Short proofs for MIU theorems

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Abstract

We study the MIU formal system, characterizing all possible theorems. We propose an algorithm to prove that a given formula t is a theorem, and based on that algorithm we prove that the number of lines of a minimum line proof is $O(\max\{n_u, \log|t|\})$ and the number of symbols of a minimum line proof is $O(\max\{n_u|t|, |t|\})$, where n_u is the number of $u's$ in the proof.

1 System MIU

We study some properties of the formal system MIU introduced in [Hof79].

Definition 1 *MIU is a formal system defined as follows:*

Alphabet: $\Sigma = \{m, i, u\}$

Axiom: mi

	Rules, where $x, y \in \Sigma^*$:	*Sequent calculus equivalent*[1]
(1)	$xi \to xiu$	$\dfrac{xi}{xiu}$
(2)	$mx \to mxx$	$\dfrac{mx}{mxx}$
(3)	$xiiiy \to xuy$	$\dfrac{xiiiy}{xuy}$
(4)	$xuuy \to xy$	$\dfrac{xuuy}{xy}$

Hofstadter proposed this system with the following puzzle where "Can you derive mu ?". The answer is no and the solution of the puzzle, given in [Hof79], shows that we cannot have a theorem where, n_i, the number of $i's$ is multiple of 3. However, we decide to go a little further and try to find a way of characterize the set of all possible theorems in MIU. In this paper any expression mx where $x \in \{i,u\}^*$ is called a well formed formulae (wff). Obviously only wffs can be theorems. The number of $i's$ and the number of $u's$ in a wff will be denoted respectively by n_i and n_u.

2 Algorithm for theorems

We now prove the following result:

Theorem 2 *All well formed formulas in MIU where $n_i = 3n+1$ or $n_i = 3n+2$ with $n \in I\!N$, are in fact theorems.*

Definition 3 *(Algorithm for theorems)Let t be a wff of MIU.*

1. *Substitute every occurrences of u in t by iii and let $k = n_i + 3n_u$*

$$\frac{mi^k}{t} \tag{1}$$

2. *Apply the following step until we get mi:*

   (a) *if k is even apply rule* (2)

$$\frac{mi^{\frac{k}{2}}}{mi^k}(2)$$

   (b) *else apply rule* (4), (3), (1) *and* (2)

$$\cfrac{\cfrac{\cfrac{\cfrac{mi^{\frac{k+3}{2}}}{mi^{k+3}}(2)}{mi^{k+3}u}(1)}{mi^k uu}(3)}{mi^k}(4).$$

Theorem 4 *The algorithm terminates if and only if n_i is $3n + x$, $x \in \{1, 2\}$ and $n \in I\!N$.*

Proof.
$\Rightarrow$
If the algorithm terminates, then t is a theorem and by [Hof79] n_i cannot be a multiple of 3.
$\Leftarrow$
After substitute every occurrence of u in t by iii then n_i has the form $3n + 1$ (respectively $3n + 2$).
If $3n + 1$, ($3n + 2$) is even by step 2a) we get $3\frac{n-1}{2} + 2$, ($3\frac{n}{2} + 1$).
If $3n + 1$, ($3n + 2$) is odd by step 2b) we get $3\frac{n}{2} + 2$, ($3\frac{n+1}{2} + 1$).
All divisions by 2 are in fact integer;
if $3n + 1$ is even then n is odd and $n - 1$ is even,
if $3n + 1$ is odd then n is even,
if $3n + 2$ is even then n is even and if $3n + 2$ is odd then n is odd and $n + 1$ is even.
So by applying k times step 2, we get 1 and the algorithm terminates.

$\diamond$

It follows that every theorem of MIU has a proof of this form. Let us call it the "normal form".

3 Bounds for lines and symbols

Now that we have characterized the theorems of MIU, Theorem 2, let us study the number of lines and symbols in a proof of a theorem.

3.1 Lines

In this section we establish an upper bound for the number of lines of a normal form proof.

Theorem 5 *An upper bound on the number of lines in a normal form proof of a theorem,* $t = mi^{3n+x}$, $x \in \{1,2\}$ $n \in I\!N$, *is* $4(d-2)+2$, *where* d *is the number of divisions by* 2 *in* n_i *that we must do to get* mi.

Proof. *For simplicity consider* $|t| = 3n + x$. *The worst case is achieved if, whenever we divide* n_i *by 2, we get a number which is not multiple of 2.*
The final 3 steps must be divisions by 2, *as the only way we may get one* i *is to divide* $n_i = 2$ *by* 2, ii *is to divide* $n_i = 4$ *by* 2 *and* $iiii$ *is to divide* $n_i = 8$ *by* 2.

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\frac{|t|+3}{2}+3}{2}+3}{\dots}}{2}}{2}}{2}}{} = 1$$

So we get the recourrence

$$\frac{|t| + \sum_{i=0}^{d-3} 2^i 3}{2^d} = 1 \Longleftrightarrow |t| + 3\sum_{i=0}^{d-3} 2^i = 2^d.$$

Whose solution is : $d = \dfrac{\ln(|t|-3)}{\ln 2} + 2$.
The number of lines we add whenever n_i *is not multiple of* 2 *is 4.*

$$\cfrac{\cfrac{\cfrac{\cfrac{mi^{\frac{n_i+3}{2}}}{mi^{n_i+3}}(2)}{mi^{n_i+3}u}(1)}{mi^{n_i}uu}(3)}{mi^{n_i}}(4)$$

So the upper bound for the number of lines is $l = 4(d-2)+2$.

◇

Corollary 6 *An upper bound on the number of lines in a normal form proof of a theorem* t *is*

$$n_u + 4\frac{\ln(|3n_u + n_i| - 3)}{\ln 2} + 3.$$

Proof. *By Theorem 5, the number of lines is* $1 + n_u + 4(\dfrac{\ln(|3n_u + n_i| - 3)}{\ln 2} + 2 - 2) + 2$. *In order to eliminate all the* u*'s we must replace them by* iii *so we need* $n_u + 1$ *lines.*

◇

As a consequence of this results we have

Theorem 7 *The number of lines in a normal form proof of a theorem* t *is*

$$O(\max\{n_u, \log|t|\}).$$

Example 8 *Lets see an worst case example. If we have* mi^{11}

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{mi}{mi^2}(2)}{mi^4}(2)}{mi^8}(2)}{mi^8u}(1)}{mi^5uu}(3)}{mi^5}(4)}{mi^{10}}(2)}{mi^{10}u}(1)}{mi^7uu}(3)}{mi^7}(4)}{mi^{14}}(2)}{mi^{14}u}(1)}{mi^{11}uu}(3)}{mi^{11}}(4)$$

every time we divide n_i *by* 2, *until we get* mi^8, *we get a number that is not multiple of* 2. *The number of lines in the proof is* 15, *and applying Corollary 6 we get*

$$n_u + 4\frac{\ln(|t|-3)}{\ln 2} + 3 = 0 + 4\frac{\ln(|11|-3)}{\ln 2} + 3 = 15.$$

3.2 Symbols

In this section we establish an upper bound for the number of symbols of a normal form proof.

Lemma 9 *The first step of our algorithm, that coresponds to the replacement of all occurrences of* u *by* iii, *spends*

$$\sum_{i=0}^{n_u-1}(|t|+2i) = \frac{2|t|+2(n_u-1)}{2}n_u = (|t|+n_u-1)n_u$$

symbols[2].

Proof. *Trivial.*

$\diamond$

Lemma 10 *Whenever* n_i *is not a multiple of* 2, *we need* $4|t|+9$ *symbols to reduce* n_i *to a multiple of* 2.

Proof. *If* n_i *is not multiple of* 2 *we must do*

$$\cfrac{\cfrac{\cfrac{mi^{n_i+3}}{mi^{n_i+3}u}(1)}{mi^{n_i}uu}(3)}{mi^{n_i}}(4)$$

so we need $|t|+|t|+2+|t|+4+|t|+3 = 4|t|+9$ *symbols.*

$\diamond$

Lemma 11 *An upper bound on the number of symbols,* n_s, *in the proof of a theorem* $t = mi^{3n+x}$, $x \in \{1,2\}$ $n \in I\!N$ *is given by*

$$\frac{8(\ln 2)|t| - 25\ln 2 + 21\ln(|t|-3) + n_l \ln 2}{\ln 2} \simeq 8|t| + n_l$$

where n_l *is the number of lines in the proof.*

Proof. *By Theorem 7 we know that* $d = \dfrac{\ln(|t|-3)}{\ln 2} + 2$ *is the number of divisions by* 2 *in* n_i

[2] We consider $n_u - 1$ because all the symbols of `mi*` will be considered in the next theorem.

needed to obtain mi. Removing the last 2 divisions to d, adding 7 (number of i's in the last 3 lines) to the result we get $k = \dfrac{\ln(|t|-3)}{\ln 2}$ *that represents the number of times we must apply lemma 9 and*

$$\begin{aligned}
n_s &= \sum_{i=0}^{k-1}(4\frac{|t|+3\frac{1-2^i}{1-2}}{2^i}+9)+7+n_l, \\
&= \sum_{i=0}^{k-1}{}^{k}(4\frac{|t|-3+3\times 2^i}{2^i}+9)+7+n_l \\
&= 4\,|t|\sum_{i=0}^{k-1}2^{-i}-12\sum_{i=0}^{k-1}{}^{k}2^{-i}+21\sum_{i=0}^{k-1}{}^{k}1+7+n_l \\
&= (4\,|t|-12)\sum_{i=0}^{k-1}2^{-i}+21(k+1)+7+n_l \\
&= (4\,|t|-12)(2-2^{-k})+21(k+1)+7+n_l \\
&= \frac{8\,(\ln 2)\,|t|-25\ln 2+21\ln\left(|t|-3\right)+n_l\ln 2}{\ln 2} \simeq 8\,|t|+nl
\end{aligned}$$

adding n_l we are adding the number of m's in the proof.

◇

Theorem 12 *An upper bound on the number of symbols in a proof of a theorem, t, is given by*

$$\begin{aligned}
&\frac{2\,|t-1|+2(n_u-1)}{2}n_u+\frac{8\,(\ln 2)\,|3n_u+n_i|-25\ln 2+21\ln\left(|3n_u+n_i|-3\right)}{\ln 2}+nl \\
&\simeq n_u|t|+n_u^2+8\,|t|+nl
\end{aligned}$$

Proof. *Consequence of Lemma 9 and 11.*

◇

As a consequence of this results we have

Theorem 13 *The number of symbols in a proof of a theorem t is*

$$O(\max\{n_u|t|,|t|\}).$$

Proof. *By Theorem 12, the number of symbols in the proof of a theorem t is*

$$O(\max\{n_u|t|,n_u^2,|3n_u+n_i|\,,\ln\left(|3n_u+n_i|-3\right),nl\}).$$

But

$$\begin{cases} \textit{if } n_u\neq 0 \textit{ then } n_u|t|>n_u^2, n_u|t|>|3n_u+n_i|>\ln\left(|3n_u+n_i|-3\right) \\ \textit{if } n_u=0 \textit{ them } \max\{n_u|t|,n_u^2,|3n_u+n_i|\,,\ln\left(|3n_u+n_i|-3\right)\}=|3n_u+n_i|\,, \end{cases}$$

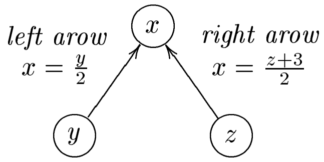
and n_l is $O(\max\{n_u,\log|t|\})$.

◇

4 The MI graph

We introduce a new representation of mi^n, $n\in I\!N$ theorem proofs. This representation is a graph based on the algorithm for theorems presented is Section 2. In the algorithm, except for mi and mii, we have 2 different ways of achieving mi^n, $n\in I\!N$ one is applying rule $2a$) of the algorithm to mi^{2n} and the other is applying rule $2b$) to mi^{2n-3}.

Definition 14 *An mi_graph is a graph* (E, V) *where* $V = \mathbb{N} \setminus \{\dot{3}\}$, *all natural numbers that are note multiple of* 3, *and* $(n, m) \in E \Leftrightarrow m = \dfrac{n}{2}$ *or* $m = \dfrac{n+3}{2}$



Part of the *mi_graph* is represented in Figure 1. We now present some proprieties of the graph.

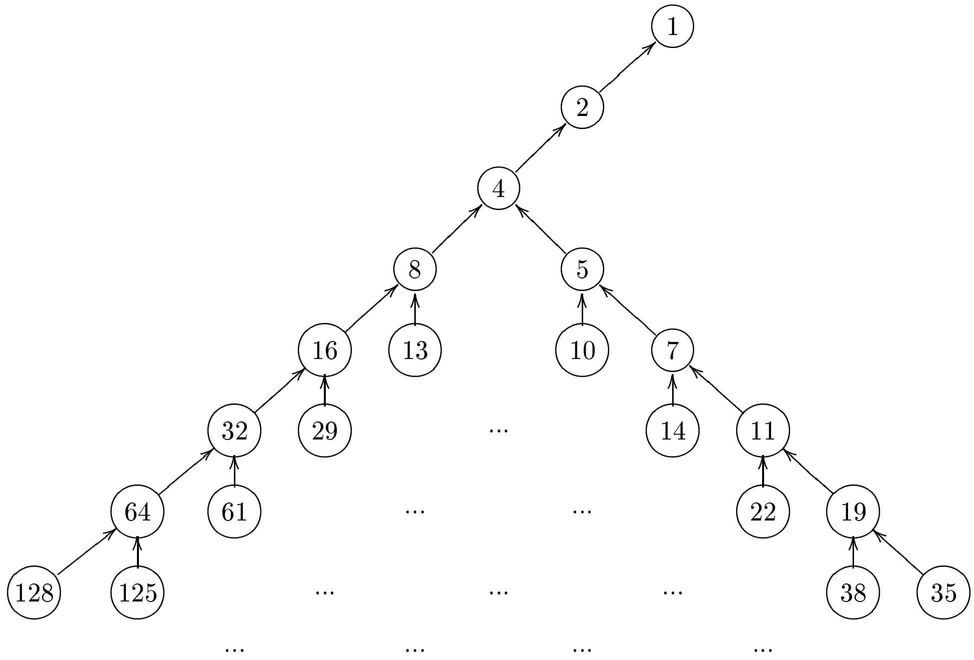


figure 1

Definition 15 *The sequence of elements at the same depth in the graph form a line.*

Lemma 16 *Some proprieties of the mi_graph:*

1. *The left-most elements in each line of the graph are powers of* 2.
2. *The right-most elements in each line* n, *except for* 1 *and* 2, *of the graph have the form* $3 + 2^n$.
3. *If the right-most element of a line is* $3 + 2^n$ *then, that line has* 2^n *elements.*
4. *Two consecutive elements of a line differ by* 3 *units.*

   Proof.

1. *All left arrows represents divisions by* 2, *so all elements are powers of* 2.
2. *By induction,*
   $n = 0; 4 = 3 + 2^0$.
   $n \Rightarrow n + 1$;
   by the induction hyphoteses the right-most element in line n *is* $3 + 2^n$ *so by a left arrow from line* $n + 1$ *to line* n *the rigth-most element of line* $n + 1$ *is* $2(3 + 2^n) - 3 = 3 + 2^{n+1}$.

3. *By induction,*
   $n = 0; 4 = 3 + 2^0$ *and this is the only element of that line.*
   $n \Rightarrow n + 1$;
   by the induction hyphoteses line n *has* 2^n *elements, as each element origin* 2 *new elements of line* $n + 1$, *in line* $n + 1$ *we have* 2^{+1} *elements.*

4. *Consider* $n > m$ *consecutive elements of the graph.*
   If $\exists x \in V : (n, x) \in E \wedge (m, x) \in E$ *then* $n = 2x$ *and* $m = 2x - 3$, *and* $n - m = 2x - 2x + 3 = 3$.
   Else $\exists x, y, z \in V : (n, x) \in E \wedge (m, y) \in E \wedge (x, z) \in E \wedge (y, z) \in E$ *so* $n - m = 2x - 3 - 2y = 4z - 3 - 4z + 6 = 3$.

$\diamond$

Based on this proprieties we are able to prove some bounds on the number of lines and symbols using the *mi_graph*.

Theorem 17 *An upper bound on the number of lines in a normal form proof of a theorem,* $t = mi^*$, *is* $4(n\_arr - 2) + 2$, *where* n_arr *is the the lenght of a path from a given element to the root.*

Proof. *Trivial*

$\diamond$

Notice that in line n of the *mi_graph* the number of lines in a proof is growing from left to right and approximating the upper bound.

With the *mi_graph* we are able to characterize a subclass of theorems mi^* for which our bounds are exact.

Theorem 18 *For theorems of the form* $mi^{3+2^n}, n \in I\!N$ *our bounds are exact.*

Proof. *Trivial, by noticing that this class of theorems are in fact the worst case for our algorithm.*

$\diamond$

Corollary 19 *For theorems of the form* mi^{3+2^n} *the bounds are in fact integers.*

Proof. $n_l = 4\dfrac{\ln(3 + 2^n - 3)}{\ln 2} + 3 = 4n + 3$.
$n_s = 8(3 + 2^n) - 25 + 21n + nl = 2^{n+3} + 25n + 2$.

$\diamond$

5 Conclusions

We have established an upper bound in the number of lines and symbols in the proof of a theorem t. However, our algorithm is not optimum, in the sense that produces the shortest proof, as we do not have the smallest proof of theorem t as result of the algorithm. Consider an example from

[Mat97a, Mat97b]

Standard proofs	Normal proofs	A shortest proof
mi	*mi*	*mi*
mii	*mii*	*mii*
miiii	*miiii*	*miiii*
miiiiiiii	*miiiiiiii*	*miiiiiiii*
miiiiiiiiiiiiiiii	*miiiiiiiiu*	*muiiiii*
muiiiiiiiiiiiii	*miiiiiuu*	*muuii*
muuiiiiiiiiii	*miiiii*	*muuiiuuii*
muuuiiiiiii	*miiiiiiiiii*	*muuiiii*
muuuiiiiiiiu	*muiiiiiii*	*muuui*
muuuiiiiuu	*muuiiii*	
muuuiiii	*muuui*	
muuuiiiiu		
muuuiuu		
muuui		

As we can see, our algorithm produces a result that is near the optimum but yet it is not the shortest proof. So, the question that remains open is if there is a better upper bound for the number of lines and symbols in the proof?

After analyzing some shortest proofs [Mat97b] we saw that one way of achieving the shortest proof using our algorithm is exploring some symmetry that can exists before replace all the u by iii.

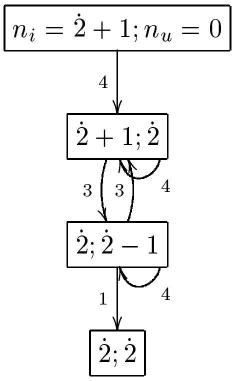
Example 20 *In theorem muuiuui before replace all the u by iii we could aply rule* (2) *and try to prove muui.*

Doing this, we can reduce the number of lines and symbols in the proof. But, exploring the possible symmetry, seems to require an exhaustive search. There are many paths for finding the symmetry.

However, there are some theorems for which the upper bound cannot be improved, our previous example, mi^{11}, is one of them. As we can see, there is not any kind of symmetry (except the one corresponding to all divisions by 2).

Theorem 21 *There is a class of theorems, $C = \{t : t = mi^{3+2n}, n \in \mathbb{N}\}$, for which our upper bound cannot be improved.*

Proof. *We have already seen, Theorem 12, that using our algorithm the bounds are exact. It is now necessary to prove that for this class of theorems there is no better algorithm. We consider the graph of possible derivations begining with mi^{3+2n}.*
For simplicity this graph is not exautive, all the derivations that are not in the graph will lead the proof to a state where the number of lines are bigger than the derivations in the graph.



Starting from mi^{3+2n}, *i.e., an odd number of* $i's$, *we will never be able to apply rule* (2), *that exploits symmetrie, unless we have an even number of* $i's$ *and* $u's$.

So as a consequence of this theorem we can state that our upper bound is optimum.

References

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