The formulae-as-types notion of construction

W. H. Howard[1]

1969

Transcribed in 2017 to LaTeX by Armando B. Matos

Dedicated to H. B. Curry on the occasion of his 80th birthday.

The following consists of notes which were privately circulated in 1969. Since they have been referred to a few times in the literature, it seems worth while to publish them. They have been rearranged for easier reading, and some inessential corrections have been made.

The ultimate goal was to develop a notion of construction suitable for the interpretation of intuitionistic mathematics. The notion of construction developed in the notes is certainly too crude for that, so the use of the word *construction* is not very appropriate. However, the terminology has been kept in order to preserve the original title and also to preserve the character of the notes. The title has a second defect; namely, a *type* should be regarded as a abstract object whereas a *formula* is the name of a type.

In Part I the ideas are illustrated for the intuitionistic propositional calculus and in Part II (page 6) they are applied to Heyting arithmetic.

I Intuitionistic propositional calculus

H. Curry (1958) has observed that there is a close correspondence between axioms of positive implicational propositional logic, on the one hand, and basic combina-

---

[1]Department of Mathematics, University of Illinois at Chicago Circle Chicago, Illinois 60680, U.S.A.

tors on the other hand. For example, the combinator $\mathsf{K} = \lambda\mathsf{X}.\lambda\mathsf{Y}.\mathsf{X}$ corresponds to the axiom $\alpha \supset (\beta \supset \alpha)$.

The following notion of construction, for positive implicational propositional logic, was motivated by Curry's observation. More precisely, Curry's observation provided *half* the motivation. The other half was provided by W. Tait's discovery of the close correspondence between cut elimination and reduction of λ-terms (W. W. Tait, 1965). It is convenient to use λ-terms rather than combinators. This corresponds to the sequent formulation of propositional logic.

1. Formulation of the sequent calculus
Let $\mathsf{P}(\supset)$ denote positive implicational propositional logic. The prime formulae of $\mathsf{P}(\supset)$ are propositional variables. If α and β are formulae, so is $\alpha \supset \beta$. A *sequent* has the form $\Gamma \rightarrow \beta$, where Γ is a (possibly empty) finite sequence of formulae and β is a formula. The axioms and rules of inference of $\mathsf{P}(\supset)$ are as follows.

(1.1) Axioms: all sequents of the form $\alpha \rightarrow \alpha$,

(1.2) $\dfrac{\Gamma, \alpha \rightarrow \beta}{\Gamma \rightarrow \alpha \supset \beta}$,

(1.3) $\dfrac{\Gamma \rightarrow \alpha \quad \Delta \rightarrow \alpha \supset \beta}{\Gamma, \Delta \rightarrow \beta}$,

(1.4) Thinning, permutation and contraction rules.

2. Type symbols, terms and constructions
By a type symbol is meant a formula of $\mathsf{P}(\supset)$. We will consider a λ-formalism in which each term has a type symbol α as a superscript (which we may not always write); the term is said to be of type α. The rules of term formation are as follows.

(2.1) Variables X^α, Y^β,... are terms.

(2.2) λ-abstraction: from F^β get $(\lambda\mathsf{X}^\alpha.\mathsf{F}^\beta)^{\alpha \supset \beta}$.

(2.3) Application: from $G^{\alpha \supset \beta}$ and H^{α} get $(G^{\alpha \supset \beta}H^{\alpha})^{\beta}$.

By a *construction* of a sequent $\Gamma \to \beta$ is meant a term F^{β} of type β such that for every free variable X^{α} occurring in F^{β} there is a corresponding occurrence of α in Γ (it being understood that the existence of k distinct free variables of the same type α in F^{β} is reflected by at least k occurrences of α in Γ). Thus X^{α} is a construction of $\alpha \to \alpha$ (but X^{α} is also a construction of $\beta, \alpha \to \alpha$). Another example: $\lambda X^{\alpha}.\lambda Y^{\beta}.X^{\alpha}$ is a construction of $\to \alpha \supset (\beta \supset \alpha)$.

3. Correspondence between derivation and terms

Clearly the axioms and rules of inference (1.1)–(1.3) of $P(\supset)$ correspond exactly to the rules (2.1)–(2.3) of term formation. A construction of $\Gamma \supset \beta$ is clearly also a construction of $\Gamma, \alpha \to \beta$ (thinning); similarly for a permutation of Γ; and the contraction $\dfrac{\Gamma, \alpha, \alpha \to \beta}{\Gamma, \alpha \to \beta}$ corresponds to replacing two distinct variables of type α by one variable of type α in the corresponding construction.

Hence:

Theorem 1 *Given any derivation of $\Gamma \to \beta$ in $P(\supset)$ we can find a construction of $\Gamma \to \beta$ and conversely.*

4. Interpretation of terms

For an interpretation in ordinary set theory let each propositional variable (i.e., prime type symbol) denote a specific set of basic objects. Then every type symbol can be taken to denote a set of things according to the rule: $\alpha \supset \beta$ denotes the set of all functions whose domain is a superset of α and whose range is a subset of β. (According as to whether the superset depends on the function in question, or whether it just depends on α, we get somewhat differing interpretations). The variables of type α are interpreted as ranging over the set α.

It is now clear, by induction on the rules (2.1)–(2.3) of term formation, how each term is to be interpreted as a function of the objects over which its free variables range. Thus the closed terms can be interpreted as a perfectly concrete set of functionals of finite type over the basic objects. This interpretation is used by H.

Laüchli in the paper he read at the Summer Conference at Buffalo in 1968: see (Laüchli, 1970) pp. 227–229.

Of course a constructivist would be interested in other interpretations; for example, interpretations related to the *calculation* of terms (i.e., reduction to irreducible form). It is easy to prove that the terms given above can he reduced to normal (i.e., irreducible) form by λ-contractions. The relation between this and cut elimination will now be discussed briefly.

5. Normalization of terms and cut elimination

Clearly the cut rule for $P(\supset)$ corresponds to the following rule of term formation: from F^{α} and G^{β} get $[F^{\alpha}/X^{\alpha}]G^{\beta}$ (the result of substituting F^{α} for the free variable X^{α} in G^{β}, where no free variable in F^{α} becomes hound in $[F^{\alpha}/X^{\alpha}]G^{\beta}$). Though we did not include substitution in our rules of term formation, the rule (2.3) (application) is just about as bad —from the viewpoint of obtaining irreducible terms. Professor Curry is fond of pointing out how to get irreducible terms: simply replace the rule (2.3) by:

from a variable X of type $\alpha_1 \supset (\alpha_2 \supset (\dots \supset (\alpha_n \supset \beta)\dots))$
and terms $F_1,\dots, F_n$ of types $\alpha_1,\dots, \alpha_n$, respectively, $\qquad\qquad$ (5.1)
get the term $XF_1\dots F_n$ of type β.

Correspondingly, replace the rule (1.3) of $P(\supset)$ by the n-premise rule

$$\frac{\Gamma_1 \to \alpha_1 \qquad \Gamma_2 \to \alpha_2 \qquad \dots \qquad \Gamma_n \to \alpha_n}{\Gamma_1, \Gamma_2, \dots, \Gamma_n,\, \alpha_1 \supset (\alpha_2 \supset (\dots \supset (\alpha_n \supset \beta)\dots)) \, \to \, \beta}. \qquad (5.2)$$

Of course (5.2) can be obtained by n applications of one of the Gentzen rules

$$\frac{\Gamma \to \alpha \qquad \beta, \Delta \to \gamma}{\alpha \supset \beta,\, \Gamma,\, \Delta \, \to \, \gamma}, \qquad (5.3)$$

with γ equal to β, and use of $\beta, \Delta \to \beta$. We could replace (5.1) by a rule of term formation corresponding to (5.3), but (5.1) seems more natural. As a modification of Theorem 1 we have:

4

Theorem 2 *Let* $P^\star(\supset)$ *be* $P(\supset)$ *with the rule (1.3) replaced by (5.2). Then, given any derivation of* $\Gamma \to \beta$ *in* $P^\star(\supset)$ *we can find an irreducible construction of* $\Gamma \to \beta$ *and conversely.*

Cut *elimination* can be taken to mean: the transformation of a proof of $\Gamma \to \beta$ in $P(\supset)$ into a proof of $\Gamma \to \beta$ in $P^\star(\supset)$. Thus cut elimination can be obtained as a consequence of the reduction of terms to normal form. As mentioned in §4, such reduction is easy to prove for the terms under discussion. Results following from cut elimination in $P(\supset)$ (e.g.) the nonderivability of Peirce's Law ($\alpha \supset \beta. \supset \alpha) \supset \alpha$ seem to be obtainable at least as easily from the normalizability of constructions.

6. Addition of $\neg$, $\wedge$ *and* $\vee$ *to* $P(\supset)$

Corresponding to each of these connectives we add certain closed prime terms to our supply of terms.

(i) For $\neg$ add a new prime formula f to $P(\supset)$. Then, for each formula α, introduce a term $A^{f \supset \alpha}$. As an exercise, the reader may wish to prove — for the resulting system — that there are no closed terms of type f. (By normalizability it is sufficient to prove this for irreducible terms). There *are* open terms of type f; for example, the variable X^f — which is a construction of $f \to f$.

(ii) For $\wedge$: add terms
$\quad B_1{}^{\alpha \supset (\beta \supset \alpha \wedge \beta)}$,
$\quad B_2{}^{\alpha \wedge \beta \supset \alpha}$, and
$\quad B_3{}^{\alpha \wedge \beta \supset \beta}$.

These are just pairing and projection functionals ($\alpha \wedge \beta$ is the type of a pair of terms of types α and β). We do not need to add a term of type $\beta \supset (\alpha \supset \alpha \wedge \beta)$ because such a term can be defined as $\lambda Y^\beta . \lambda X^\alpha . B_1 X^\alpha Y^\beta$.

In connection with the theory of reducibility of constructions it is useful to postulate the contraction
$\quad B_2(B_1 FG)$ contr F and
$\quad B_3(B_1 FG)$ contr G.

Then we get the following theorem.

5

Theorem 3 *Every closed irreducible term of type* $\alpha \wedge \beta$ *has the form* $B_1 F^\alpha G^\beta$, *where* B_1 *is as above.*

[*Note added 1979.* This treatment of $\neg$ and $\wedge$ does not seem to be appropriate for cut elimination in Gentzen's sequent calculus. As P. Martin-Löf soon pointed out to me, it is appropriate for D. Prawitz's theory of Gentzen's system of natural deduction. See Prawitz (1965). The terms $A^{f \supset \alpha}$ and $B_1^{\alpha \supset (\beta \supset \alpha \wedge \beta)}$ correspond to the inference rules $\dfrac{f}{\alpha}$ and $\dfrac{\alpha \quad \beta}{\alpha \wedge \beta}$, respectively, while $B_2^{\alpha \wedge \beta \supset \alpha}$ and $B_3^{\alpha \wedge \beta \supset \beta}$ correspond to $\dfrac{\alpha \wedge \beta}{\alpha}$ and $\dfrac{\alpha \wedge \beta}{\beta}$, respectively.]

(iii) For $\vee$: there are two possibilities, corresponding to the discussion of weak existence and strong existence in §12, below. Corresponding to the case of weak existence we add terms $C_1^{\alpha \supset \alpha \vee \beta}$, $C_2^{\beta \supset \alpha \vee \beta}$, and $C_3^{\alpha \vee \beta \supset (\alpha \supset \gamma. \supset (\beta \supset \gamma. \supset \gamma))}$. It is useful to postulate the contractions

$\quad\quad C_3(C_1 M)FG \;$ contr FM and

$\quad\quad C_3(C_2 N)FG \;$ contr GN

for all terms M, N, F, G of types α, β, $\alpha \supset \gamma$, $\beta \supset \gamma$, respectively. Then we get the following theorem.

Theorem 4 *Every closed irreducible term of type* $\alpha \vee \beta$ *has the form* $C_1 F^\alpha$ *or* $C_2 G^\beta$, *where* C_1 *and* C_2 *are as above.*

II Heyting arithmetic

We will be concerned mainly with the subsystem $H(\supset, \wedge, \forall)$ of Heyting arithmetic obtained by omitting $\vee$ and $\exists$. As is well-known, $\neg\alpha$ can be defined as $\alpha \supset 0 = 1$. In §12 we will make some remarks about the question of including $\exists$. Of course $\vee$ can be defined by use of $\exists$. The variables belonging to $H(\supset, \wedge, \forall)$ will be called *number variables.*

7. Constructions

Our constructions will be terms built up from prime terms by means of rules of

term formation as indicated in (ii)–(iv), below. Every term is supplied with a unique type symbol. The numerical terms — namely, the terms belonging to $H(\supset, \wedge, \forall)$ — have type 0.

(i) *Type symbols.* The prime type symbols are: 0 and every equation of $H(\supset, \wedge, \forall)$. From these we generate all type symbols by the following two rules.

    (a) From α and β get $\alpha \supset \beta$ and $\alpha \wedge \beta$.

    (b) From α and a number variable x get $\forall x\, \alpha$.

(ii) *Prime terms.* These are:

    (a) number variables x, y,...; constants 0 and 1; function symbols for plus and times,

    (b) variables X^{α}, Y^{β},...,

    (c) certain special terms, mentioned in §8, below, corresponding to axioms and rules of inference of $H(\supset, \wedge, \forall)$.

(iii) *λ-abstraction:*

    (a) From F^{β} get $(\lambda X^{\alpha}.F^{\beta})^{\alpha \supset \beta}$ as in §2.

    (b) If x does not occur free in the type symbol of any free variable of F, form $(\lambda x.F^{\beta})^{\forall x\, \beta}$.

(iv) *Application:*

    (a) From F^{α} and $G^{\alpha \supset \beta}$ form $(GF)^{\beta}$ as in §2.

    (b) From $G^{\forall x\, \alpha(x)}$ and t of type 0 form $G(t)^{\alpha(t)}$.

8. Special terms

(i) terms of types $x + 0 = x$ and $x + (y + 1) = (x + y) + 1$,

(ii) terms of types $x \cdot 0 = 0$ and $x \cdot (y + 1) = x \cdot y + x$,

(iii) a term of type $x = x$,

(iv) a term of type $x = y \supset t(x) = t(y)$ for each term $t(x)$ of type 0,

(v) the terms B_1, B_2, and B_3 discussed in §6(ii),

(vi) a term $R^{\forall y\, \beta(y)}$ for each $\beta(y)$ of the form
$$\alpha(0) \supset [\forall x\, (\alpha(x) \supset \alpha(x+1)) \supset \alpha(y)],$$
also a term $R^{\beta(n)}$ for each numeral n.

9. Constructions and derivations in $H(\supset, \wedge, \forall)$

Define constructions as in §2. As in the case of $P(\supset)$ — see §3 — the axioms of $H(\supset, \wedge, \forall)$ correspond to the existence of certain terms, and the rules of inference correspond to rules of term formation. In particular $\forall x\, \alpha(x) \to \alpha(t)$ has the construction $Y^{\forall x\, \alpha(x)}t$. If F is a construction of $\Gamma \to \alpha$, then $\lambda x.F$ is a construction of $\Gamma \to \forall x\, \alpha$. If $G(X^{\alpha(t)})$ is a construction of $\Gamma, \alpha(y) \to \beta$, then $G(Y^{\forall x\, \alpha(x)}t)$ is a construction of $\Gamma, \forall x\, \alpha(x) \to \beta$. Thus we obtain:

Theorem 5 *Given any derivation of $\Gamma \to \beta$ in $H(\supset, \wedge, \forall)$ we can find a construction of $\Gamma \to \beta$ and conversely.*

10. Interpretation of terms of $H(\supset, \wedge, \forall)$

We extend the discussion of §4 as follows.

(i) We interpret each closed formula α as a set $\alpha^\star$ in the following manner. If α is a closed equation, then $\alpha^\star$ is the singleton set $\{1\}$ if the equation is true and the set $\{0\}$ if the equation is false. If $\alpha(n)^\star$ has been defined for each numeral n, then define $(\forall x\, \alpha(x))^\star$ as the set of all functions f such $f(n) \in \alpha(n)^\star$ for every n. Define $(\alpha \supset \beta)^\star$ in terms of $\alpha^\star$ and $\beta^\star$ as in §4. Define $(\alpha \wedge \beta)^\star$ as the Cartesian product of $\alpha^\star$ and $\beta^\star$.

(ii) To each term we associate an object $F^\star$ by use of the following clauses. It should be noted that when F is closed, then the type symbol of F is a closed formula α; and $F^\star$ is an element of $\alpha^\star$.

(a) Suppose F has the form $G(x_1, \ldots, x_k)$, where $x_1, \ldots, x_k$ are the free number variables of F. Assuming that $G(n_1, \ldots, n_k)^\star$ has been defined for all numer-

8

als $n_1, \ldots, n_k$, we define $F^\star$ to be the mapping which sends each k-tuple $n_1, \ldots,$ n_k into $G(n_1, \ldots, n_k)^\star$.

(b) Suppose F has no free number variables. If F has the form $\forall x\, G(x)$, then define $F^\star$ as the mapping which takes each numeral n into $G(n)^\star$. If F does not have this form, then the free variables of F will have types $\beta_1, \ldots, \beta_k$ which have been interpreted as sets $\beta_1^\star, \ldots, \beta_k^\star$ in (i), above, and F will be a mapping from the Cartesian product of $\beta_1^\star, \ldots, \beta_k^\star$ into $\alpha^\star$.

In particular, if F has the form $\lambda Y^\beta.G$, then $F^\star$ is defined in terms of $G^\star$ by the usual interpretation of λ-abstraction.

To he systematic we must allow free variables, here and in (a), to occur vacuously.

(c) If F has the form $R_n^{\beta(n)}$ as in §8(vi), then $F^\star$ is defined by primitive recursion on n. If F is $R^{\forall y\, \beta(y)}$, then $F^\star$ is the mapping which takes each numeral n into $(R_n^{\beta(n)})^\star$.

11. Normalization of terms

For the theory of reducibility of terms we postulate the following contraction schemes

(i)
$$(\lambda X.F(X))^{\alpha \supset \beta}\, G \quad \text{contr} \quad F(G)^\beta$$
$$(\lambda X.F(X))^{\forall x\, \alpha(x)}\, t \quad \text{contr} \quad F(t)^{\alpha(t)}$$

(ii)
$$B_2(B_1 F G) \quad \text{contr} \quad F$$
$$B_3(B_1 F G) \quad \text{contr} \quad G$$

(iii)
$$R_0^{\beta(0)} F G \quad \text{contr} \quad F$$
$$R_{n+1}^{\beta(n+1)} F G \quad \text{contr} \quad G n (R_n^{\beta(n)} F G)$$
$$R^{\forall x\, \beta(x)}\, n \quad \text{contr} \quad R_n^\beta(n).$$

Also we consider the steps in the calculation of closed numerical terms to be reduction steps.

By the method of Tait (1967) it is easy to prove:

Theorem 6 *Every term can be reduced to irreducible form.*

Theorem 3 of §6 extends to the present terms. By use of Theorem 3 it is easy to

show that there is no irreducible construction of a sequent of the form $\rightarrow m = n$ with $m \neq n$. Thus by finitistic reasoning it can be proved that Theorem 6 implies the consistency of $H(\supset, \wedge, \forall)$.

12. Introduction of $\exists$

How are we to handle the existential quantifier in our theory of constructions? This appears to be a nontrivial question. The following two alternatives suggest themselves.

(i) *Weak existence.* In the formulation of $H(\supset, \wedge, \forall)$ take the sequents

$$\rightarrow \alpha(t) \supset \exists y\, \alpha(y)$$
$$\rightarrow \exists y\, \alpha(y) \supset (\forall x\, (\alpha(x) \supset \beta) \supset \beta)$$

(x not free in β) to be axioms and introduce new prime terms C_1, C_2 of types $\alpha(t) \supset \exists y\, \alpha(y)$ and $\exists y\, \alpha(y) \supset (\forall x\, (\alpha(x) \supset \beta) \supset \beta)$, respectively. Postulate the contraction

$C_2(C_1 F)G$ contr GtF

for all terms F and G of types $\alpha(t)$ and $\forall x\, (\alpha(x) \supset \beta) \supset \beta$, respectively. Theorems 3, 5 and 6 extend to $H(\supset, \wedge, \forall, \exists)$.

Corresponding to Theorem 4 we have:

Theorem 7 *Every closed irreducible term of type $\exists y\, \alpha(y)$ has the form* $C_1^{\alpha(n) \supset \exists y\, \alpha(y)} F^{\alpha(n)}$.

Thus from an irreducible construction of $\rightarrow \exists y\, \alpha(y)$ we get a numeral n and an irreducible construction of $\rightarrow \alpha(n)$ (assuming $\exists y\, \alpha(y)$ closed).

(ii) *Strong existence (choice operators).* It is natural to interpret an object of type $\exists y\, \alpha(y)$ as a pair $(t, F^{\alpha(t)})$. Thus, in §10, $(\exists y\, \alpha(y))^*$ would be defined as the set of all pairs (n, Z) such that $Z \in \alpha(n)^*$. Hence introduce projection operator P_1 and P_2 which give the required components t and $F^{\alpha(t)}$ when applied to a pair $(t, F^{\alpha(t)})$ regarded as an object of type $\exists y\, \alpha(y)$. The operators P_1 and P_2 can be considered to have types $\exists y\, \alpha(y) \supset \emptyset$ and $X^{\exists y \alpha(y)} \alpha(P_1 X^{\exists y \alpha(y)})$, respectively. This takes us outside the formalism of $H(\supset, \wedge, \forall, \exists)$; the type symbols of P_1 and P_2

are not formulae of $H(\supset, \wedge, \forall, \exists)$. Nevertheless, it is clear what meaning we are to attach to P_1 and P_2. Namely, P_1 applies to an object F of type $\exists y\, \alpha(y)$ and yields an object $P_1 F$ of type 0; and P_2 when applied to F yields an object of type $\alpha(P_1 F)$.

To illustrate the use of P_1 as a choice operator, let $\alpha(x, y)$ be a formula of $H(\supset, \wedge, \forall, \exists)$ with free variables x and y, and let F be a construction of type $\exists y\, \alpha(x, y)$. Then $\lambda x . P_1 F$ is a term ϕ satisfying $\forall x\, \alpha(x, \phi(x))$.

In general a type symbol of the form $\forall X^\alpha \beta$ arises by abstraction from a term F^β, where the variable X^α occurs in β (as well as in other locations in F). A theory of such terms would extend the theory developed in the preceding sections. In the latter theory the only variables occurring in the type symbols have type 0 (they are, namely, numerical variables of $H(\supset, \wedge, \forall, \exists)$).

It might be of interest to know the answer to the following question. Let G be a closed term obtained by extending our supply of constructions in the manner just described. Suppose G has type α where α is a formula of $H(\supset, \wedge, \forall, \exists)$. Must there be a derivation of $\rightarrow \alpha$ in $H(\supset, \wedge, \forall, \exists)$?

13. Infinite constructions

As is well-known, we do not have *cut elimination* for $H(\supset, \wedge, \forall)$ unless the axiom of mathematical induction is replaced by an ω-rule. There is no difficulty in developing a theory of constructions for the ω-rule version of $H(\supset, \wedge, \forall)$. In fact, if one uses Tait's notion of infinite terms (Tait, 1967), one gets a very simple theory (modulo the question of handling infinite terms constructively).

REFERENCES

[1] Curry, H. B. and Feys, R. (1958).
    Combinatory Logic, North-Holland, Amsterdam.

[2] Lauchli, H. (1970).
    An abstract notion of realizability for which intuitionistic predicate calcu-
    lus is complete. In "Intuitionism and Proof Theory. Proceedings of the
    Summer Conference at Buffalo N.Y." (Eds. G. Myhill, A. Kino, R. E.
    Vesley) pp. 227–234. North-Holland, Amsterdam.

[3] Prawitz, D. (1965).
    Natural Deduction, Almqvist and Wiksell, Stockholm.

[4] Tait, W. W. (1965).
    Infinitely long terms of transfinite type. In "Formal Systems and Recursive
    Functions" (Eds. J. N. Crossley and M. A. E. Dummett), North-Holland,
    Amsterdam.

[5] Tait, W. W. (1967).
    Intensional interpretations of functionals of finite type. Journal of Symbolic
    Logic 32, 198–212.