SRL transformations can grow as fast as any primitive recursive function

May 2022

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SRL computations can grow as fast as any primitive recursive function in the sense that

For any positive integer *k* there are positive SRL programs with outputs larger than $2 \uparrow n$.

The proof is constructive: the corresponding SRL programs are described.

Note. See Knuth's notation [Knu76].

<u>Note.</u> The Ackermann function $a(m, n) = [2 \uparrow^{m-2} (n+3) - 3]$, see

[MP80, MP95], is not primitive recursive and thus can not be the output of a SRL computation.

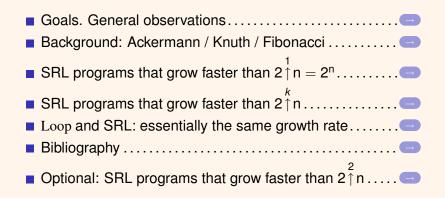
The "other direction" of the inequality,

Primitive recursive functions can grow as fast as any SRL transformation

is simpler to prove and is not discussed here.

For that purpose a small overhead simulation technique, for instance represent $x \in \mathbb{Z}$ by a pair of non-negative integers can be used.

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- For any positive integer *k*, there are SRL programs *P* with outputs that grow faster than $2 \uparrow n$, where the input values of $P(\overline{x})$ are either 0 or *n*, $n \in \mathbb{N}$. Thus $|\overline{x}| \le n$.
- The programs *P* are explicitly described, as a function of *k*.

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The SRL language is a very simple, but non-trivial, reversible total language, whose programs define bijections $\mathbb{Z}^k \to \mathbb{Z}^k$ for some positive integer *k*.

See [Mat03, MRP18, PPR16, Per14].



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SRL computation $P : \mathbb{Z}^n \to \mathbb{Z}^n$, bijection

PR computation $Q : \mathbb{N}^n \to \mathbb{N}$, function





1 SRL programs without the "dec" instruction.



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- 2 Non-negative SRL inputs.
- 3 A particular output of the SRL computation is selected.

Property. The contents of any SRL register never decreases — and thus is never negative.

k "2 \square *n*" notation / Knuth's hyperpower notation / Fibonacci sequences.

A notation used in this report:

$$2 \overset{k}{\square} n = 2^{2^{-\frac{2^n}{n}}}$$
 where the number of 2's is k.

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For instance, $2 \stackrel{1}{=} n = 2^n$.

Right associativity of exponentiation is assumed.

Knuth "hyperpower" notation [Knu76]

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 $a \stackrel{m}{\uparrow} n = \underbrace{a \stackrel{m-1}{\uparrow} a \stackrel{m-1}{\uparrow} \cdots \stackrel{m-1}{\uparrow} a}_{n a'S}$ for $m \ge 2$.

For instance,

$$2 \uparrow^2 n = 2 \uparrow 2 \cdots \uparrow 2$$
 number of 2's is *n*

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A simple property



Property. For $a, m \ge 1, n \ge 2$:

$$a^{m} n = a^{m-1} [a^{m} (n-1)]$$

(Assuming right associativity of exponentiation, the square brackets may be removed.)

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(Assuming right associativity of exponentiation, the square brackets may be removed.)

In the sequel: a = 2, and we rename *m* and *n* as *k* and *m*. For instance

$$2 \stackrel{k}{\uparrow} m = 2 \stackrel{k-1}{\uparrow} [2 \stackrel{k}{\uparrow} (m-1)]$$

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A recursive definition

$$a(m,n) = \begin{cases} n+1 & \text{if } m = 0\\ a(m-1,1) & \text{if } m \ge 1 \text{ and } n = 0\\ a(m-1,a(m,n-1)) & \text{if } m \ge 1 \text{ and } n \ge 1 \end{cases}$$



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A closed-form expression, see [MP95]

$$a(m,n) = 2 \stackrel{m-2}{\uparrow} (n+3) - 3$$



Theorem. The Ackermann function a(m, n) is not primitive recursive. \Box



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Theorem. The diagonal Ackermann function d(m) = a(m, m) is not primitive recursive. \Box



Fibonacci sequences Definition.

$$\begin{cases}
F_0(x,y) = x \\
F_1(x,y) = y \\
F_n(x,y) = F_{n-1}(x,y) + F_{n-2}(x,y)
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Note that $F_{n-2}(x, y) = F_n(x, y) - F_{n-1}(x, y)$. Thus $F_n(x, y)$ is defined for every $n \in \mathbb{Z}$ (given x and y).



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For x = 0, y = 1: $\frac{n: \dots -3 -2 -1 1 0 1 2 3 4 5 6 7 8 \dots}{F_n(0,1): \dots -3 2 -1 1 0 1 1 2 3 5 8 13 21 \dots}$

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Closed-form expression of $F_n(0, 1)$

$$F_n(0,1) = \frac{1}{\sqrt{5}}(\phi^n - \hat{\phi}^n)$$

= round(\phi^n/(\sqrt{5})) for n \ge 0

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where

•
$$\phi = (1 + \sqrt{5})/2,$$

• $\hat{\phi} = (1 - \sqrt{5})/2,$
• round(x) = $\lfloor x + 0.5 \rfloor.$

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\triangleright Programs that grow faster than 2 \uparrow n

SRL programs: lower bound $2 \uparrow n = 2^n$

A SRL program:

Q(n, a, b): for n(for b(inc a); for a(inc b)).

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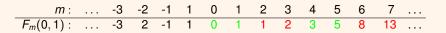
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Initial values a = 0, b = 1. Some final values a', b':

$$\begin{array}{c|cccc} n & a' & b' \\ \hline 0 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 3 & 5 \end{array} \qquad (n' = n)$$



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Recall Fibonacci...

(Proof ahead...)

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Recall Fibonacci...

$$\begin{cases} a' = F_{2n}(0,1) \\ b' = F_{2n+1}(0,1) \\ n' = n \end{cases}$$
(§)

(Proof ahead...)

Example n = 100.

 $\begin{cases} a' = 280571172992510140037611932413038677189525 \\ b' = 453973694165307953197296969697410619233826 \\ n' = 100 \end{cases}$

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Step of the Fibonacci sequence:

$$\begin{cases} a' = b \\ b' = a + b \end{cases} \quad \text{matrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

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Loop body of program "for n(for b(inc a); for a(inc b))":

$$\begin{cases} a'' = a+b \\ b'' = a+2b \end{cases} \quad \text{matrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \times \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$



Theorem 1. Let $R(n, a, b) = \operatorname{inc} b$; for $n(\operatorname{for} b(\operatorname{inc} a)$; for $a(\operatorname{inc} b))$. After the computation R(n, 0, 0) the final contents of a and b satisfy

$$a'(n) > 2^n$$
 for $n \ge 4$
 $b'(n) > 2^n$ for $n \ge 3$.



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The main result of this report: for every positive integer *k* there are SRL programs that grow faster than lower bound $2 \uparrow^{k} n$

Theorem 2. For every $k \ge 1$ there is a SRL program using k + 2 registers such that, if all the registers are initialized with $n \ge 2$, then all the registers have a final contents of at least $2 \uparrow n$. \Box

(Thus the registers a and b are also initialized with n.)



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$$2 \stackrel{k}{\square} n = 2^{2 \cdot \dots \cdot 2^n} \qquad \qquad \#(2^*s) = k$$

= 2 \cdot 2 \ldots \cdot 2 \cdot n \qquad \pm #(2^*s) = k
\ge 2 \cdot 2 \ldots \cdot 2 \cdot 2 \qquad for n \ge 2, \pm #(2^*s) = k + 1
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where #(2's) denotes the number of 2's.

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Proof



The proof is by induction on k

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The proof is by induction on k

Statement of the theorem

 $\forall k \in \mathbb{N}^+, \ \exists (\mathsf{SRL program} \ P : \mathbb{Z}^{k+2} \to \mathbb{Z}^{k+2}) : \\ \forall n \in \mathbb{N}^+, n \ge 2 : \ P(\overline{n})|_{\mathsf{all}} \ge 2 \stackrel{k}{\uparrow} n$



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 $P(\overline{n})|_{all}$: the final contents of all the registers, when all the initial contents are n.

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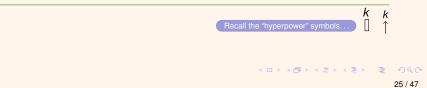
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Recall that T(n, a, b) = inc b; inc n; inc n; inc n; inc n; Q(n, a, b); Q(a, b, n)

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Recall that T(n, a, b) = inc b; inc n; inc n; inc n; inc n; Q(n, a, b); Q(a, b, n) $\boxed{k = 1}$ Let n = n, a = 0, b = 0 be the initial values. We have seen that $T(n, n, n) \ge T(n, 0, 0) \ge 2^n = 2 \boxed{1} n \ge 2^{\uparrow} n$

for every output of T.

Proof, case k = 1, continuation







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Summary of the proof

Let Q(n, a, b)="for n(for b(inc a); for a(inc b))"

Q
$$(n, 0, 1)$$
: $a' = F_{2n}(0, 1), b' = F_{2n+1}(0, 1), n' = n.$

■
$$Q(n,0,1)|_{a,b} \ge \frac{1}{\sqrt{5}}(\phi^{2n} - \hat{\phi}^{2n}) \ge 2^{r}$$

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$$T(n,n,n)|_{all} \geq 2^n = 2 \uparrow n$$

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$$T(n,n,n)|_{all} \ge 2^n = 2 \uparrow n$$

Thus the program *P* of the statement may be T(n, a, b). QED

$k \Rightarrow k + 1$

Assume that all the k + 2 registers \overline{x} have the initial contents n. IH, induction hypothesis: $P(\overline{x})|_{all} \ge 2 \uparrow^k n$.



 $k \Rightarrow k + 1$

Assume that all the k + 2 registers \overline{x} have the initial contents n. IH, induction hypothesis: $P(\overline{x})|_{all} \ge 2 \uparrow^{k} n$.

Consider the sequence $U(m, \overline{x}) =$ "for $m(P(\overline{x}))$ ", which, with the initial contents of the new register *m* also equal to n is (semantically)

$$U(n, \overline{x}) = \underbrace{\underbrace{P; \dots P}_{n-1}}^{n}; P(\overline{x}).$$
 Compare with

$$2 \stackrel{k+1}{\uparrow} n = \underbrace{2 \stackrel{k}{\uparrow} 2 \stackrel{k}{\uparrow} \dots 2 \stackrel{k}{\uparrow} 2}_{n2's}$$

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Usual convention: *U* is executed from left to tight, $2 \stackrel{k+1}{\uparrow} n$ is interpreted from right to left.

(Recall: the initial contents of every element of \overline{x} is *n*.)

Leftmost $P(\overline{x})$ is executed first and (by the IH): $P(\overline{x}) \ge 2 \stackrel{k}{\uparrow} n$.

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(Recall: the initial contents of every element of \bar{x} is *n*.) Leftmost $P(\bar{x})$ is executed first and (by the IH): $P(\bar{x}) \ge 2 \uparrow^k n$. $P(\bar{x}) \ge 2 \uparrow^k n \ge 2 \uparrow^k 2$ (for $n \ge 2$) and " $2 \uparrow^k 2$ " is also at the right of the expression $2 \uparrow^{k+1} n$.

(Recall: the initial contents of every element of \overline{x} is *n*.)

Leftmost $P(\bar{x})$ is executed first and (by the IH): $P(\bar{x}) \ge 2 \uparrow n$. $P(\bar{x}) \ge 2 \uparrow n \ge 2 \uparrow 2$ (for $n \ge 2$) and " $2 \uparrow 2$ " is also at the right of the expression $2 \uparrow n$.

The leftmost sequence " $(P; P)(\overline{x})$ " satisfies $(P; P)(\overline{x}) \ge P(2\uparrow 2)$ (The second *P* receives all inputs > 2 $\uparrow 2$)

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(Recall: the initial contents of every element of \overline{x} is *n*.)

Leftmost $P(\bar{x})$ is executed first and (by the IH): $P(\bar{x}) \ge 2 \uparrow n$. $P(\bar{x}) \ge 2 \uparrow n \ge 2 \uparrow 2$ (for $n \ge 2$) and " $2 \uparrow 2$ " is also at the right of the expression $2 \uparrow n$.

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 $(P; P)(\overline{x}) \ge P(2\uparrow 2)$ (IH)
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... and this is exactly the rightmost sequence with three 2's of $2 \stackrel{k+1}{\uparrow} n = \underbrace{2 \stackrel{k}{\uparrow} 2 \stackrel{k}{\uparrow} \cdots 2 \stackrel{k}{\uparrow} 2 \stackrel{k}{\uparrow} 2 \stackrel{k}{\uparrow} 2}_{ (n+1)}$

... and so on... More formally, use induction on *k*. We get

$$U(m,\overline{\mathbf{x}}) \ge 2 \stackrel{k+1}{\uparrow} \mathbf{n},$$
 (*)

assuming that the k + 3 parameters (m, \overline{x}) are initialized with n

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This finishes the proof by induction on k. \Box



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Loop and SRL: the same lower bounds

In Loop and SRL programs, the (absolute value of the) maximum contents of the registers is essentially the same.

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the lowest upper bounds of PR functions and of (the final register contents of) SRL programs are essentially the same.

The theorem...



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Primitive recursive (PR) functions and the SRL transformations have essentially the same maximum growth rate in the sense that

For every $k \ge 1$ there is a PR function f(n) that grows faster than $2 \uparrow n$.

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- No SRL transformation grows faster than $2^{''}_{\uparrow} n$. □



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Theorem 3.

Primitive recursive (PR) functions and the SRL transformations have essentially the same maximum growth rate in the sense that

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- For every $k \ge 1$ there is a SRL program using k + 2 registers such that, if all the registers are initialized with $n \ge 2$, then all their final contents are at least $2 \uparrow n$.
- No PR function f(n) grows faster than the diagonal Ackermann function $2 \stackrel{n}{\uparrow} n$.

■ No SRL transformation grows faster than $2 \uparrow n$. \Box

The end

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> Optional section



This is optional material: Another proof of the existence of SRL programs with lower bound $\geq 2 \stackrel{k}{\Box} n^{"}$.

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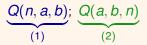
This is optional material: Another proof of the existence of SRL programs with lower bound $\geq 2 \stackrel{k}{\square} n^{"}$.

The program $\text{Hyper}_k(n, a, b)$:

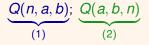
Line	Instruction
	inc b;
1	Q(n, a, b);
2	Q(a, b, n);
3	Q(n, a, b);
4	Q(a, b, n);
k	$\begin{cases} k \text{ even: } Q(a, b, n) \\ k \text{ odd: } Q(n, a, b) \end{cases}$



The sequence:



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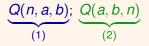


Lower bounds of (1) and (2) for initial value $n \ge 4$:

$$(1) \begin{cases} n' = n \\ a' > 2^{n} \\ b' > 2^{n} \end{cases} (2) \begin{cases} a'' = a' \\ b'' > 2^{a'} \\ n'' > 2^{a'} \end{cases}$$

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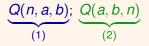
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For *b*" we get

$$b^{\prime\prime}>2^{a^\prime}\Rightarrow b^{\prime\prime}>2^{(2^n)}$$

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For b" we get

$$b^{\prime\prime}>2^{a^\prime}\Rightarrow b^{\prime\prime}>2^{(2^n)}$$

Similarly: $a'' > 2^n$, $n'' > 2^{(2^n)}$.

Recalling the notation...



$$2 \overset{k}{\square} n = 2^{2 \overset{n}{\dots}^{2^{n}}}$$
$$= 2 \uparrow 2 \dots \uparrow 2 \uparrow n$$
$$\geq 2 \overset{2}{\uparrow} k \quad \text{for } n \geq 2$$

where the number of 2's is k

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Recalling the notation...



$$2 \overset{k}{\square} n = 2^{2 \cdot 2^{2''}}$$
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where the number of 2's is k

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More generally,

$$\begin{array}{rcl}
a^{\uparrow}n &=& a^{\uparrow}n = a^{n} \\
a^{m} &=& \underbrace{a^{\uparrow}n}_{n a \uparrow} a^{m-1} \cdots \uparrow a}_{n a \cdot S} & \text{for } m \geq 2.
\end{array}$$

SRL The theorem...



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Theorem 4.

For every positive integer k, there is a SRL program Pr(n, a, b) such that, in the computation Pr(n, 0, 0) and for every $n \ge 0$, the final contents of the registers satisfy

$$n'(n), a'(n), b'(n) > 2 \overset{k}{[]} n$$

The proof, I: a sequence



Lower bounds.

	Sequence of instructions	mem[n]	mem[<i>a</i>]	mem[b]
	inc b;	4	0	1
1	for n(for b(inc a); for a(inc b));			
	for <i>b</i> (inc <i>a</i>); for <i>b</i> (inc <i>n</i>)	2 🛛 n	1 2 🛛 <i>n</i>	2 🛛 <i>n</i>
2	for a(for n(inc b); for b(inc n));			
	for <i>b</i> (inc <i>a</i>); for <i>b</i> (inc <i>n</i>)	2 🛛 n	2 [] n	2 2 🛛 n
3	for n(for b(inc a); for a(inc b));			
	for <i>b</i> (inc <i>a</i>); for <i>b</i> (inc <i>n</i>)	3 2 🛛 n	3 2 🛛 n	3 2 🛛 n



Proof: generalize the previous program! Bottom lines for k odd:

		mem[<i>n</i>]	mem[<i>a</i>]	mem[b]
k	 for <i>n</i> (for <i>b</i> (inc <i>a</i>); for <i>a</i> (inc <i>b</i>)); for <i>b</i> (inc <i>a</i>); for <i>b</i> (inc <i>n</i>)			
		2 🛛 n	2 🛛 n	2 🛛 n

Bottom lines for *k* even:

		mem[<i>n</i>]	mem[<i>a</i>]	mem[b]
k	for $a(\text{for } n(\text{inc } b); \text{ for } b(\text{inc } n));$	•••	•••	
		k	k	k
	for <i>b</i> (inc <i>a</i>); for <i>b</i> (inc <i>n</i>)	2 🛛 n	2 🛛 n	2 🛛 n

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