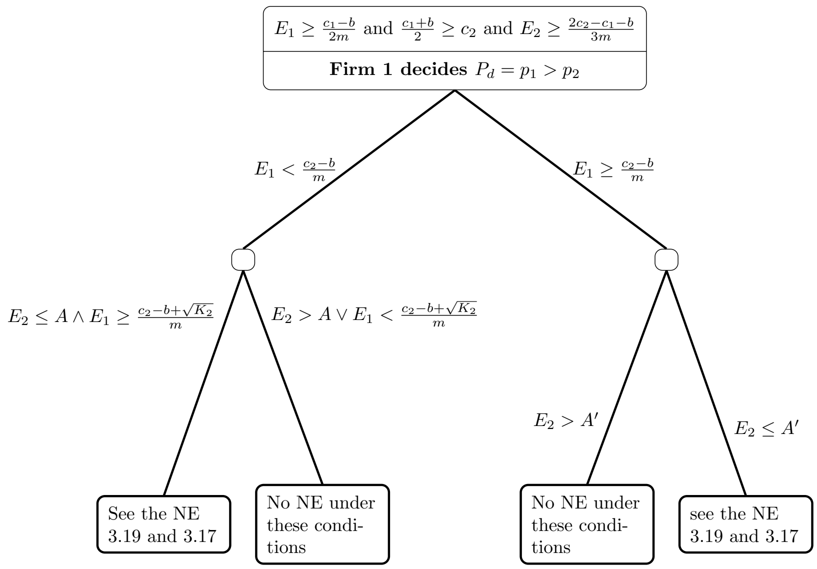

Figure 3.5: Firm 1 decides $P_d = p_1 > p_2$.

3.3.2. Firm 1 monopolizes

Now we are going to establish the conditions that make Firm 1 monopolize the market as an equilibrium (case (b) in Figure 3.1). Here, the market outcome is an ϵ-equilibrium.

Notice that in this case, $p_1 < c_2$ (otherwise, Firm 2 would have advantage in participating in the market) and it is required $E_1 \geq \frac{c_2-b}{m}$ ($\frac{c_1-b}{2m} < \frac{c_1-b}{m}$), otherwise the demand is not intersected by Firm 1's bid. Hence the best bid for Firm 1 is $(q_1, p_1) = \left(\frac{c_2-\varepsilon-b}{m}, c_2 - \varepsilon\right)$ with $\varepsilon > 0$. Firm 2 bids $(q_2, p_2) = (E_2, c_2)$. Any other strategy from Firm 2 leads to the same or less profit.

1. Consider $E_2 \geq \frac{c_2-b}{m}$. If Firm 1 chooses $\tilde{p_1} > c_2$, the produced quantity is zero, and therefore, Firm 1 will not choose to make a bid different from $p_1 = c_2 - \varepsilon$. Hence the equilibrium is:

   Strategies 15.

$$\text{Firm 1: } s_1^{NE} = \left(q_1 \in \left[\frac{c_2 - \varepsilon - b}{m}, E_1\right], c_2 - \varepsilon\right) \tag{3.20}$$

$$\text{Firm 2: } s_2^{NE} = (E_2, c_2) \tag{3.21}$$

   or

   Strategies 16.

$$\text{Firm 1: } s_1^{NE} = \left(\frac{c_2 - b}{m} - \varepsilon, p_1 \in [c_1, c_2 - \varepsilon]\right) \tag{3.22}$$

Firm 2: see Equation 3.21

2. Consider $E_2 < \frac{c_2-b}{m}$. Then Firm 1 can bid $\tilde{p_1} > c_2$, producing a non zero quantity. If Firm 1 has incentive to change its strategy it will be to:

$$(\tilde{q_1}, \tilde{p_1}) = \left(\frac{c_1 - b - E_2 m}{2m}, \frac{c_1 + b + E_2 m}{2}\right).$$

However, for this new bid to make sense, we need

$$\tilde{p_1} = \frac{c_1 + b + E_2 m}{2} > c_2,$$

which depends on the instance of our problem.
Remark: $\frac{2c_2-c_1-b}{m} < \frac{c_2-b}{m} \Leftrightarrow c_2 > c_1$. Using this the following cases are possible.

(a) Consider $E_2 < \frac{2c_2-c_1-b}{m} \Leftrightarrow p_1 = \frac{c_1+b+E_2 m}{2} > c_2$. Is

$$\Pi_1\left(\frac{\tilde{p_1}-b}{m} - E_2, \tilde{p_1}, E_2, c_2\right) = \left(\frac{c_1+b+E_2 m}{2} - c_1\right)\left(\frac{c_1 - b - E_2 m}{2m}\right)$$

higher than

$$\lim_{\varepsilon\to 0} \Pi_1\left(\frac{c_2-\varepsilon-b}{m}, c_2-\varepsilon, E_2, c_2\right) = (c_2 - c_1)\left(\frac{c_2-b}{m}\right)?$$

Not when Equation 3.23 holds (see AppendixD for a proof).

$$E_2 \in \left[\frac{c_1 - b + 2\sqrt{(c_1-c_2)(c_2-b)}}{m}, \frac{2c_2-c_1-b}{m}\right]. \tag{3.23}$$

Hence, the equilibrium is given by:
Strategies 17.

Firm 1: see Equation 3.20

Firm 2: see Equation 3.21

or
Strategies 18.

Firm 1: see Equation 3.22

Firm 2: see Equation 3.21

(b) Consider $E_2 \geq \frac{2c_2-c_1-b}{m} \Leftrightarrow p_1 = \frac{c_1+b+E_2 m}{2} \leq c_2$. In this case, Firm 1 does not have stimulus to change its behavior and hence, the equilibrium is given by Equations 3.20 to 3.22.

Let $B' = \frac{2c_2-c_1-b}{m}$. We have the decision tree of Figure 3.6 corresponding to this case.

Since , with the assumptions of this section, $\frac{c_2-b}{m} \geq \frac{2c_2-c_1-b}{m} \geq A'$ yields, the decision tree makes sense.

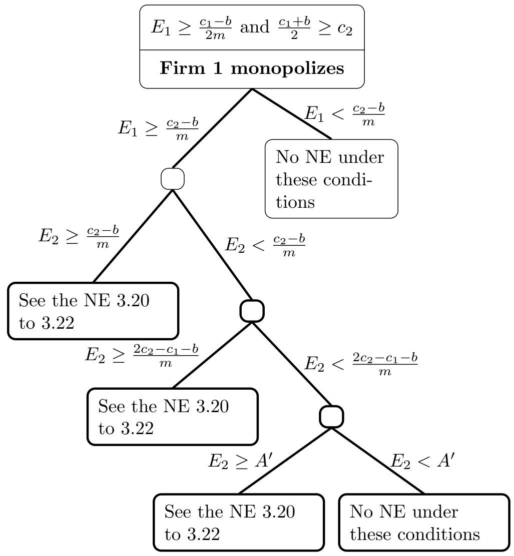

Figure 3.6: Firm 1 monopolizes.

3.3.3. Firm 2 decides $P_d = p_2 > p_1$

Finally, we will search for equilibria where Firm 2 decides the market clearing price, in other words, $p_1 < p_2 = P_d$ (case (d) and (e) of Figure 3.1).

Note that we have to impose $E_1 < \frac{c_2-b}{m}$, due to the fact that if $E_1 \geq \frac{c_2-b}{m}$, Firm 1 has incentive to monopolize the market (since it has sufficient capacity for that purpose). So, we consider $\frac{c_1-b}{2m} \leq E_1 < \frac{c_2-b}{m}$.

Let us assume $p_1 < c_2$. From Proposition 5, the best strategy for Firm 2 is $p_2^* = \frac{c_2+b+q_1^*m}{2}$ and $q_2^* \geq \frac{c_2-b-q_1^*m}{2m}$, where $q_1^* = E_1$.

For this strategy there are the following requirements (the price of the duopoly optimum makes sense and Firm 2 has the production capacity of playing the stationary point of the duopoly optimum):

1. $p_2^* = \frac{c_2+b+E_1m}{2} \geq c_2 \Leftrightarrow E_1 \leq \frac{c_2-b}{m}$, which holds by our assumption;
2. $E_2 \geq \frac{c_2-b-E_1m}{2m}$ which depends on the instance of our problem.

We have to study each of these cases.

1. Suppose $E_2 \geq \frac{c_2-b-E_1m}{2m}$, then Firm 2 plays the stationary optimum

$$\left(q_2^* \geq \frac{c_2-b-E_1m}{2m}, p_2^* = \frac{c_2+b+E_1m}{2}\right)$$

   and Firm 1 $(E_1, p_1 < c_2)$.
   Obviously Firm 2 will not have incentive in reconsidering another proposal, but Firm 1 may have advantage in choosing a higher price $\tilde{p_1} = \frac{c_1+b+q_2^*m}{2} > p_2^*$, such that $\Pi_1\left(q_1^*, p_1 < c_2, q_2^*, p_2^*\right) < \Pi_1\left(\frac{\tilde{p_1}-b}{m} - q_2^*, \tilde{p_1} > p_2^*, q_2^*, p_2^*\right)$. Is $\tilde{p_1} = \frac{c_1+b+q_2^*m}{2} > p_2^* = \frac{c_2+b+E_1m}{2}$?

$$\frac{c_1+b+q_2^*m}{2} > \frac{c_2+b+E_1m}{2} \Leftrightarrow q_2^* < \frac{c_2-c_1+E_1m}{m}$$

which depends on the instance of our problem. Thus, these cases have to be considered separately:

(a) Suppose $\frac{c_2-b-E_1m}{2m} \geq \frac{c_2-c_1+E_1m}{m} \Leftrightarrow E_1 \leq \frac{2c_1-c_2-b}{3m}$. However, $E_1 \geq \frac{c_1-b}{2m} \geq \frac{2c_1-c_2-b}{3m} \Leftrightarrow c_2 \geq \frac{c_1+b}{2}$, which by assumption does not hold. So this case never happens.

(b) Suppose $E_1 > \frac{2c_1-c_2-b}{3m}$ and $q_2^* = \frac{c_2-b-E_1m}{2m}$, then Firm 1 has benefit in increasing its proposal price since:

$$\Pi_1\left(\frac{p_1-b}{m}-q_2^*, \frac{c_1+b+q_2^*m}{2}, q_2^*, p_2^*\right) > \Pi_1\left(E_1, p_1 < c_2, q_2^*, p_1^*\right)$$
$$\Leftrightarrow \frac{-1}{16m}\left(2c_1-c_2-b+E_1m\right)^2 > \left(c_2+b+E_1m-2c_1\right)\frac{E_2}{2}$$
$$\Leftrightarrow E_1 \neq \frac{2c_1-b-c_2}{3m}.$$

(c) Suppose $E_1 > \frac{2c_1-c_2-b}{3m}$ and $q_2^* > \frac{c_2-b-E_1m}{2m}$, then as we already did before, there is an equilibrium if $E_2 \geq \frac{c_1-b+\sqrt{K_1}}{m}$, where $K_1 = -2m^2E_1^2 + \left(-2mc_2-2bm+4mc_1\right)E_1$: **Strategies 19.**

Firm 1: see Equation 3.5

$$\textit{Firm 2: } s_2^{NE} = \left(q_2 \in \left[\frac{c_1-b+\sqrt{K_1}}{m}, E_2\right], \frac{c_2+b+E_1m}{2}\right) \tag{3.24}$$

2. Suppose $E_2 < \frac{c_2-b-E_1m}{2m}$. In this case $(q_1^*, p_1^*) = (E_1, p_1 < c_2)$ and $(q_2^*, p_2^*) = (E_2, (E_1+E_2)m+b)$.

Firm 2 will not change this behavior, so let us see when Firm 1 has advantage in increasing p_1^* to $\tilde{p_1}$. For the purpose we need

$$\tilde{p_1} = \frac{c_1+b+E_2m}{2} > p_2^* = (E_1+E_2)m+b \Leftrightarrow E_2 > \frac{c_1-b}{m} - 2E_1$$

and $E_2 > \frac{c_1-b}{m} - 2E_1$ is true, since $E_2 > 0$ and

$$\frac{c_1-b}{m} - 2E_1 \leq 0 \Leftrightarrow E_1 \geq \frac{c_1-b}{2m}.$$

So, $\tilde{p_1} = \frac{c_1+b+E_2m}{2} > p_2^* = (E_1+E_2)m+b$.
Is

$$\Pi_1\left(\frac{\tilde{p_1}-b}{m}-E_2, \frac{c_1+b+E_2m}{2}, E_2, p_2^*\right) = \frac{-1}{4m}\left(-c_1+b+E_2m\right)^2$$

higher than

$$\Pi_1\left(E_1, p_1 < c_2, E_2, p_2^*\right) = E_1\left((E_1+E_2)m+b-c_1\right)?$$

Yes, see AppendixE. Therefore Firm 1 has stimulus in changing its strategy.

In short, we can summarize this Nash equilibria with the decision tree of Figure 3.7.

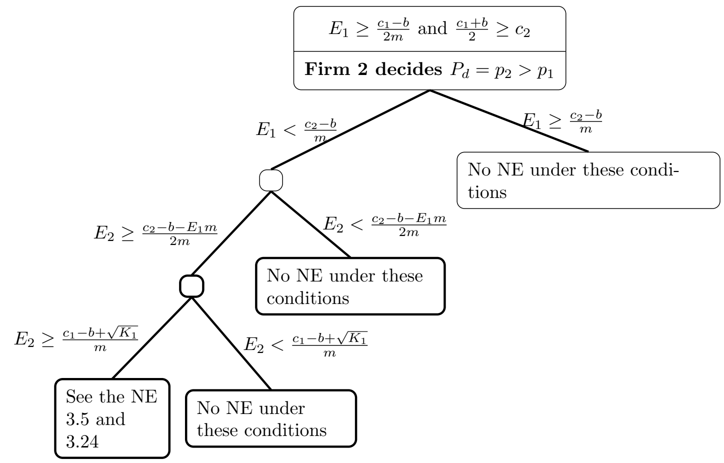

Figure 3.7: Firm 2 decides $P_d = p_2 > p_1$.

3.4. Both firms participate in the market: $E_1 < \frac{c_1-b}{2m}$ *and* $\frac{c_1+b}{2} \geq c_2$

The NE of this section are going to be the ones from cases (c), (d) and (e) of Figure 3.1. Since $E_1 < \frac{c_1-b}{2m} < \frac{c_2-b}{m}$, Firm 1 does not have capacity to monopolize the market.

3.4.1. Firm 2 decides $P_d = p_2 > p_1$

We start with the case in which $P_d = p_2 > p_1$ (case (e) of Figure 3.1). As Proposition 5 states, Firm 2 plays the stationary duopoly optimum $(q_2^*, p_2^*) = \left(\frac{c_2-b-q_1^*m}{2m}, \frac{c_2+b+q_1^*m}{2}\right)$ and q_1^* will be as large as possible such that:

$$p_2^* = \frac{c_2+b+q_1^*m}{2} \geq c_2 \Leftrightarrow q_1^* \leq \frac{c_2-b}{m}$$

and

$$q_2^* = \frac{c_2-b-q_1^*m}{2m} \geq 0 \Leftrightarrow q_1^* \leq \frac{c_2-b}{m}$$

thus $q_1^* = E_1$. Note that $q_2^* > 0$, since $q_2^* = \frac{c_2-b-E_1m}{2m} > \frac{c_2-b}{2m} - \frac{c_1-b}{4m} = \frac{2c_2-b-c_1}{4m} > 0 \Leftrightarrow c_2 < \frac{b+c_1}{2}$. An important fact is that $p_2^* = \frac{c_2+b+E_1^*m}{2} > c_2$, since this is equivalent to $E_1 < \frac{c_2-b}{m}$ which is true. On the other hand $q_2^* = \frac{c_2-b-E_1m}{2m} \leq E_2$ depends on the instance of our problem, so Firm 2 may have to play the extreme point of the duopoly optimum.

1. Suppose $E_2 \geq \frac{c_2-b-E_1m}{2m}$. Firm 1 plays $(q_1^* = E_1, p_1^* < c_2)$ and Firm 2 plays $\left(q_2^* = \frac{c_2-b-E_1m}{2m}, p_2^* = \frac{c_2+b+E}{2}\right.$ It is easy to see that Firm 2 does not have advantage in choosing other strategy. Let us see if Firm 1 will change its behavior to $\tilde{p_1} = \frac{2c_1+c_2+b-E_1m}{4}$. In that case, $\tilde{p_1}$ must be higher than p_2^*:

$$\tilde{p_1} = \frac{2c_1+c_2+b-E_1m}{4} > \frac{c_2+b+E_1m}{2} = p_2^* \Leftrightarrow E_1 > \frac{2c_1-c_2-b}{3m}$$

and $\frac{2c_1-c_2-b}{3m} \leq \frac{c_1-b}{2m}$, which depends on the instance of our problem.

(a) Let $E_1 \leq \frac{2c_1 - c_2 - b}{3m}$. Then, Firm 1 does not have stimulus in changing its behavior unilaterally. Here, the Nash equilibrium is given by:
Strategies 20.

Firm 1: see Equation 3.5

Firm 2: see Equation 3.6

(b) Let $E_1 > \frac{2c_1 - c_2 - b}{3m} \Leftrightarrow p_1 = \frac{2c_1 + c_2 + b - E_1 m}{4} > \frac{c_2 + b + E_1 m}{2} = p_2^*$. We have:

$$\Pi_1 \left(\frac{\tilde{p_1} - b}{m} - q_2^*, \tilde{p_1}, q_2^*, p_2^* \right) = \frac{-1}{16m} \left(c_2 + b - 2c_1 - E_1 m \right)^2$$

which is larger than the profit

$$\Pi_1 \left(E_1, p_1 < c_2, q_2^*, p_2^* \right) = \frac{E_1}{2} \left(c_2 + b + E_1 m - 2c_1 \right)$$

if $E_1 \neq \frac{2c_1 - b - c_2}{3m}$. Therefore, Firm 1 has advantage in changing its strategy. However, like we already did in the Section 3.3.3, Firm 2 can pick $q_2^* > \frac{c_2 - b - E_1 m}{2m}$ sufficiently large such that Firm 1 does not have advantage in changing its strategy. This case is completely analogous to the one treated in Section 3.3.3, let $K_1 = -2E_1^2 m^2 + (-2c_2 m + 4c_1 m - 2bm)E_1$. If $E_2 \geq \frac{c_1 - b + \sqrt{K_1}}{m}$ we have the Nash equilibrium:
Strategies 21.

Firm 1: see Equation 3.5

Firm 2: see Equation 3.24

2. Suppose $E_2 < \frac{c_2 - b - E_1 m}{2m}$. Here Firm 2's duopoly optimum is an extreme point, $q_2^* = E_2$, $p_2^* = (E_2 + E_1)\, m + b$ and $q_1^* = E_1$. Notice that $p_2^* = (E_2 + E_1)\, m + b > c_2$, since both firms are playing smaller quantities than the last case (the market clearing price increases when the market clearing quantity decreases).
Clearly, Firm 2 will not change its strategy, but Firm 1 may have incentive in choosing a higher price $\tilde{p_1} > p_2^*$ and that requires:

$$\tilde{p_1} = \frac{c_1 + b + E_2 m}{2} > p_2^* = (E_1 + E_2)\, m + b \Leftrightarrow E_1 > \frac{c_1 - b - E_2 m}{2m}$$

which depends on the instance of our problem.

(a) Let $E_1 \leq \frac{c_1 - b - E_2 m}{2m}$. Then, Firm 1 will not change its behavior. The Nash equilibrium is given by:
Strategies 22.

Firm 1: see Equation 3.9

Firm 2: see Equation 3.10

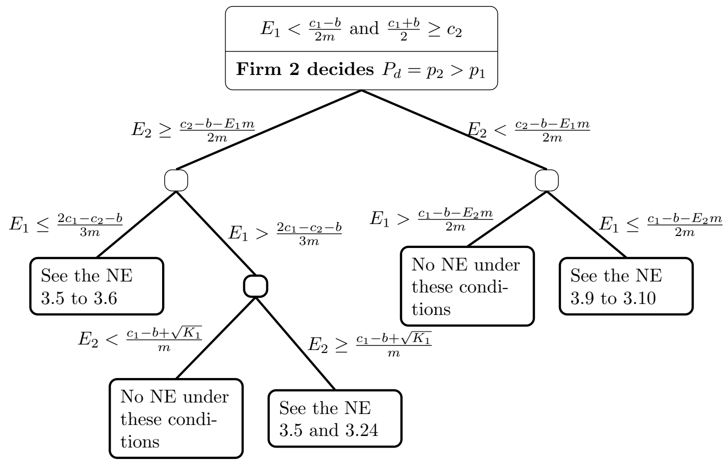

Figure 3.8: Firm 2 decides $P_d = p_2 > p_1$.

(b) Let $E_1 > \frac{c_1-b-E_2m}{2m}$. Is

$$\Pi_1\left(\frac{p_1-b}{m} - E_2, \frac{c_1+b+E_2m}{2}, E_2, p_2^*\right)$$

higher than

$$\Pi_1\left(E_1, p_1 < c_2, E_2, \left(E_1+E_2\right)m + b\right)?$$

This is equivalent to solving:

$$\frac{-1}{4m}\left(-c_1+b+E_2m\right)^2 < \left(\left(E_1+E_2\right)m + b - c_1\right)E_1$$

$$\Leftrightarrow -mE_1^2 + \left(c_1-b-E_2m\right)E_1 - \frac{1}{4m}\left(-c_1+b+E_2m\right)^2 < 0$$

$$\Leftrightarrow E_1 \neq \frac{c_1-b-E_2m}{2m},$$

so Firm 1 has advantage in changing its strategy.

From the above we reach the decision tree of Figure 3.8.

3.4.2. Firm 1 decides $P_d = p_1 > p_2$

Firm 1 wants to play the stationary point $(q_1^*, p_1^*) = \left(\frac{c_1-b-q_2^*m}{2m}, \frac{c_1+b+q_2^*m}{2}\right)$, which requires:

$$\frac{c_1+b+q_2^*m}{2} > c_2 \Leftrightarrow q_2^* < \frac{2c_2-c_1-b}{m}$$

thus $q_2^* = \begin{cases} \frac{2c_2-c_1-b}{m} - \varepsilon & E_2 \geq \frac{2c_2-c_1-b}{m} \\ E_2 & otherwise \end{cases}$. Let us note that: $\frac{c_1-b-q_2^*m}{2m} \leq E_1$ depends on the instance of our problem.

1. Suppose $E_2 \geq \frac{2c_2-c_1-b}{m}$, then $q_2^* = \frac{2c_2-c_1-b}{m} - \varepsilon$.
   (a) Let $E_1 \geq \frac{c_1-b-q_2^* m}{2m} = \frac{c_1-c_2}{m} - \frac{\varepsilon}{2}$. Hence, Firm 1 can play (q_1^*, p_1^*) and Firm 2 can play $\left(\frac{2c_2-c_1-b}{m} - \varepsilon, c_2\right)$.
   Will Firm 1 decrease the price $p_1 < c_2$? This means:

$$\Pi_1(E_1, c_1, q_2^*, c_2) = (c_2 - c_1) E_1 \geq \lim_{\varepsilon \to 0} \Pi_1(q_1^*, p_1^*, q_2^*, c_2)$$

$$\Leftrightarrow (c_2 - c_1) E_1 \geq \frac{-1(c_1 - c_2)^2}{m}$$

$$\Leftrightarrow E_1 \geq \frac{c_1 - c_2}{m}.$$

   Since $E_1 \geq \frac{c_1-b-q_2^* m}{2m} = \frac{c_1-c_2}{m} - \frac{\varepsilon}{2}$, Firm 1 will change its behavior.
   (b) Let $E_1 < \frac{c_1-c_2}{m}$ then $q_1^* = E_1$, $p_1^* = (E_1 + q_2) m + b$ and

$$q_2 = \begin{cases} \frac{c_2-b}{m} - E_1 - \varepsilon & E_2 \geq \frac{c_2-b}{m} - E_1 \\ E_2 & \textit{otherwise.} \end{cases}$$

   If $E_2 \geq \frac{c_2-b}{m} - E_1$, Firm 2 will change its strategy as with the present one, its profit is almost zero and increasing p_2 from c_2 to $\tilde{p_2} = \frac{c_2+b+E_1 m}{2}$ ($> c_2 \Leftrightarrow E_1 < \frac{c_2-b}{m}$ which holds by assumption), Firm 2's profit is higher: $\frac{-1}{4m}(-c_2 + b + E_1 m)^2 > 0$. Hence, let $E_2 < \frac{c_2-b}{m} - E_1$ that implies $q_2^* = E_2$. In this case, it is easy to see that Firm 1 does not have advantage in changing its strategy:

$$\Pi_1(E_1, (E_1 + E_2) m + b, E_2, c_2) \geq \Pi_1(E_1, p_1 < c_2, E_2, c_2)$$

$$\Leftrightarrow ((E_1 + E_2) m + b - c_1) E_1 \geq (c_2 - c_1) E_1$$

$$\Leftrightarrow E_2 \leq \frac{c_2 - b}{m} - E_1$$

   which holds.
   Now, we are going to see under which conditions Firm 2 does not have incentive to change its behavior. First of all, if Firm 2 changes its strategy, it will be with $\tilde{p_2} = \frac{c_2+b+E_1 m}{2}$, requiring

$$\tilde{p_2} > p_1^* = (E_1 + E_2) m + b \Leftrightarrow E_2 > \frac{c_2 - b - E_1 m}{2m}$$

   which depends on our instance.
   If $E_2 > \frac{c_2-b-E_1 m}{2m}$, Firm 2 chooses this new strategy, see AppendixF.
   If $E_2 \leq \frac{c_2-b-E_1 m}{2m}$, we have the NE:
   Strategies 23.

Firm 1: see Equation 3.11

Firm 2: see Equation 3.12

2. Suppose $E_2 < \frac{2c_2-c_1-b}{m}$, which implies $q_2^* = E_2$.

   (a) Let $E_1 \geq \frac{c_1-b-E_2m}{2m}$. So, Firm 1 can play $(q_1^*, p_1^*) = \left(q_1^* \geq \frac{c_1-b-E_2m}{2m}, \frac{c_1+b+E_2m}{2}\right)$. Firm 1 does not have advantage in decreasing the price to $\tilde{p_1} < c_2$ when $E_2 < \frac{c_1-b+2\sqrt{E_1m(c_1-c_2)}}{m}$, see AppendixG.
   Finally, we have to check if Firm 2 has advantage in increasing the price $\tilde{p_2} > p_1^*$. In that case, $\tilde{p_2} = \frac{c_2+b+q_1^*m}{2}$ which requires:

$$\tilde{p_2} = \frac{c_2 + b + q_1^* m}{2} > p_1^* = \frac{c_1 + b + E_2 m}{2} \Leftrightarrow q_1^* < \frac{c_1 - c_2}{m} + E_2.$$

   Note that $\frac{c_1-c_2}{m} + E_2 \leq \frac{c_1-b-E_2m}{2m} \Leftrightarrow E_2 \leq \frac{2c_2-c_1-b}{3m}$. In this way, if $E_2 \leq \frac{2c_2-c_1-b}{3m} \Rightarrow q_1^* \geq \frac{c_1-c_2}{m} + E_2$ and we have the NE:
   Strategies 24.

Firm 1: see Equation 3.16

Firm 2: see Equation 3.17

   Let $E_2 > \frac{2c_2-c_1-b}{3m} \Leftrightarrow \frac{c_1-c_2}{m} + E_2 > \frac{c_1-b-E_2m}{2m}$, thus q_1^* need to be $q_1^* < \frac{c_1-c_2}{m} + E_2 \Leftrightarrow p_2 = \frac{c_2+b+q_1^*m}{2} > p_1^* = \frac{c_1+b+E_2m}{2}$. Will this new proposal price for Firm 2 increases its profit?
   If $E_1 > \frac{c_2-b+\sqrt{K_2}}{m}$, the answer is no (see AppendixI) and thus we have the following NE:
   Strategies 25.

$$\text{Firm 1: } s_1^{NE} = \left(q_1 \in \left[\frac{c_2 - b + \sqrt{K_2}}{m}, E_1\right], \frac{c_1 + b + E_2 m}{2}\right) \tag{3.25}$$

Firm 2: see Equation 3.17

   (b) Let $E_1 < \frac{c_1-b-mE_2}{2m}$, then $q_1^* = E_1$, $p_1^* = (E_1 + q_2)\, m + b$ and

$$q_2 = \begin{cases} \frac{c_2-b}{m} - E_1 - \varepsilon & E_2 \geq \frac{c_2-b}{m} - E_1 \\ E_2 & \textit{otherwise.} \end{cases}$$

   If $E_2 \geq \frac{c_2-b}{m} - E_1$ then $E_1 \geq \frac{c_2-b}{m} - E_2$. Since $E_1 < \frac{c_1-b-mE_2}{2m}$, then $\frac{c_2-b}{m} - E_2 < \frac{c_1-b-mE_2}{2m} \Leftrightarrow E_2 > \frac{2c_2-c_1-b}{m}$ and we previously assumed $E_2 < \frac{2c_2-c_1-b}{m}$. Thus $E_2 < \frac{c_2-b}{m} - E_1$, implies $q_2^* = E_2$. Firm 1 does not change its strategy, since:

$$\Pi_1 \left(E_1, (E_1 + E_2)\, m + b, E_2, c_2\right) \geq \Pi_1 \left(E_1, p_1 < c_2, E_2, c_2\right)$$
$$\Leftrightarrow \left((E_2 + E_1)\, m + b - c_1\right) E_1 \geq \Pi_1 = (c_2 - c_1)\, E_1$$
$$\Leftrightarrow E_2 \leq \frac{c_2 - b}{m} - E_1$$

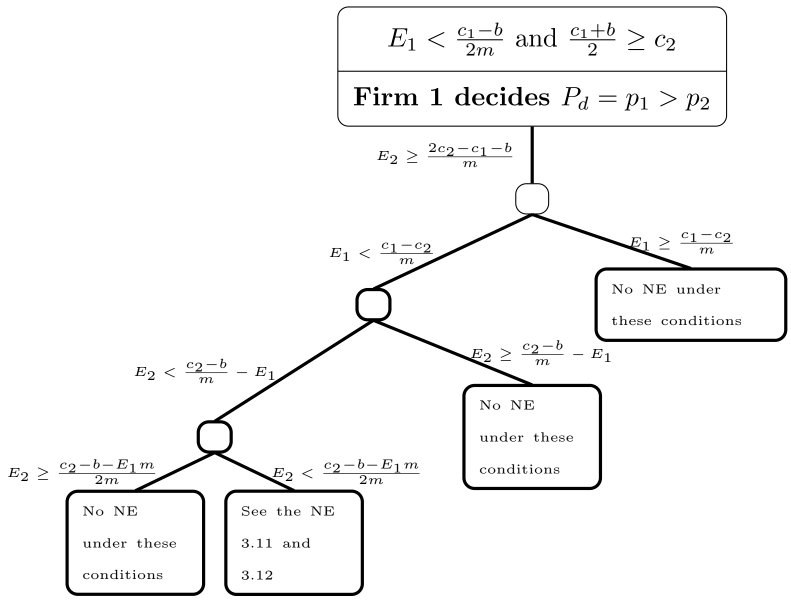

Figure 3.9: Firm 1 decides $P_d = p_1 > p_2$.

and the last inequality holds.
Firm 2 does not change its strategy (note that $\frac{c_2+b+E_1m}{2} > (E_1+E_2)\,m+b \Leftrightarrow E_2 > \frac{c_2-b}{2m} - \frac{E_1}{2}$) when $E_2 = \frac{c_2-b-E_1m}{2m}$, since

$$\Pi_2\left(E_1,(E_1+E_2)\,m+b,E_2,c_2\right) \geq$$

$$\geq \Pi_2\left(E_1,(E_1+E_2)\,m+b,\frac{p_2-b}{m}-E_1,\frac{c_2+b+E_1m}{2}\right)$$

$$\Leftrightarrow \left((E_2+E_1)\,m+b-c_2\right)E_2 \geq \frac{-1}{4m}\left(E_1m+b-c_2\right)^2$$

$$\Leftrightarrow E_2 = \frac{c_2-b-E_1m}{2m}$$

Thus, for $E_2 \leq \frac{c_2-b}{2m} - \frac{E_1}{2}$, we have the NE:

Strategies 26.

Firm 1: see Equation 3.11

Firm 2: see Equation 3.12

Proceeding as before, we have the decision tree of Figures 3.9 and 3.10.

In AppendixI there are the trees with all the possible equilibria in pure strategies.

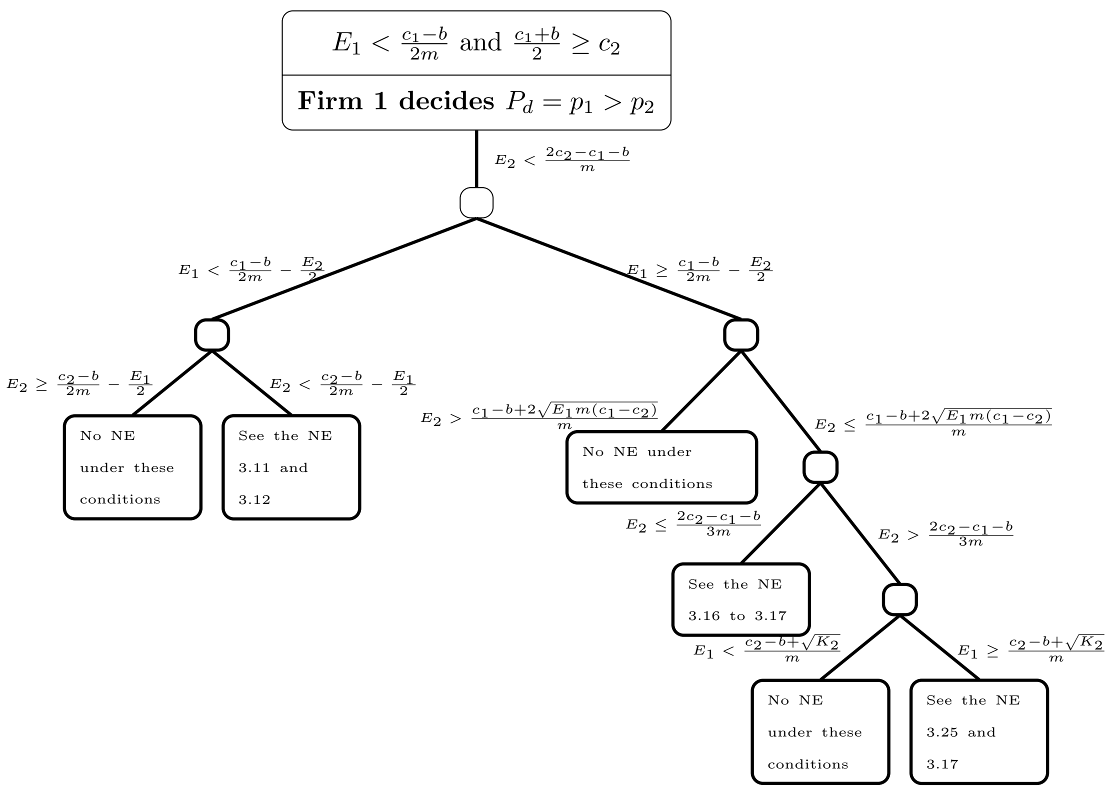

Figure 3.10: Firm 1 decides $P_d = p_1 > p_2$.

4. Discussion and conclusions

In the Iberian duopoly market model, the demand and the production costs are linear. As Proposition 5 suggests, there are five types of Nash equilibria. Instances where Firm 2 participates in the market with infinitesimal quantity ε will be considered as a monopoly for Firm 1.

When Firm 1 monopolizes the market in an equilibrium, the selected prices may be $\frac{c_1+b}{2}$, $E_1 m + b$ or $c_2 - \varepsilon$, depending on the efficiency/competitiveness of Firm 2. For a high marginal cost c_2, Firm 1 bids the monopoly optimum: $p_1 = \frac{c_1+b}{2}$ or $p_1 = E_1 m + b$. Furthermore, for c_2 closer to c_1, if Firm 2's capacity E_2 is large enough, Firm 1 monopolizes bidding $p_1 = c_2 - \varepsilon$; otherwise, for a limited capacity E_1, Firm 1 may prefer to bid a higher price, sharing the market with Firm 2. For c_2 even closer to c_1, the market clearing price in an equilibrium may be decided by either one of the firms. Firm 1 decides P_d, which means $P_d = p_1 > p_2$, if the capacity of Firm 2 is limited. The case of Firm 2 deciding P_d, meaning $P_d = p_2 > p_1$, is analogous.

In some competitive situations, there are two NE: one case with $P_d = p_1 > p_2$ and another with $P_d = p_2 > p_1$. By simulating random instances it was observed that Firm 1 has a high profit in the equilibrium $P_d = p_2 > p_1$, while Firm 2 benefits when $P_d = p_1 > p_2$ and there is no rational way of deciding among them. We have discretized the space of strategies S_1 and S_2 to compute NE in mixed strategies in these cases. It was observed that a combination of these two equilibria leads to a new NE in mixed strategies. However, the new equilibrium does not benefit either of the firms comparatively to the pure NE.

In conclusion, this work completely classified the NE that may occur in a duopoly day-ahead market. This helps understand the sensitivity of the outcomes to the instances' parameters and the diversity of equilibria that may arise. Furthermore, it illustrates the rational strategies of each firm.

5. Acknowledgments

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AppendixA. Firm 2 changes its strategy

Assume that $E_1 \geq \frac{c_1-b}{2m}$, $\frac{c_1+b}{2} \geq c_2$ and $E_2 > \frac{2c_2-c_1-b}{3m}$, and that Firm 1 is playing $\left(q_1^* = \frac{c_1-b-q_2^* m}{2m}, p_1^* = \frac{c_1+b+q_2^* m}{2}\right)$. By Equation 3.13, since $E_2 > \frac{2c_2-c_1-b}{3m}$ then $q_2^* > \frac{2c_2-c_1-b}{3m}$. Let us prove that Firm 2 will change its strategy $(q_2^*, p_2^* = c_2)$ to $\left(\tilde{q_2} = \frac{\tilde{p_2}-b}{m} - q_1^*, \tilde{p_2} = \frac{b+c_1+2c_2-q_2^* m}{4}\right)$:

$$\Pi_2\left(q_2^*, p_2^*, q_1^*, p_1^*\right) < \Pi_2\left(\tilde{q_2}, \tilde{p_2}, q_1^*, p_1^*\right)$$
$$\Leftrightarrow \frac{q_2^*}{2}\left(c_1 + b + q_2^* m - 2c_2\right) < \frac{-1}{16m}\left(b + c_1 - 2c_2 - q_2^* m\right)^2$$
$$\Leftrightarrow \frac{9}{2}m\left(q_2^*\right)^2 + 3q_2^*\left(c_1 + b - 2c_2\right) + \frac{1}{2m}\left(c_1 + b - 2c_2\right)^2 < 0$$
$$\Leftrightarrow q_2^* \neq \frac{2c_2 - b - c_1}{3m}$$

for this reason, Firm 2 benefits from changing its strategy.

AppendixB. Firm 1 does not change its strategy

Assume that $E_1 \geq \frac{c_1-b}{2m}$, $\frac{c_1+b}{2} \geq c_2$, $E_2 \leq \frac{2c_2-c_1-b}{3m}$ and $E_1 \geq \frac{c_2-b}{m}$, and that Firm 2 is playing $(q_2^* = E_2, p_2^* = c_2)$. Let us prove in which conditions Firm 1 does not change its strategy $\left(q_1^* = \frac{c_1-b-E_2 m}{2m}, p_1^* = \frac{c_1+b+E_2 m}{2}\right)$ to $\left(\tilde{q_1} = \frac{c_2-\varepsilon-b}{m}, \tilde{p_1} = c_2 - \varepsilon\right)$:

$$\Pi_1\left(q_1^*, p_1^*, q_2^*, p_2^*\right) \geq \lim_{\varepsilon\to 0} \Pi_1\left(\tilde{q_1}, \tilde{p_1}, q_2^*, p_2^*\right)$$
$$\Leftrightarrow \frac{-\left(b + E_2 m - c_1\right)^2}{4m} \geq \lim_{\varepsilon\to 0}\left(c_2 - c_1\right)\left(\frac{c_2 - b}{m} - \varepsilon\right)$$
$$\Leftrightarrow \frac{-1}{4m}\left(\left(b - c_1\right)^2 + 2E_2 m\left(b - c_1\right) + E_2^2 m^2\right) - \left(c_2 - c_1\right)\left(\frac{c_2 - b}{m}\right) > 0$$
$$\Rightarrow E_2^2 m\frac{-1}{4} - E_2\frac{1}{2}\left(b - c_1\right) - \frac{1}{4m}\left(b - c_1\right)^2 - \left(c_2 - c_1\right)\left(\frac{c_2 - b}{m}\right) = 0$$
$$\Leftrightarrow E_2 = \frac{c_1 - b \pm 2\sqrt{\left(c_2 - b\right)\left(c_1 - c_2\right)}}{m}$$

Nextm this solution is studied:

- $\frac{c_1-b-2\sqrt{(c_2-b)(c_1-c_2)}}{m} > \frac{c_1-b+2\sqrt{(c_2-b)(c_1-c_2)}}{m} \Leftrightarrow -2 < 2$;
- $\frac{c_1-b+2\sqrt{(c_2-b)(c_1-c_2)}}{m} < 0 \Leftrightarrow \sqrt{\left(c_1 - c_2\right)\left(c_2 - b\right)} > \frac{b-c_1}{2} \Leftrightarrow \left(c_1 - c_2\right)\left(c_2 - b\right) > \frac{(b-c_1)^2}{4} \Leftrightarrow -c_2^2 + c_2\left(c_1 + b\right) - c_1 b - \frac{(b-c_1)^2}{4} > 0$, note that $-c_2^2 + c_2\left(c_1 + b\right) - c_1 b - \frac{(b-c_1)^2}{4} = 0 \Leftrightarrow c_2 = \frac{c_1+b}{2}$ thus, $0 \leq \frac{c_1-b+2\sqrt{(c_2-b)(c_1-c_2)}}{m} \leq \frac{c_1-b-2\sqrt{(c_2-b)(c_1-c_2)}}{m}$;

- $\frac{c_1-b-2\sqrt{(c_2-b)(c_1-c_2)}}{m} > \frac{2c_2-c_1-b}{3m} \Leftrightarrow 4c_1 - 2b - 2c_2 - 6\sqrt{(c_1-c_2)(c_2-b)} < 0 \Leftrightarrow \underbrace{-3\sqrt{(c_1-c_2)(c_2-b)}}_{<0} < \underbrace{b+c_2-2c_1}_{>0}$;
- $\frac{c_1-b+2\sqrt{(c_1-c_2)(c_2-b)}}{m} < \frac{2c_2-c_1-b}{3m} \Leftrightarrow 3\sqrt{(c_1-c_2)(c_2-b)} > c_2+b-2c_1 \Leftrightarrow -10c_2^2 + c_2(7b+13c_1) - 5c_1b - b^2 - 4c_1^2 > 0 \Rightarrow c_2 \in \left[\frac{b+4c_1}{5}, \frac{b+c_1}{2}\right]$.

Therefore, Firm 1 does not change its strategy when

$$E_2 \in \left[0, \min\left(\frac{c_1-b+2\sqrt{(c_1-c_2)(c_2-b)}}{m}, \frac{2c_2-c_1-b}{3m}\right)\right].$$

AppendixC. Firm 2 changes its strategy

Assume that $E_1 \geq \frac{c_1-b}{2m}$, $\frac{c_1+b}{2} \geq c_2$ and $E_2 \geq \frac{2c_2-c_1-b}{3m}$, and that Firm 1 is playing $\left(q_1^* > \frac{c_1-b-q_2^*m}{2m}, p_1^* = \frac{c_1+b+q_2^*m}{2}\right)$, where q_2^* is given by Equation 3.13. Let us prove in which conditions Firm 2 changes its strategy $(q_2^*, p_2^* = c_2)$ to $\left(\tilde{q_2} = \frac{\tilde{p_2}-b}{m} - q_1^*, \tilde{p_2} = \frac{c_2-b-q_1^*m}{2m}\right)$:

$$\Pi_2(q_1^*, p_1^*, \tilde{q_2}, \tilde{p_2}) = \left(\frac{c_2+b+q_1^*m}{2} - c_2\right)\left(\frac{c_2-b-q_1^*m}{2m}\right)$$

is higher than

$$\Pi_2(q_1^*, p_1^*, q_2^*, p_2^*) = \frac{q_2^*}{2}(c_1+b+q_2^*m-2c_2)$$

$$\Leftrightarrow \frac{-1}{4m}(-c_2+b+q_1^*m)^2 > \frac{q_2^*}{2}(c_1+b+q_2^*m-2c_2)$$

$$\Leftrightarrow \frac{-m}{4}(q_1^*)^2 + q_1^*\frac{c_2-b}{2} - \frac{1}{4}\left(\frac{b-c_2}{m}\right)^2 - \frac{1}{2}q_2^*(c_1+b+q_2^*m-2c_2) > 0$$

let $K_2 = -2(q_2^*m)^2 + q_2^*(-2mc_1 - 2bm + 4c_2m)$

$$\Leftrightarrow q_1^* \in \left[0, \frac{c_2-b+\sqrt{K_2}}{m}\right[\cup \left]\frac{c_2-b-\sqrt{K_2}}{m}, \infty\right[$$

Note that for $q_2^* \leq \frac{2c_2-b-c_1}{m}$, K_2 is non negative and :

$$\frac{c_2-b+\sqrt{K_2}}{m} > \frac{c_1-c_2}{m} + q_2^*$$

$$\Leftrightarrow -3(mq_2^*)^2 + q_2^*(-2bm + 4c_2m - 2mc_1 - 2m(c_1-2c_2+b)) - (c_1-2c_2+b)^2 < 0$$

$$\Leftrightarrow q_2^* \in \left[0, \frac{2c_2-b-c_1}{3m}\right[\cup \left]\frac{2c_2-b-c_1}{m}, \infty\right[$$

but clearly

$$\frac{2c_2-b-c_1}{3m} \leq q_2^* \leq \frac{2c_2-b-c_1}{m}$$

so $q_1^* < \frac{c_2-b+\sqrt{K_2}}{m}$ is the condition for Firm 2 to change its strategy.

AppendixD. Firm 1 does not change its strategy

Assume that $E_1 \geq \frac{c_2-b}{m}$, $\frac{c_1+b}{2} \geq c_2$, $E_1 \geq \frac{c_2-b}{m}$, $E_2 < \frac{c_2-b}{m}$ and $E_2 < \frac{2c_2-c_1-b}{m}$, and that Firm 2 is playing $(q_2^* = E_2, p_2^* = c_2)$. Let us prove in which conditions Firm 1 does not change its strategy $\left(q_1^* = \frac{c_2-\varepsilon-b}{m}, p_1^* = c_2 - \varepsilon\right)$ to $\left(\tilde{q_1} = \frac{\tilde{p_1}-b}{m} - E_2, \tilde{p_1} = \frac{c_1+b+E_2m}{2}\right)$:

$$\Pi_1\left(\tilde{q_1}, \tilde{p_1}, q_2^*, p_2^*\right) = \left(\frac{c_1+b+E_2m}{2} - c_1\right)\left(\frac{c_1-b-E_2m}{2m}\right)$$

higher than

$$\lim_{\varepsilon\to 0} \Pi_1\left(\frac{c_2-\varepsilon-b}{m}, c_2-\varepsilon, q_2^*, p_2^*\right) = (c_2-c_1)\left(\frac{c_2-b}{m}\right)$$

is equivalent to:

$$\frac{-1}{4m}\left(b+E_2m-c_1\right)^2 > (c_2-c_1)\left(\frac{c_2-b}{m}\right)$$

$$\Leftrightarrow \left(E_2m\right)^2 + 2E_2m\left(b-c_1\right) + \left(b-c_1\right)^2 - 4m\left(c_1-c_2\right)\left(\frac{c_2-b}{m}\right) > 0$$

$$\Rightarrow E_2 \in \left[0, \frac{c_1-b+2\sqrt{(c_1-c_2)(c_2-b)}}{m}\right] \cup \left[\frac{c_1-b-2\sqrt{(c_1-c_2)(c_2-b)}}{m}, \infty\right] \quad \text{(D.1)}$$

Let us study this solution:

1. $\frac{c_1-b-2\sqrt{(c_1-c_2)(c_2-b)}}{m} > \frac{2c_2-c_1-b}{m} \Leftrightarrow \underbrace{c_2-c_1}_{>0} > \underbrace{-\sqrt{(c_1-c_2)(c_2-b)}}_{<0}$
2. $\frac{c_1-b+2\sqrt{(c_1-c_2)(c_2-b)}}{m} < \frac{2c_2-c_1-b}{m} \Leftrightarrow \sqrt{(c_1-c_2)(c_2-b)} > c_2-c_1 \Leftrightarrow -2c_2^2+c_2(3c_1+b)-bc_1-c_1^2 > 0 \Rightarrow c_2 \in \left[c_1, \frac{c_1+b}{2}\right]$

If

$$E_2 \in \left[\frac{c_1-b+2\sqrt{(c_1-c_2)(c_2-b)}}{m}, \frac{2c_2-c_1-b}{m}\right]$$

holds then Firm 1 does not change its bid.

AppendixE. Firm 1 changes its strategy

Assume that $\frac{c_1-b}{2m} \leq E_1 < \frac{c_2-b}{m}$, $\frac{c_1+b}{2} \geq c_2$ and $E_2 < \frac{c_2-b-E_1m}{2m}$, and that Firm 2 is playing $(q_2^* = E_2, p_2^* = (E_2+E_1)m+b)$. Let us prove that Firm 1 will change its strategy $(q_1^* = E_1, p_1^* < c_2)$ to $\left(\tilde{q_1} = \frac{\tilde{p_1}-b}{m} - E_2, \tilde{p_1} = \frac{c_1+b+E_2m}{2}\right)$:

$$\Pi_1\left(\tilde{q_1}, \tilde{p_1}, q_2^*, p_2^*\right) = \frac{-1}{4m}\left(-c_1+b+E_2m\right)^2$$

higher than

$$\Pi_1\left(q_1^*, p_1^*, q_2^*, p_2^*\right) = E_1\left(\left(E_1+E_2\right)m+b-c_1\right)$$

is equivalent to

$$\frac{-1}{4m}\left(-c_1+b+E_2m\right)^2 \leq E_1\left(\left(E_1+E_2\right)m+b\right)$$

$$\Leftrightarrow -mE_1^2+E_1\left(c_1-b-E_2m\right)-\frac{1}{4m}\left(-c_1+b+E_2m\right)^2 \leq 0$$

$$\Rightarrow E_1=\frac{c_1-b-E_2m}{2m}=\frac{c_1-b}{2m}-\frac{E_2}{2}$$

which never occurs since $E_1 \geq \frac{c_1-b}{2m}$.

AppendixF. Firm 2 changes its strategy

Assume that $E_1<\frac{c_1-c_2}{m}$, $\frac{c_1+b}{2}\geq c_2$, $E_2\geq\frac{2c_2-c_1-b}{m}$ and $\frac{c_2-b-E_1m}{2m}<E_2<\frac{c_2-b}{m}-E_1$, and that Firm 1 is playing $(q_1^*=E_1, p_1^*=(E_1+E_2)m+b)$. Let us prove that Firm 2 changes its strategy $(q_2^*=E_2, p_2^*=c_2)$ to $\left(\tilde{q}_2=\frac{\tilde{p_2}-b}{m}-E_1, \tilde{p}_2=\frac{c_2+b+E_1m}{2}\right)$:

$$\Pi_2\left(q_1^*,p_1^*,q_2^*,p_2^*\right)\geq\Pi_2\left(q_1^*,p_1^*,\tilde{q}_2,\tilde{p}_2\right)$$

$$\Leftrightarrow \left(mE_1+E_2m+b-c_2\right)E_2\geq\frac{-1}{4m}\left(-c_2+b+E_1m\right)^2$$

$$\Leftrightarrow E_2^2m+\left(E_1m+b-c_2\right)E_2+\frac{1}{4}\left(\frac{E_1m+b-c_2}{m}\right)^2\geq 0$$

$$\Leftrightarrow E_2=\frac{c_2-b-E_1m}{2m}$$

thus Firm 2 will choose this new strategy.

AppendixG. Firm 1 does not change its strategy

Assume that $\frac{c_1-b-E_2m}{2m}\leq E_1<\frac{c_1-b}{2m}$, $\frac{c_1+b}{2}\geq c_2$ and $E_2<\frac{2c_2-c_1-b}{m}$, and that Firm 2 is playing $(q_2^*=E_2, p_2^*=c_2)$. Let us see in which conditions Firm 1 does not change its strategy $\left(q_1^*\geq\frac{c_1-b-E_2m}{2m}, p_1^*=\frac{c_1+b+E_2m}{2}\right)$ to $(\tilde{q}_1=E_1, \tilde{p_1}<c_2)$:

$$\Pi_1\left(q_1^*,p_1^*,q_2^*,p_2^*\right)\leq\Pi_1\left(\tilde{q}_1,\tilde{p_1},q_2^*,p_2^*\right)$$

$$\Leftrightarrow \frac{-1}{4m}\left(-c_1+b+q_2^*m\right)^2\leq\left(c_2-c_1\right)E_1$$

$$q_2^*\in\left[\frac{c_1-b+2\sqrt{E_1m\left(c_1-c_2\right)}}{m},\frac{c_1-b-2\sqrt{E_1m\left(c_1-c_2\right)}}{m}\right]$$

Note:

- $\frac{c_1-b-2\sqrt{E_1m(c_1-c_2)}}{m}>\frac{2c_2-c_1-b}{m}\Leftrightarrow\underbrace{-2\sqrt{E_1m\left(c_1-c_2\right)}}_{<0}<\underbrace{2\left(c_2-c_2\right)}_{>0}$;

- $\frac{c_1-b+2\sqrt{E_1 m(c_1-c_2)}}{m} > \frac{2c_2-c_1-b}{m} \Leftrightarrow E_1 < \frac{c_1-c_2}{m}$ which never happens because by assumption $E_1 \geq \frac{c_1-b}{2m} - \frac{E_2}{2} \geq \frac{c_1-b}{2m} - \frac{2c_2-c_1-b}{2m} = \frac{c_1-c_2}{m}$.

So if $E_2 < \frac{c_1-b+2\sqrt{E_1 m(c_1-c_2)}}{m}$, Firm 1 does not change its strategy.

AppendixH. Firm 2 does not change its strategy

Assume that $\frac{c_1-b-E_2m}{2m} \leq E_1 < \frac{c_1-b}{2m}$, $\frac{c_1+b}{2} \geq c_2$ and $\frac{2c_2-c_1-b}{3m} < E_2 < \frac{2c_2-c_1-b}{m}$, and that Firm 1 is playing $\left(q_1^* \geq \frac{c_1-b-E_2m}{2m}, p_1^* = \frac{c_1+b+E_2m}{2}\right)$ with $q_1^* < \frac{c_1-c_2}{m} + E_2$. Let us see in which conditions Firm 2 does not change its $(q_2^* = E_2, p_2^* = c_2)$ to

$$\left(\tilde{q_2} = \frac{\tilde{p_2}-b}{m} - q_1^*, \tilde{p_2} = \frac{c_2+b+q_1^*m}{2}\right),$$

so:

$$\Pi_2\left(q_1^*, p_1^*, \tilde{q_2}, \tilde{p_2}\right) \geq \Pi_2\left(q_1^*, p_1^*, q_2^*, p_2^*\right)$$

$$\Leftrightarrow \frac{-1}{4m}\left(-c_2+b+q_1^*m\right)^2 \geq \left(\frac{c_1+b+E_2m}{2} - c_2\right) E_2$$

let $K_2 = -2E_2^2m^2 + \left(-2mc_1 - 2bm + 4mc_2\right) E_2$

$$\Leftrightarrow q_1^* \in \left[0, \frac{c_2-b+\sqrt{K_2}}{m}\right] \cup \left[\frac{c_2-b-\sqrt{K_2}}{m}, \infty\right[$$

Some facts about this solution:

- $\frac{c_1-c_2}{m} + E_2 < \frac{c_2-b-\sqrt{K_2}}{m} \Leftrightarrow \underbrace{b+c_1-2c_2+E_2m}_{>0} > -\sqrt{K_2}$ which holds since $b + c_1 - 2c_2 + E_2m > 0 \Leftrightarrow E_2 < \frac{2c_2-c_1-b}{m}$;
- $\frac{c_1-c_2}{m} + E_2 < \frac{c_2-b+\sqrt{K_2}}{m} \Leftrightarrow 3E_2^2m^2 + \left(2\left(-2c_2+c_1+b\right)m + 2mc_1 + 2bm - 4mc_2\right) E_2 + \left(-2c_2+c_1+b\right)^2 > 0 \Leftrightarrow E_2 \in \left[0, \frac{2c_2-c_1-b}{3m}\right[\cup \left]\frac{2c_2-c_1-b}{m}, \infty\right[$.

Then, $q_1^* < \frac{c_2-b+\sqrt{K_2}}{m}$ is the strongest condition. If $E_1 > \frac{c_2-b+\sqrt{K_2}}{m}$ Firm 2 does not change its strategy.

AppendixI. There is always an equilibrium

First of all, we made a complete study of the Nash equilibria under the following conditions: $E_1 \geq \frac{c_1-b}{2m} \wedge \frac{c_1+b}{2} < c_2$ (Section 3.1) and $E_1 < \frac{c_1-b}{2m} \wedge \frac{c_1+b}{2} < c_2$ (Section 3.2). In the two remaining cases, $E_1 \geq \frac{c_1-b}{2m} \wedge \frac{c_1+b}{2} \geq c_2$ (Section 3.3) and $E_1 < \frac{c_1-b}{2m} \wedge \frac{c_1+b}{2} \geq c_2$ (Section 3.4), we still have to intersect the conditions of the equilibria found.

In order to achieve the purpose of summarizing all the classification made so far, the leafs of the decision trees in Figures 3.4, 3.9 and 3.10 will be analyzed using the information of the remaining trees:

1. Let $E_1 \geq \frac{c_1-b}{2m}$, $\frac{c_1+b}{2} \geq c_2$, $E_1 < \frac{c_2-b}{m}$, $E_1 \geq \frac{(b+c_2-2c_1)^2}{9m(c_1-c_2)}$, $E_2 \leq A$. The decision tree in Figure 3.4 already provides an equilibrium in this case. Neither of the decision trees in Figures 3.5 and 3.6 have an equilibrium under these conditions. However, tree 3.7 adds the equilibrium of Equations 3.5 and 3.24 to this case. Note that:

$$A \leq \frac{c_1 - b + \sqrt{K_1}}{m}$$

$$\Leftrightarrow E_1 \in \{0\} \cup \left[\frac{c_2-b}{m}, \infty\right[$$

   thus $A > \frac{c_1-b+\sqrt{K_1}}{m}$. So, $E_2 \geq \frac{c_1-b+\sqrt{K_1}}{m}$ depends on the instance;
2. Let $E_1 \geq \frac{c_1-b}{2m}$, $\frac{c_1+b}{2} \geq c_2$, $E_1 < \frac{c_2-b}{m}$, $E_1 \geq \frac{(b+c_2-2c_1)^2}{9m(c_1-c_2)}$, $E_2 > A$. Neither of the trees in Figures 3.4, 3.5 and 3.6 have an equilibrium in this case. So, it is expected that the decision tree of Figure 3.7 has an equilibrium under these conditions. Remember that $E_2 > A > \frac{c_1-b+\sqrt{K_1}}{m}$, so the Equations 3.5 and 3.24 give us a NE;
3. Let $E_1 \geq \frac{c_1-b}{2m}$, $\frac{c_1+b}{2} \geq c_2$, $E_1 < \frac{c_2-b}{m}$, $E_1 < \frac{(b+c_2-2c_1)^2}{9m(c_1-c_2)}$, $E_2 > B$, $E_2 > A$. As before, there is only the NE of Equations 3.5 and 3.24;
4. Let $E_1 \geq \frac{c_1-b}{2m}$, $\frac{c_1+b}{2} \geq c_2$, $E_1 < \frac{c_2-b}{m}$, $E_1 < \frac{(b+c_2-2c_1)^2}{9m(c_1-c_2)}$, $E_2 > B$, $E_2 \leq A$, $E_2 \geq \frac{c_1-b+\sqrt{K_1}}{m}$, $E_1 \geq \frac{c_2-b+\sqrt{K_2}}{m}$. It is obviously that in this case we have the NE of Equations 3.19 and 3.17 and of Equations 3.5 and 3.24. It should be stressed that there are instances with this conditions;
5. Let $E_1 \geq \frac{c_1-b}{2m}$, $\frac{c_1+b}{2} \geq c_2$, $E_1 < \frac{c_2-b}{m}$, $E_1 < \frac{(b+c_2-2c_1)^2}{9m(c_1-c_2)}$, $E_2 > B$, $E_2 \leq A$, $E_2 \geq \frac{c_1-b+\sqrt{K_1}}{m}$, $E_1 < \frac{c_2-b+\sqrt{K_2}}{m}$. Clearly the only NE in pure strategies is the one given by Equations 3.5 and 3.24. Again, there is instances in this conditions.
6. Let $E_1 \geq \frac{c_1-b}{2m}$, $\frac{c_1+b}{2} \geq c_2$, $E_1 < \frac{c_2-b}{m}$, $E_1 < \frac{(b+c_2-2c_1)^2}{9m(c_1-c_2)}$, $E_2 > B$, $E_2 \leq A$, $E_2 < \frac{c_1-b+\sqrt{K_1}}{m}$. We will prove that this conditions imply $E_1 \geq \frac{c_2-b+\sqrt{K_2}}{m}$, and thus, Equations 3.19 and 3.17, give us an equilibrium.
   First, $E_2 < \frac{c_1-b+\sqrt{K_1}}{m} \leq \frac{c_2-c_1+E_1 m}{m}$ since:

$$\frac{c_1 - b + \sqrt{K_1}}{m} \leq \frac{c_2 - c_1 + E_1 m}{m}$$

$$\Leftrightarrow E_1 \in \left[\frac{2c_1 - c_2 - b}{3m}, \frac{2c_1 - c_2 - b}{m}\right]$$

   which holds. Note that:

$$\frac{c_2 - c_1 + E_1 m}{m} \geq \frac{c_2 - b + \sqrt{K_2}}{m}$$

$$\Leftrightarrow E_2 \in \left[B, \frac{2c_2 - c_1 - b}{m}\right]$$

which holds, since $E_2 > B$ and $E_2 \leq A < \frac{2c_2 - c_1 - b}{m}$:

$$A < \frac{2c_2 - c_1 - b}{m} \Leftrightarrow E_1 > \frac{c_1 - c_2}{m}$$

and $E_1 \geq \frac{c_1-b}{2m} > \frac{c_1-c_2}{m}$.
Second, $E_1 \geq \frac{c_1 - c_2 + E_2 m}{m} \geq \frac{c_2 - b + \sqrt{K_2}}{m}$ since:

$$E_1 \geq \frac{c_1 - c_2 + E_2 m}{m}$$

$$\Leftrightarrow E_2 \leq \frac{c_2 - c_1 + E_1 m}{m}$$

which holds;

7. Let $E_1 \geq \frac{c_1-b}{2m}$, $\frac{c_1+b}{2} \geq c_2$, $E_1 < \frac{c_2-b}{m}$, $E_1 < \frac{(b+c_2-2c_1)^2}{9m(c_1-c_2)}$, $E_2 \leq B < A$. In this case, $E_2 \geq \frac{c_1-b+\sqrt{K_2}}{m}$ depends on the instance, and thus we can have the NE of Equations 3.5 and 3.24, beyond the equilibrium of Equations 3.16 and 3.17;
8. Let $E_1 \geq \frac{c_1-b}{2m}$, $\frac{c_1+b}{2} \geq c_2$, $E_1 \geq \frac{c_2-b}{m}$, $c_2 \geq \frac{b+4c_1}{5}$, $E_2 \leq A' \leq B$. In this case, Equations 3.16 and 3.17 give an equilibrium. If $E_2 = A'$, there is also the NE of Equations 3.20 to 3.22;
9. Let $E_1 \geq \frac{c_1-b}{2m}$, $\frac{c_1+b}{2} \geq c_2$, $E_1 \geq \frac{c_2-b}{m}$, $c_2 \geq \frac{b+4c_1}{5}$, $E_2 > A'$. Here, there is a single NE given by the Equations 3.20 to 3.22;
10. Let $E_1 \geq \frac{c_1-b}{2m}$, $\frac{c_1+b}{2} \geq c_2$, $E_1 \geq \frac{c_2-b}{m}$, $c_2 < \frac{b+4c_1}{5}$, $E_2 > B$. Since $A' > B$ depends on the value of E_2, we can have at least one of the NE given by Equations 3.19 and 3.17 or 3.20 and 3.22;
11. Let $E_1 \geq \frac{c_1-b}{2m}$, $\frac{c_1+b}{2} \geq c_2$, $E_1 \geq \frac{c_2-b}{m}$, $c_2 < \frac{b+4c_1}{5}$, $E_2 \leq B$. In this case just one NE in pure strategies exists and it is given by Equations 3.16 and 3.17;
12. Let $E_1 < \frac{c_1-b}{2m}$, $\frac{c_1+b}{2} \geq c_2$, $E_2 < \frac{2c_2-c_1-b}{m}$, $E_1 < \frac{c_1-b}{2m} - \frac{E_2}{2}$ and $E_2 \geq \frac{c_2-b}{2m} - \frac{E_1}{2}$. In the decision tree of Figure 3.10 there is no NE. So, the decision tree of Figure 3.8 must have an equilibrium for this case. In this way we need to prove that this conditions imply $E_1 \leq \frac{2c_1-c_2-b}{3m}$.
Note that:

$$E_2 \geq \frac{c_2 - b}{2m} - \frac{E_1}{2} \Leftrightarrow E_1 \geq \frac{c_2 - b}{m} - 2E_2$$

in this way

$$\frac{c_2 - b}{m} - 2E_2 < \frac{c_1 - b}{2m} - \frac{E_2}{2} \Leftrightarrow E_2 > \frac{2c_2 - c_1 - b}{3m}$$

thus

$$E_1 < \frac{c_1 - b}{2m} - \frac{E_2}{2} < \frac{c_1 - b}{2m} - \frac{2c_2 - c_1 - b}{6m} = \frac{2c_1 - b - c_2}{3m}$$

hence, we are in the conditions of the NE 3.5 to 3.6;
13. Let $E_1 < \frac{c_1-b}{2m}$, $\frac{c_1+b}{2} \geq c_2$, $E_2 < \frac{2c_2-c_1-b}{m}$, $E_1 < \frac{c_1-b}{2m} - \frac{E_2}{2}$ and $E_2 < \frac{c_2-b}{2m} - \frac{E_1}{2}$. Apart from the NE of Equations 3.11 and 3.12, there is also the equilibrium of Equations 3.9 and 3.10;

14. Let $E_1 < \frac{c_1-b}{2m}$, $\frac{c_1+b}{2} \geq c_2$, $E_2 < \frac{2c_2-c_1-b}{m}$, $E_1 \geq \frac{c_1-b}{2m} - \frac{E_2}{2}$ and $E_2 > A$.

    First of all, it will be proved that $A = \frac{c_1-b+2\sqrt{E_1 m(c_1-c_2)}}{m} > \frac{c_2-b-E_1 m}{2m}$, which implies $E_2 > \frac{c_2-b-E_1 m}{2m}$:

$$\frac{c_1 - b + 2\sqrt{E_1 m\,(c_1 - c_2)}}{m} > \frac{c_2 - b - E_1 m}{2m}$$

$$\Leftrightarrow 4\sqrt{E_1 m\,(c_1 - c_2)} < \underbrace{c_2 - 2c_1 + b - E_1 m}_{>0}$$

    let $\gamma = 2c_1^2 - 5c_2 c_1 + b + c_1 + 3c_2^2 - c_2 b$

$$\Leftrightarrow E_1 \in \left[0, \frac{6c_1 - 7c_2 + b + 4\sqrt{\gamma}}{m}\right[\cup \left]\frac{6c_1 - 7c_2 + b - 4\sqrt{\gamma}}{m}, \infty\right[$$

    note that $\gamma < 0 \Leftrightarrow c_2 \in \left]c_1, \frac{2c_1+b}{3}\right[$, so if $c_2 < \frac{2c_1+b}{3}$ our proof is over, on the other hand, if $c_2 \geq \frac{2c_1+b}{3}$:

$$\frac{c_1 - b}{2m} \leq \frac{6c_1 - 7c_2 + b + 4\sqrt{\gamma}}{m}$$

$$\Leftrightarrow \underbrace{14c_2 - 11c_1 - 3b}_{>0} \geq 8\sqrt{\gamma}$$

$$\Leftrightarrow c_2 \in \left[0, \frac{c_1 + b}{2}\right] \cup \left[\frac{9b - 7c_1}{2}, \infty\right[$$

    therefore, $E_1 < \frac{c_1-b}{2m} \leq \frac{6c_1-7c_2+b+4\sqrt{\gamma}}{m}$ which ends our proof.
    If $E_1 \leq \frac{2c_1-c_2-b}{3m}$ we are in the conditions of the NE 3.5 to 3.6.
    If $E_1 > \frac{2c_1-c_2-b}{3m}$, note that:

$$E_2 > \frac{c_1 - b + 2\sqrt{E_1 m\,(c_1 - c_2)}}{m} \geq \frac{c_1 - b + \sqrt{K_1}}{m}$$

    since,

$$\frac{c_1 - b + 2\sqrt{E_1 m\,(c_1 - c_2)}}{m} \geq \frac{c_1 - b + \sqrt{K_1}}{m}$$

$$\Leftrightarrow 2E_1^2 m^2 + (4m\,(c_1 - c_2) + 2mc_c + 2bm - 4mc_1)\,E_1 \leq 0$$

$$\Leftrightarrow E_1 \in \left[0, \frac{c_2 - b}{m}\right]$$

    which holds. Thus, we are in the conditions of the NE 3.5 and 3.24;

15. Let $E_1 < \frac{c_1-b}{2m}$, $\frac{c_1+b}{2} \geq c_2$, $E_2 < \frac{2c_2-c_1-b}{m}$, $E_1 \geq \frac{c_1-b}{2m} - \frac{E_2}{2}$ and $E_2 \leq A$, $E_2 \leq \frac{2c_2-c_1-b}{3m}$. The decision tree of Figure 3.10 already provides the equilibrium of Equations 3.16 and 3.17. Depending on our instance, the Decision Tree 3.8 can give a new equilibrium;

16. Let $E_1 < \frac{c_1-b}{2m}$, $\frac{c_1+b}{2} \geq c_2$, $E_2 < \frac{2c_2-c_1-b}{m}$, $E_1 \geq \frac{c_1-b}{2m} - \frac{E_2}{2}$ and $E_2 \leq A$, $E_2 > \frac{2c_2-c_1-b}{3m}$, $E_1 < \frac{c_2-b+\sqrt{K_2}}{m}$.
As before, it will be proved that $E_2 \geq \frac{c_2-b-E_1m}{2m}$.
If $E_1 \leq \frac{2c_1-c_2-b}{3m}$, then $E_2 \geq \frac{c_1-b}{m} - 2E_1 \geq \frac{c_2-b}{m} - \frac{E_1}{2}$ since:

$$\frac{c_1-b}{m} - 2E_1 \geq \frac{c_2-b}{m} - \frac{E_1}{2}$$

$$\Leftrightarrow \frac{2c_1-c_2-b}{3m} \geq E_1$$

thus, we are in the conditions of the NE 3.5 to 3.6.
If $E_1 > \frac{2c_1-c_2-b}{3m}$, then $E_2 > \frac{2c_2-c_1-b}{3m} > \frac{c_2-b}{2m} - \frac{E_1}{2}$ since

$$\frac{2c_2-c_1-b}{3m} > \frac{c_2-b}{2m} - \frac{E_1}{2}$$

$$\Leftrightarrow E_1 > \frac{2c_1-c_2-b}{3m}$$

as we intended to prove. We also have $E_2 \geq \frac{c_1-b+\sqrt{K_1}}{m}$, since

$$\frac{c_1-b+\sqrt{K_1}}{m} < \frac{2c_2-c_1-b}{3m}$$

$$\Leftrightarrow 9K_1 > (2c_2-4c_1+2b)^2$$

$$\Leftrightarrow E_1 \in \left]\frac{2c_1-c_2-b}{3m}, 2\left(\frac{2c_1-c_2-b}{3m}\right)\right[,$$

which holds since:

$$2\left(\frac{2c_1-c_2-b}{3m}\right) > \frac{c_1-b}{2m}$$

$$\Leftrightarrow 5c_1 < 4c_2 + b$$

So, we are in the conditions of the NE 3.5 and 3.24.

17. Let $E_1 < \frac{c_1-b}{2m}$, $\frac{c_1+b}{2} \geq c_2$, $E_2 < \frac{2c_2-c_1-b}{m}$, $E_1 \geq \frac{c_1-b}{2m} - \frac{E_2}{2}$ and $E_2 \leq A$, $E_2 > \frac{2c_2-c_1-b}{3m}$, $E_1 \geq \frac{c_2-b+\sqrt{K_2}}{m}$. The decision tree of Figure 3.10 has an equilibrium under this conditions. Similarly to the last case, $E_2 \geq \frac{c_2-b-E_1m}{2m}$, and thus, depending on the value of E_1, there is an equilibrium in Equations 3.5 and 3.6 or 3.5 and 3.24;
18. Let $E_1 < \frac{c_1-b}{2m}$, $\frac{c_1+b}{2} \geq c_2$, $E_2 \geq \frac{2c_2-c_1-b}{m}$, $E_1 \geq \frac{c_1-c_2}{m}$.
Note that:

$$\frac{2c_2-c_1-b}{m} > \frac{c_2-b-E_1m}{2m} \Leftrightarrow E_1 > \frac{2c_1-3c_2+b}{m},$$

which holds, since $\frac{2c_1-3c_2+b}{m} \leq \frac{c_1-c_2}{m} \Leftrightarrow \frac{c_1+b}{2} \geq c_2$ and $E_1 > \frac{c_1-c_2}{m}$. Thus $E_2 \geq \frac{2c_2-c_1-b}{m} > \frac{c_2-b-E_1m}{2m}$.
If $E_1 \leq \frac{2c_1-c_2-b}{3m}$, we are in the conditions of the NE 3.5 to 3.6.

If $E_1 > \frac{2c_1-c_2-b}{3m}$, since

$$E_2 \geq \frac{2c_2-c_1-b}{m} \geq \frac{c_2-c_1}{m} + E_1 \Leftrightarrow \frac{c_2-b}{m} \geq E_1$$

which holds, and

$$E_2 \geq \frac{2c_2-c_1-b}{m} \geq \frac{c_2-c_1}{m} + E_1 \geq \frac{c_1-b+\sqrt{K_1}}{m},$$

we are in the conditions of the NE 3.5 and 3.24;

19. Let $E_1 < \frac{c_1-b}{2m}$, $\frac{c_1+b}{2} \geq c_2$, $E_2 \geq \frac{2c_2-c_1-b}{m}$, $E_1 < \frac{c_1-c_2}{m}$, $E_2 \geq \frac{c_2-b}{m} - E_1$. Note that:

$$E_1 < \frac{c_1-c_2}{m} \leq \frac{2c_1-c_2-b}{3m}$$

and

$$E_2 \geq \frac{c_2-b-E_1m}{m} \geq \frac{c_2-b-E_1m}{2m}$$

so, we are in the conditions of the NE 3.5 to 3.6.

20. Let $E_1 < \frac{c_1-b}{2m}$, $\frac{c_1+b}{2} \geq c_2$, $E_2 \geq \frac{2c_2-c_1-b}{m}$, $E_1 < \frac{c_1-c_2}{m}$, $E_2 < \frac{c_2-b}{m} - E_1$, $E_2 < \frac{c_2-b-E_1m}{2m}$. Apart from the NE of Equations 3.11 and 3.12, the decision tree of Figure 3.8 also adds the NE of Equations 3.9 and 3.10. In order to demonstrate this we need to prove that this conditions imply $E_1 \leq \frac{c_1-b-E_2m}{2m}$.
Note that $E_2 < \frac{c_2-b-E_1m}{2m} \Leftrightarrow E_1 < \frac{c_2-b-2mE_2}{m}$ and:

$$\frac{c_2-b-2mE_2}{m} < \frac{c_1-b-E_2m}{2m}$$

$$\Leftrightarrow E_2 > \frac{2c_2-c_1-b}{3m}$$

which holds, since $E_2 \geq \frac{2c_2-c_1-b}{m}$;

21. Let $E_1 < \frac{c_1-b}{2m}$, $\frac{c_1+b}{2} \geq c_2$, $E_2 \geq \frac{2c_2-c_1-b}{m}$, $E_1 < \frac{c_1-c_2}{m}$, $E_2 < \frac{c_2-b}{m} - E_1$, $E_2 \geq \frac{c_2-b-E_1m}{2m}$. Note that:

$$\frac{c_1-c_2}{m} \leq \frac{2c_1-c_2-b}{3m} \Leftrightarrow \frac{c_1+b}{2} \geq c_2$$

thus, $E_1 < \frac{c_1-c_2}{m} \leq \frac{2c_1-c_2-b}{3m}$. Since, by assumption $E_2 \geq \frac{c_2-b-E_1m}{2m}$, we are in the conditions of the NE 3.5 to 3.6.

In conclusion, the Nash equilibria were completely classified for this small sized example. The resulting Global decision trees are in Figures I.1, I.2, I.3 and I.4.

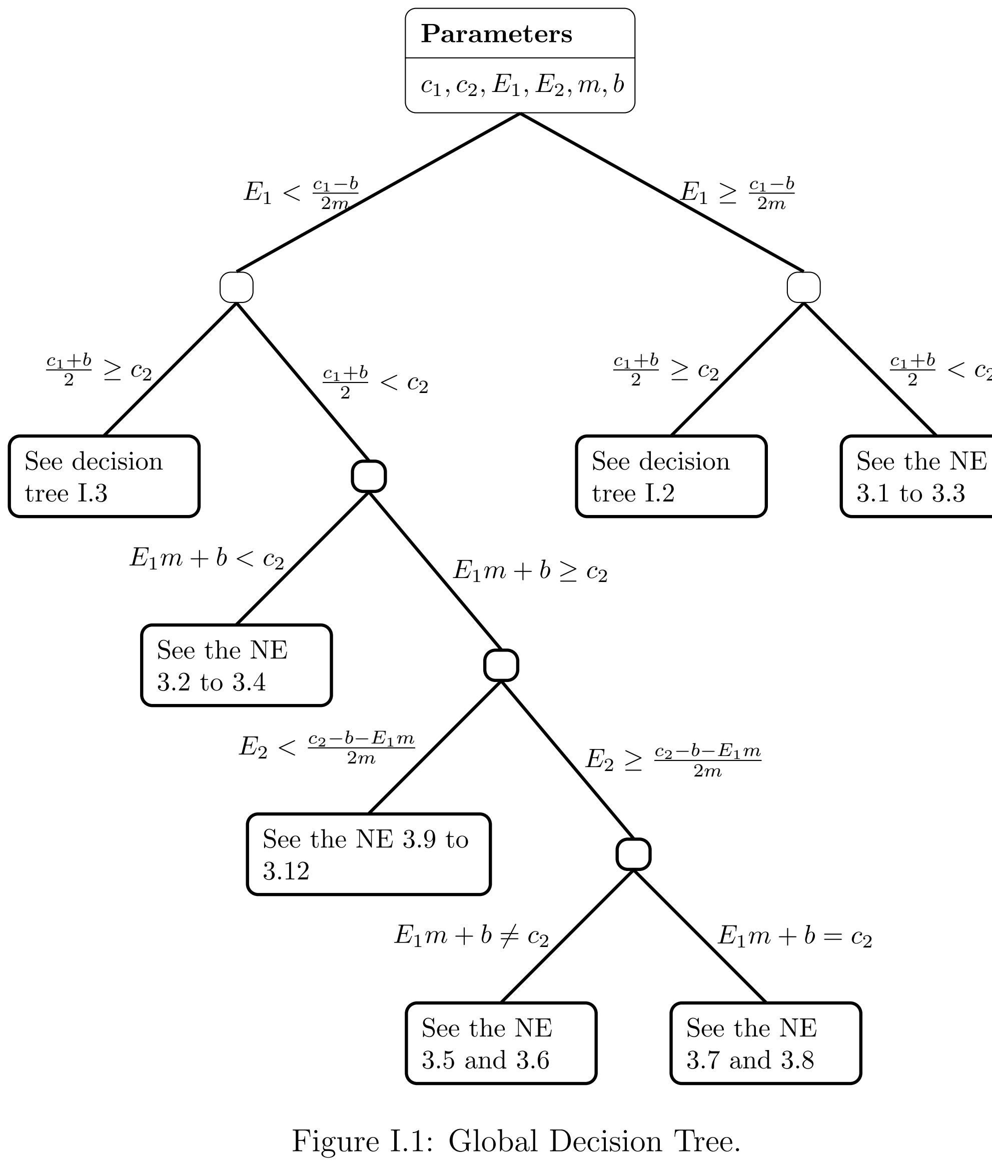

Figure I.1: Global Decision Tree.

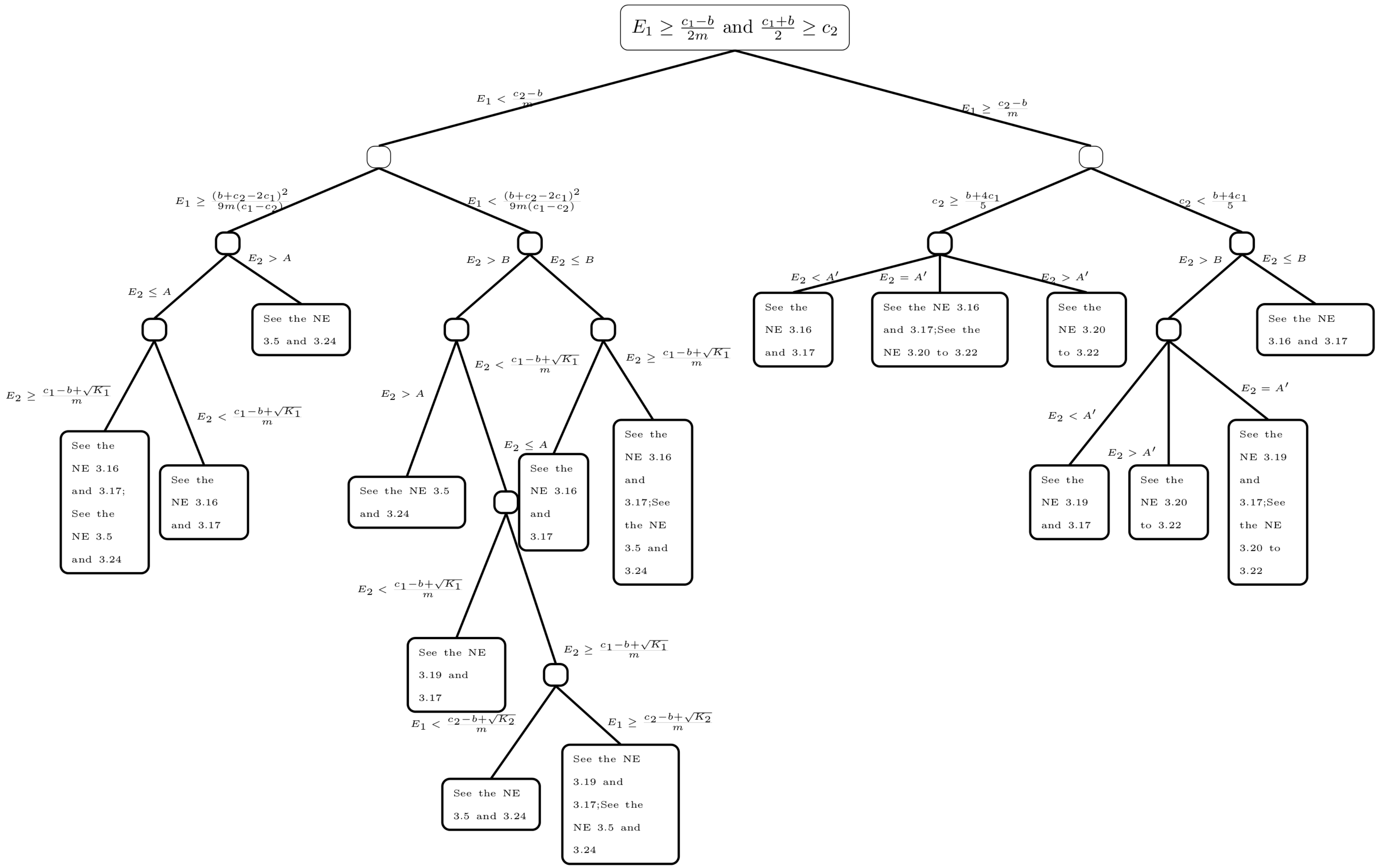

Figure I.2: Global Decision Tree - Part A.

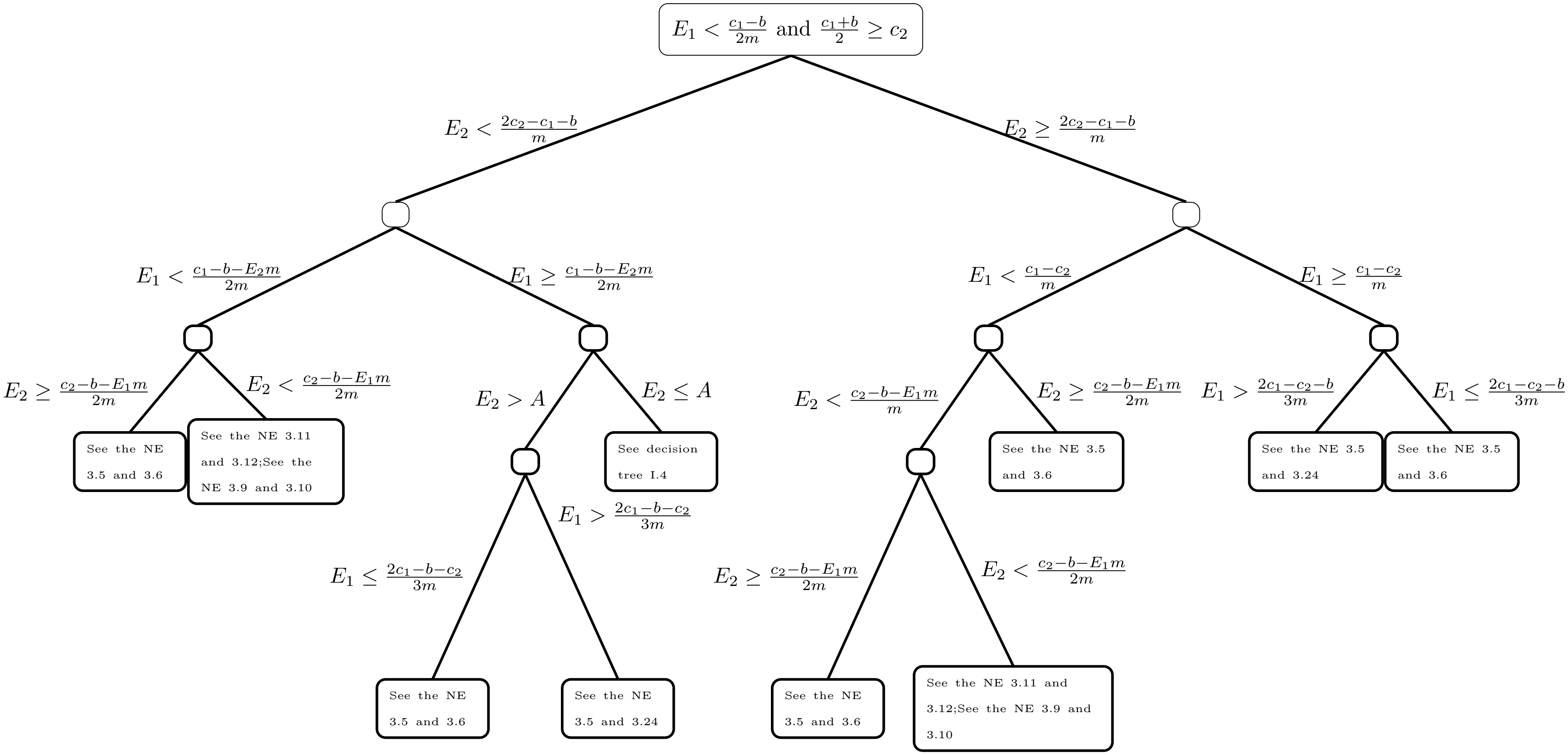

Figure I.3: Global Decision Tree - Part B.

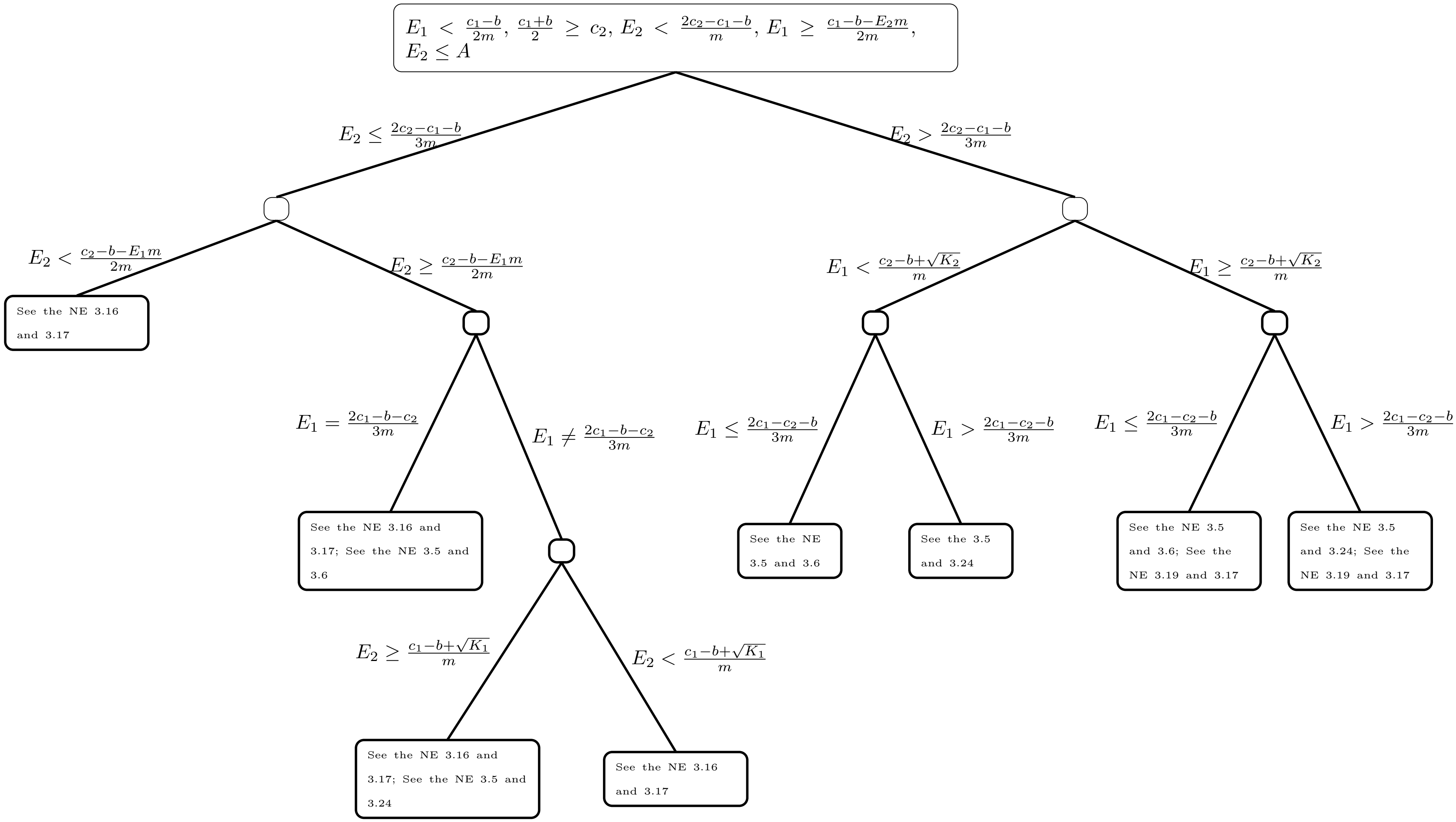

Figure I.4: Global Decision Tree - Part C.