Lower bounds for deterministic communication complexity

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- ⋆ Briefly review the concept of deterministic communication complexity (DCC)

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- ⋆ Explain two methods for obtaining DCC lower bounds: “fooling sets” and “log-rank”
- ⋆ Compare those lower bounds

Outline

Basics

Two lower bound techniques

How good is the log-rank lower bound? History. . .

How do “fooling sets” and “log-rank” compare?

Further study. . .

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Goal: $\mathcal{A}$ and $\mathcal{B}$ want to compute $f(x, y)$

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  - ...
- ... until $\mathcal{A}$ (or $\mathcal{B}$) knows $f(x, y)$;
  she (he) sends that value to $\mathcal{B}$ ($\mathcal{A}$)

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  a partition of M_f in monochromatic rectangles
- The converse is not true

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communication complexity of f

$$\mathrm{CC}(f) = \min_P \{\mathrm{CC}_P(f)\}$$

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- The "leaves of a protocol" correspond to a partition of M_f into monochromatic rectangles.
- A monochromatic set $S \subset M_f$ is a fooling set if

$$\begin{array}{ccc} (x,y) \in S & \wedge & (x',y') \in S \\ & \Longrightarrow & \\ (S[x,y'] \neq m) & \vee & (S[x',y] \neq m) \end{array}$$

(thus two distinct elements of a fooling must disagree on lines <u>and</u> on columns)

Fooling set method (cont.)

The fooling set is blue, $\{(x, y), (x', y'), \ldots\}$

$$
\begin{array}{cc}
 & \begin{array}{ccccc} & y & & y' & \end{array} \\
\begin{array}{c} \\ x \\ \\ x' \\ \\ \end{array} &
\begin{bmatrix}
\ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & 1 & \ldots & 1 & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & 0 & \ldots & 1 & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots
\end{bmatrix}
\end{array}
$$

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- ...$\mathrm{CC}(f) \geq \log |S|$

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Fooling set lowerbound

$$\mathrm{CC}(f) \geq \log(\lceil |S| + 1 \rceil)$$

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Let R_i' be the matrix with the dimensions of M_f and containing the rectangle R_i (1's):

$$M_i[k,l] = \begin{cases} 1 & \text{if } (k,l) \text{ is in } R_i \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The rank method (cont)

$+ \qquad\qquad +$

$$+ \qquad = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

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\log(\operatorname{rank}(M_f)) &\leq \log m \leq \mathrm{CC}(f)
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Log-rank lowerbound

$$\mathrm{CC}(f) \geq \lceil \log(2 \times \mathrm{rank}(M_f) - 1) \rceil$$

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- To compute the rank, matrix M_f in considered over the field $\mathbb{Q}$ (or $\mathbb{R}$)

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CONJ1:

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  Because linear algebra is a very well developed branch of mathematics
- The log-rank conjecture is related to another conjecture: relation between the chromatic number of a graph and the rank of its adjacency matrix (Lovász, Saks).

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The conjecture (as we stated it) may still be true!

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FSM - “fooling set” method
LRM - “log-rank” method

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- LRM is almost always much better than FSM:
  for almost all functions f: $\mathrm{r}(f) \approx n$ and $\mathrm{fs}(f) \leq \log n + \log 10$
- LRM cannot be much worse than the FSM:
  $\forall f: \ \mathrm{fs}(f) \leq 2\mathrm{r}(f)$

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  There is a function f such that for n large enough we have $\mathrm{CC}(f) = n$ and $\mathrm{fs}(f) = O(\log n)$.
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  for almost all functions f: $\mathrm{r}(f) \approx n$ and $\mathrm{fs}(f) \leq \log n + \log 10$
- LRM cannot be much worse than the FSM:
  $\forall f: \ \mathrm{fs}(f) \leq 2\mathrm{r}(f)$
- LRM can be (slightly) worse than FSM:
  There is a function f such that $\mathrm{fs}(f) = n$ and $\mathrm{r}(f) = (\log 3)/2n \approx 0.79n$

Comparisons (cont.)

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About these results see also

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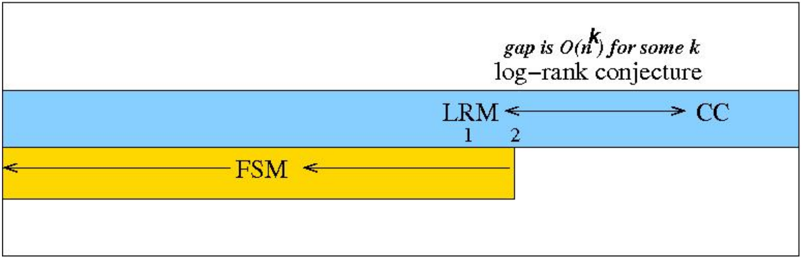
Summary. . .



Outline

Basics

Two lower bound techniques

How good is the log-rank lower bound? History...

How do “fooling sets” and “log-rank” compare?

Further study...

Comments and further study…

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- Study lower bounds for uniform communication complexity.

the end