Ackermann and the superpowers

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Original version 1980, published in "ACM SIGACT News"
Modified in October 20, 2012
Modified in January 23, 2016
(working paper)

Abstract

The Ackermann function $a(m,n)$ is a classical example of a total recursive function which is not primitive recursive. It grows faster than any primitive recursive function. It is usually defined by a general recurrence together with two "boundary" conditions. In this paper we obtain a closed form of $a(m,n)$, which involves the Knuth superpower notation, namely $a(m,n) = 2 \overset{m-2}{\uparrow} (n+3) - 3$. Generalized Ackermann functions, that is functions satisfying only the general recurrence and one of the boundary conditions are also studied. In particular, we show that the function $2 \overset{m-2}{\uparrow} (n+2) - 2$ also belongs to the "Ackermann class".

1 Introduction and definitions

The "arrow" or "superpower" notation has been introduced by Knuth [1] as a convenient way of expressing very large numbers. It is based on the infinite sequence of operators:

$$+, *, \uparrow, \ldots$$

We shall see that the arrow notation is closely related to the Ackermann function (see, for instance, [2]).

1.1 The Superpowers

Let us begin with the following sequence of integer operators, where all the operators are right associative.

$$\begin{array}{rcll} a \times n & = & a + a + \cdots + a & (n\ a\text{'s}) \\ a \uparrow n & = & a \times a \times \cdots \times a & (n\ a\text{'s}) \\ a \overset{2}{\uparrow} n & = & a \uparrow a \uparrow \cdots \uparrow a & (n\ a\text{'s}) \end{array}$$

In general we define $a \overset{m}{\uparrow} n$ as

Definition 1

$$a \overset{m}{\uparrow} n = \underbrace{a \overset{m-1}{\uparrow} a \overset{m-1}{\uparrow} \cdots \overset{m-1}{\uparrow} a}_{n\ a\text{'s}}$$

The operator $\overset{m}{\uparrow}$ is not associative for $m \geq 1$. For instance, it is well known that the usual exponentiation operator is not associative. We have for instance

$$19683 = (3^3)^3 \neq 3^{(3^3)} = 7625597484987$$

In the absence of parenthesis, the implicit order of evaluation is *from right to left.* The right hand side in Definition 1 should be interpreted as

$$\underbrace{a \overset{m-1}{\uparrow} (a \overset{m-1}{\uparrow} (\cdots (a \overset{m-1}{\uparrow} a) \cdots))}_{n\ a\text{'s}}$$

Definition 1 may be extended for the cases $m = -2$, $m = -1$ and $m = 0$

$$a \overset{-2}{\uparrow} n = n + 1, \quad a \overset{-1}{\uparrow} n = a + n, \quad a \overset{0}{\uparrow} n = a \times n$$

We see that sums and products are also superpowers.

It should be noted that Definition 1 is only valid for $m \geq 0$. We have, for instance, $a \overset{-1}{\uparrow} 3 = a + 3$; this *is not* equal to

$$a \overset{-2}{\uparrow} (a \overset{-2}{\uparrow} a) = a + 2$$

Theorem 1 *For every $m \geq -1$, $n \geq 2$ we have*

$$a \overset{m}{\uparrow} n = a \overset{m-1}{\uparrow} a \overset{m}{\uparrow} (n-1)$$

Proof

$$a \overset{m}{\uparrow} n = \underbrace{a \overset{m-1}{\uparrow} a \cdots \overset{m-1}{\uparrow} a}_{n\ a\text{'s}} = a \overset{m-1}{\uparrow} \underbrace{(a \overset{m-1}{\uparrow} \cdots \overset{m-1}{\uparrow} a)}_{n-1\ a\text{'s}} = a \overset{m-1}{\uparrow} a \overset{m}{\uparrow} (n-1)$$

Let us denote $a \overset{m}{\uparrow} n$ by $f(a, m, n)$ and rephrase Theorem 1 as

$$f(a, m, n) = f(a, m-1, f(a, m, n-1))$$

For a fixed value of a, this corresponds exactly the general recurrence of the definition of the Ackermann function.

2 The Ackermann function and its generalizations

This function is the classical example of a total recursive function which is not primitive recursive. It grows faster than any primitive recursive function. The reason can be explained in intuitive terms as follows. In order to express the Ackermann function with the methods used for the construction of primitive recursive functions one would need to apply primitive recursion a *variable* number of times, that is a number of times dependent on the parameters of the function. Of course, this is not possible.

2.1 Definitions

The Ackermann function $a(m,n)$ is usually defined as (for all non-negative integer values of m and n)

$$a(m,n) = \begin{cases} n+1 & \text{if } m=0 \\ a(m-1,1) & \text{if } m \geq 1 \text{ and } n=0 \\ a(m-1,a(m,n-1)) & \text{if } m \geq 1 \text{ and } n \geq 1 \end{cases}$$

The proof that $a(m,n)$ is a total function can be based on the (total) ordering relation between pairs of nonnegative integers: $(m,n) \prec (m',n')$ iff $m < m'$ or $[m = m'$ and $n < n']$.

In order to characterize a class of functions that include the Ackermann function as a special case let us begin by defining the Ackermann functional τ as

$$\tau(f(m,n)) = f(m-1, f(m,n-1))$$

Definition 2 *The Ackermann class of functions $\mathcal{A}$ is the set of all functions $f(m,n)$ that, for all large enough values of m and n, are fixed points of τ. In other words all $f(m,n)$ such that*

$$\exists m_0\, \exists n_0\, \forall m \geq m_0\, \forall n \geq n_0 : \quad f(m,n) = \tau(f(m,n)) = f(m-1, f(m,n-1))$$

Clearly, a, the Ackermann function, is a member of $\mathcal{A}$. Other conditions may specify particular functions of $\mathcal{A}$. These are called *boundary conditions*. For the Ackermann function the boundary conditions are (see the definition of $a(m,n)$)

$$\begin{aligned} &a(0,n) = n+1 \\ &a(m,0) = a(m-1,1) \quad \text{for } m \geq 1 \end{aligned}$$

2.2 Some generalized Ackermann functions

The following is an example of a generalized Ackermann function (This is a direct consequence of Theorem 1).

$$a \overset{m}{\uparrow} n$$

Next theorem generalizes this result.

Theorem 2 *If $n+k \geq 2$ and $m+\alpha \geq 1$ then*

$$a \overset{m+\alpha}{\uparrow} (n+k) - k$$

is a generalized Ackermann function.

Proof. Using Theorem 1 we get

$$\begin{aligned} f(m,n) &= a \overset{m+\alpha}{\uparrow} (n+k) - k \\ &= a \overset{m+\alpha-1}{\uparrow} (a \overset{m+\alpha}{\uparrow} (n+k-1)) - k \end{aligned}$$

and

$$\begin{aligned} f(m-1, f(n, m-1)) &= a \overset{m-1+\alpha}{\uparrow} (f(m, n-1) + k) - k \\ &= a \overset{m-1+\alpha}{\uparrow} (a \overset{m+\alpha}{\uparrow} (n-1+k) - k + k) - k \\ &= f(m, n-1) \end{aligned}$$

The function given above, $a \overset{m}{\uparrow} n$, corresponds to the case $\alpha = k = 0$. We do not know whether the characterization of Theorem 2 includes all functions in class $\mathcal{A}$.

3 Boundary conditions for the Ackermann function

As the Ackermann function $a(m,n)$ belongs to the Ackermann class, let us see if it is identical to some function referred in Theorem 2. This search will be successful and we will obtain a closed form for the Ackermann function.

One boundary condition

We begin by imposing the boundary condition

$$a(m,0) = a(m-1,1) \quad \text{for } m \geq 1$$

to the class of functions mentioned in Theorem 2. As $n=0$, we get

$$a \overset{m+\alpha}{\uparrow} k - k = a \overset{m-1+\alpha}{\uparrow} (k+1) - k$$

for $k \geq 2$ and $m+\alpha \geq -1$. Removing $-k$ in both members and applying Theorem 1 to the left side we get

$$a \overset{m+\alpha-1}{\uparrow} (a \overset{m+\alpha}{\uparrow} (k-1)) = a \overset{m-1+\alpha}{\uparrow} (k+1)$$

This equality implies that, for $a \geq 2$ (the case $a=1$ is trivial)

$$a \overset{m+\alpha}{\uparrow} (k-1) = k+1$$

This must be true for every m such that $m+\alpha \geq -1$ (a and k are constants) so that we get 2 solutions: $k = 2$ and $a = 3$

$$3 \overset{m+\alpha-1}{\uparrow} 1 \equiv 3 \text{ for } m + \alpha \geq 0$$

and $k = 3$ and $a = 2$

$$2 \overset{m+\alpha-1}{\uparrow} 2 \equiv 4 \text{ for } m + \alpha \geq -1$$

We express these results in the following theorem

Theorem 3 *The functions*

$$2 \overset{m+\alpha}{\uparrow} (n+3) - 3 \textit{ with } m + \alpha \geq -1$$

and

$$3 \overset{m+\alpha}{\uparrow} (n+2) - 2 \textit{ with } m + \alpha \geq 0$$

belong to the Ackermann class and satisfy the boundary condition $a(m, 0) = a(m-1, 1)$ *for* $m \geq 1$.

The other boundary condition

We have now 2 candidates for the Ackermann function. Do they satisfy the other boundary condition, $f(0, n) = n + 1$?

The equality

$$2 \overset{m+\alpha}{\uparrow} (n+3) - 3 = n + 1$$

is satisfiable with $\alpha = -2$

$$(n+3) + 1 - 3 = n + 1$$

The equality

$$3 \overset{m+\alpha}{\uparrow} (n+2) - 2 = n + 1$$

is also satisfiable with $\alpha = -2$. However, for $\alpha = -2$ and $m = -1$, the condition $m + \alpha \geq 0$ is not true.

As promised, we got a closed form for the Ackermann function.

Theorem 4 *The Ackermann function is*

$$2 \overset{m-2}{\uparrow} (n+3) - 3$$

With this result let us check the values taken by the Ackermann function for $0 \leq m \leq 4$.

$$\begin{aligned}
a(0,n) &= n + 3 + 1 - 3 = n + 1 \\
a(1,n) &= n + 3 + 2 - 3 = n + 2 \\
a(2,n) &= 2n + 6 - 3 = 2n + 3 \\
a(3,n) &= 2^{n+3} - 3 \\
a(4,n) &= 2 \overset{2}{\uparrow} (n+3) - 3
\end{aligned}$$

$m \backslash n$	0	1	2	3	4
0	1 (1)	2 (2)	3 (3)	4 (4)	5 (5)
1	2 (3)	3 (4)	4 (5)	5 (6)	6 (7)
2	3 (4)	5 (7)	7 (10)	9 (13)	11 (16)
3	5 (7)	13 (25)	29 (79)	61 (241)	125 (727)

Figure 1: Some values of $a(m,n)$ (the Ackermann function) and of $g(m,n)$ (between parenthesis).

Consider for instance the value of $a(4,1)$

$$a(4,1) = 2 \overset{2}{\uparrow} 4 - 3 = 2 \uparrow (2 \uparrow (2 \uparrow 2)) - 3 = 2^{2^4} - 3 = 65533$$

Even with a fast computer, this value takes a long time to compute if the direct definition of $a(m,n)$ is used. With the superpower closed form the computation is immediate.

The values of $a(m,n)$ quickly become incredibly large. For instance, the value $a(4,4)$ is about 2 raised to a number much larger than any conceivable physical quantity (the observable number of atoms in the universe is about 10^{80}; we are talking about 2 raised to a number that is about 10^{20000}).

3.1 Another function in the Ackermann class

Let us now consider the following function mentioned in Theorem 3

$$3 \overset{m-2}{\uparrow} (n+2) - 2$$

It is easy to see that it can be characterized by

$$g(m,n) = \begin{cases} n+1 & \text{if } m = 0 \\ 3 & \text{if } m = 1 \text{ and } n = 0 \\ g(m-1,1) & \text{if } m \geq 2 \text{ and } n = 0 \\ g(m-1, g(m, n-1)) & \text{if } m \geq 1 \text{ and } n \geq 1 \end{cases}$$

The following, slightly simpler, characterization is equivalent for all positive values of m

$$g(m,n) = \begin{cases} n+3 & \text{if } m = 1 \\ g(m-1,1) & \text{if } n = 0 \\ g(m-1, g(m, n-1)) & \text{if } m \geq 2 \text{ and } n \geq 1 \end{cases}$$

It is interesting to notice that the functions $a(m,n)$ and $g(m,n)$ have different asymptotic behaviours: $g(m,n)$ grows faster than $a(m,n)$ both with m and n (see also Figure 1). However, their recursive definitions are almost identical, differing only at the point $m = 1$, $n = 0$.

Below is a computer program written in Haskell for the computation of some of these values.

```
-- a(m,n) computed directly
a 0 n = n+1
a m 0 = a (m-1) 1
a m n = a (m-1) (a m (n-1))

-- a(m,n) computed with hyperpowers
ah m n = hyp 2 (m-2) (n+3) - 3

-- g(m,n) computed with hyperpowers
g  m n = hyp 3 (m-2) (n+2) - 2

-- hyperpower function  a ^^p n
hyp a (-2) n = n+1
hyp a (-1) n = a+n
hyp a   0  n = a*n
hyp a   p  n = foldr h 1 (take n (repeat a))
    where h = \x y -> hyp x (p-1) y

-- test for functions a, ah, and g
-- [a  m n | m<-[0..3], n<-[0..4] ]
-- [ah m n | m<-[0..3], n<-[0..4] ]
-- [g  m n | m<-[0..3], n<-[0..4] ]
```

4 Open problems

In the literature other forms of the Ackermann function have been used[1]. Further research is needed to fully clarify the relationship between the Ackermann functions and the superpowers. How are the boundary conditions of the definitions related to their expressiveness in superpower notation? For instance, can the following function expressed in closed form with superpowers?

$$h(m,n) = \begin{cases} n+4 & \text{if } m=0 \\ h(m-1,1) & \text{if } n=0 \\ h(m-1,g(m,n-1)) & \text{if } m \geq 1 \text{ and } n \geq 1 \end{cases}$$

References

[1] Knuth, *Mathematics and Computer Science: coping with finiteness*, Science, vol 194, 17 December 1976.

[2] Hermes, *Enumerability, Decidability, Computability*, Springer–Verlag, 1969.

[1]Just as an example, in the handouts of the course Ma/CS 117b, the function $A(m,n)$ is defined (with argument interchange) as $A(m,0) = 2$ for $m \geq 1$, $A(0,n) = n+2$, $A(m,n) = A(m-1, A(m,n-1))$.