# Trying to solve Peg Solitaire by a simple, depth-first search

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**Abstract**

In this paper we invite the reader to test his skillness in previewing the behaviour of a simple, depth-first search algorithm that solves (or tries to solve) the Peg-solitaire game; the test is based on a few questions about the search tree involving aspects such as the number of nodes explored by the algorithm, the influence of the ordering of the peg moves on the efficiency of the computation and the "average branching factor" of internal nodes. We refrain from presenting at this moment the conclusions of this paper, so that the reader can fully enjoy the experiment he is now going to take part in.

## 1 Introduction

The efficiency of depth-first search algorithms and the use of heuristics to speed up the search have always been an important practical issue in Artificial Intelligence. This paper concerns the use of a simple depth-first algorithm for the old game of Peg Solitaire and some of its variants. This is a very well known game; it is briefly explained in Appendix 1. Reference [BCG82] gives very interesting "structured" ways for solving the game. But here we are not interested in intelligent ways to solve the game but rather in the behaviour of a "stupid" program, i.e. one that does not use any heuristic to guide the search.

Figure 1 contains a listing of a straightforward recursive function written in an informal language, that tries to solve the game of Peg Solitaire. The function uses a simple depth-first search *without any heuristics*. The pegs are considered in a top to bottom, left to right order and all possible moves are tried.

Does the program solve the game in useful time? If yes, how many board configurations are visited? If not, what strategy should we use? What is the average "branching factor" (as defined

```
boolean solve(int pegs){
  int x, y;
  direction d;            // by order: NORTH, EAST, SOUTH, WEST
  for x=1 to 7
    for y=1 to 7
      if (x,y) is occupied                  // (has a peg then)
      for d=NORTH to WEST{
        if (x,y)+(dx[d],dy[d]) is occupied
           (x,y)+(2*dx[d],2*dy[d]) is empty //  (without a peg)
        then{
          move;
          if solve(pegs-1)
            return TRUE
          undo the move;
        }
      }
  return(FALSE);
}
```

Figure 1: A depth-first function that solves Peg-solitaire. The external call is "`solve(32)`" The board is an external 2-dimensional array initialized with the starting configuration of Solitaire. Each cell of the board may contain either "occupied", "empty" or "wall". The arrays "`dx`" and "`dy`" specify for each direction the corresponding unit vector; for instance, `dx`[`EAST`] = `1`, `dy`[`EAST`] = `0`. A comment starts with "`//`" and extends to the end of the line.

below)?

We invite the reader to test his skillness in previewing the behaviour of this simple, depth-first search by answering a few questions similar to these. It is very important to try to answer all the questions **before looking at the answers**. Do not turn to the answer of a question until you have made a genuine effort to solve it! The interested reader may find in [ABM98] listing of several programs so that he can to check our results or, by suitable changes, to continue the experimental research on this area.

Before presenting the questions in next Section, let us characterize more precisely the search tree associated with an algorithm. The search space of the game may be represented by a directed

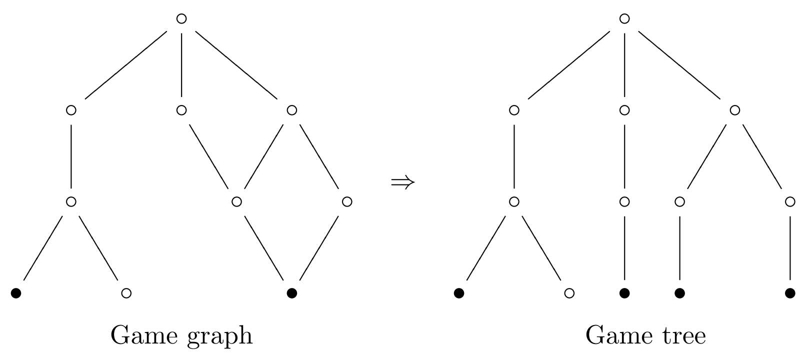


Figure 2: The graph of the game (at left) is seen by the search algorithm as a tree (at right).

graph. The graph is acyclic because every move reduces the number of pegs by one. It is not however a tree as the reader can easily check.

Our program, being as simple as possible, does not detect repetitions of board configurations so that the *graph* should be expanded to a tree as illustrated in Figure 1.

In our case, the game graph has no cycles and the maximum depth of a node equals the number of pegs in the initial configuration minus 1.

We now define several parameters related to a search. The definitions are illustrated with the game tree represented at the right side of Figure 1 where terminal nodes marked "•" correspond to solutions.

- $d$: Depth of the search (or height of the tree). In this game it is one less the number of pegs. In the example $d = 3$; in the solitaire game, $d = 33 - 1 = 32$.
- $s$: Number of nodes visited until the first solution is found. In the example, assuming a left to right, depth-first search, we have $s = 4$ corresponding to the path from the tree to the leftmost solution ("•").
- $m_s = s - 1$: Number of moves made until the first solution is found. A move corresponds to a branch of the tree. In the example, $m_s = 3$.
- $t$: Total number of nodes visited. In the example, $t = 13$.
- $m_t = t - 1$: Total number of moves which equals the number of branches in the game tree. In the example, $m_t = 12$.
- $i$, $e$: Number of internal and external nodes, respectively. In the example, $i = 8$, $e = 5$.
- $b$: Average branching factor (number of sons) or simply *branching* of internal nodes. In the example, for the complete search, $b = (3+1+2+1+1+2+1+1)/8 = 12/8 = 1.5$. Average

branching can be defined for every partial search, for instance, until finding the first solution, as

$$b = \frac{\text{Total number of moves taken so far}}{\text{Number of nonterminal nodes visited so far}}$$

– $s$: Number of solutions, corresponding in the game graph to the number of paths from the root to a "•" node. In the example, $s = 4$.

The number of "holes" is 33 and the number of possible boards (if there are 33 pegs available) equals $= t = 2^{33} = 8\,589\,934\,592$.

An upper bound on the number of possible moves from a board with $n$ pegs is

$$m \leq m_u = 4^{n-1} \times n!$$

Proof: each peg has at most 4 legal moves.

The time of a computation is essentially dependent on the number of moves done. Current PC's (using the GNU C compiler under the Linux operating system) typically examine between $10^5$ and $10^6$ moves per second.

The rest of this paper is organized as follows. Sections 2 and 3 contain, respectively, 6 questions about a simple depth-first search for the solution of Peg Solitaire and the corresponding answers. In Sections 4 you can read about your performance which is based on the number of correct answers. Finally some conclusions are presented in Section 5.

# 2 The questions

## Computation time

**Question 1 (Computation time)** *The program in Appendix 2*

1. *Takes less than half second to compute a solution, exploring about 20 000 nodes.*
2. *Takes about ten minutes to compute a solution, exploring about 700 million nodes.*
3. *Takes about one hour to compute a solution, exploring about 4,000 million nodes.*
4. *Does not terminate within ten hours time, after exploring more than $2 \times 10^9$ nodes.*

## Complete searches

**Question 2 (A complete search)** *The number $m_t$ of moves made during a complete search of the solitaire game and the corresponding number $s$ of solutions satisfy (select the strongest statement)*

1. $m_t \geq 10^4$, $s \geq 10^2$
2. $m_t \geq 10^5$, $s \geq 10^3$
3. $m_t \geq 10^7$, $s \geq 10^4$
4. $m_t \geq 10^9$, $s \geq 10^5$

## Ordering of the directions in the search

The program scans the board in a top to bottom, left to right direction. For each peg it finds, the possible moves are tried in the following order: North (N), East (E), South (S) and West (W). A move is possible (and executed) if the two next places in the corresponding direction are respectively occupied and vacant. There is no special reason for using this particular order; we have tested the program behaviour with all $24 = 4!$ possible sequences of directions. Recall that the original configuration of the solitaire game is symmetrical.

**Question 3 (Influence of order)** *For the 24 possible orderings of the 4 directions considered for the move of a peg (see the programs in [ABM98]),*

1. *The number of nodes examined is always the same for each of the 24 orderings.*
2. *There are exactly 24 possible numbers of examined nodes.*
3. *There are exactly 6 possible numbers of examined nodes.*
4. *There are exactly 3 possible numbers of examined nodes.*

*What are the best (corresponding to less nodes explored) direction orderings?*

## Average number of branches

Recall that, for each search node, the branching is defined as the total number of moves taken so far divided by the number of nonterminal nodes visited so far; terminal nodes are not considered.

**Question 4 (Branching)** *For both the fireplace and the solitaire games, the branching $b$ of a first solution search satisfies*

1. $2.02 \leq b \leq 2.18$
2. $1.40 \leq b \leq 1.94$
3. $4.16 \leq b \leq 4.40$
4. $16.0 \leq b \leq 31.8$

### Number of nodes at a specific level

Consider a complete search in the game “fireplace”. Obviously, there is only one node at depth 0 corresponding to the initial confiuration; at depth 10 (number of pegs minus 1) there are as many nodes as there are solutions to the problem (8).

**Question 5 (Number of nodes as a function of depth)** *In a complete search in the game “fireplace”, how many nodes are there at depths 0, 1, 2,. . . ,10?*

1. *1, 5, 20, 80, 350, 1272, 2532, 4860, 5854, 846, 8.*
2. *1, 5, 814, 927, 1011, 1218, 1415, 2223, 2776, 4311, 8.*
3. *1, 5, 17, 138, 217, 516, 1337, 1629, 1890, 3412, 8.*
4. *1, 5, 31, 90, 274, 618, 1840, 2017, 3099, 1126, 8.*

### Number of nodes as a function of the number of sons, at a specific level

In this question the reader should guess the total number of configurations of a certain game – “fireplace” – at a certain depth – when there are 5 pegs remaining – that have 0 sons (terminal nodes), 1 son,. . . , 7 sons.

**Question 6 (Number of nodes/number of sons)** *Consider a complete search in the game “fireplace”. At depth 6, that is, in boards with 5 pegs, the number of configurations having 0, 1,. . . , 7 sons (there are not nodes with more than 7 nodes) are respectively*

1. *101, 811, 321, 118, 44, 31, 14, 2.*
2. *110, 990, 894, 146, 356, 0, 32, 4.*
3. *80, 112, 229, 384, 128, 61,31, 1.*
4. *15, 97, 244, 1199, 889, 244,98, 7.*

## 3 The answers

### Computation time

**Question 1 (Computation time, correct answer: 1)**

*The program took less than half second, after making 20 278 moves.*

| Problem | Pegs | Nodes visited until first solution |
|---|---|---|
| Cross | 6 | 12 |
| Cross⋆ | 6 | 12 |
| Plus | 9 | 75 |
| Plus⋆ | 9 | 76 |
| Fireplace | 11 | 5 941 |
| Fireplace⋆ | 11 | 5 941 |
| Up-arrow | 17 | 17 998 001 |
| Up-arrow⋆ | 17 | 17 998 001 |
| Pyramid | 16 | 797 378 |
| Pyramid⋆ | 16 | 797 379 |
| Diamond | 24 | 8 528 473 |
| Diamond⋆ | 24 | 8 528 474 |
| Solitaire | 32 | 20 278 |
| Solitaire⋆ | 32 | 20 279 |

Figure 3: Number of nodes explored when finding the first solution

**Surprise:** To my great surprise, the program solves solitaire almost immediately. ⋄

The quite unexpected results corresponding to the number of nodes visited until the first solution is found (for Peg solitaire and the other versions of the game) are summarized in Figure 3.

**Surprise:** Solitaire, usually considered the most challenging problem, is not the hardest for our program. "Diamond" takes much more time to solve. ⋄

**Surprise:** The challenging versions of a game (names ending with "⋆" in Figure 3) are almost as hard as the non-challenging ones. In fact, the difference in the number of nodes visited never exceeds 1! ⋄

## Complete searches

### Question 2 (A complete search, correct answer: 4)

*There are more than $2 \times 10^9$ nodes accessible from the initial position. The number of solutions exceeds 690 000.*

Figure 3 shows the number of accessible nodes and the number of solutions for 3 of the problems.

| **Problem** | **Pegs** | **Nodes visited** | **No. of solutions** |
|---|---|---|---|
| Cross | 6 | 32 | 4 |
| Plus | 9 | 580 | 32 |
| Fireplace | 11 | 15 827 | 8 |
| Pyramid | 16 | 735 033 270 | 10 142 448 |
| Solitaire | 32 | $> 33 \times 10^9$ | $> 3\,400\,000$ |

Figure 4: Complete search from the initial position: number of nodes visited and total number of solutions.

## Ordering of the directions in the search

**Question 3 (Influence of order, correct answer: 4)**

*The correct answer is: There are exactly 3 possible numbers of examined nodes. See Figure 3*

Figure 3 shows that there are 3 possible number of moves: $a = 20\,275$, $b = 20\,278$ and $c = 7\,667\,769$. We have

– $a$ occurs for the orderings NSEW, NSWE, SNEW, SNWE, NWSE and SENW.

– $b$ occurs for the orderings NESW, NEWS, ENWS, ENSW, NWES and ESNW.

– $c$ occurs for the other 12 orderings.

**Surprise:** The initial configuration of solitaire is symmetric and all 4 directions are equivalent. Some people think that, due to the symmetry of the initial configuration, the search is not influenced by the ordering of the directions. However that is false and the reason is that the pegs are considered in a non-symmetric sequence – top to bottom, left to right – and symmetry is quickly destroyed. $\diamond$

## Average number of branches

**Question 4 (Branching, correct answer: 1)**

*For both the fireplace and the solitaire games, the branching $b$ of a first solution search satisfies*

$$2.02 \leq b \leq 2.18$$

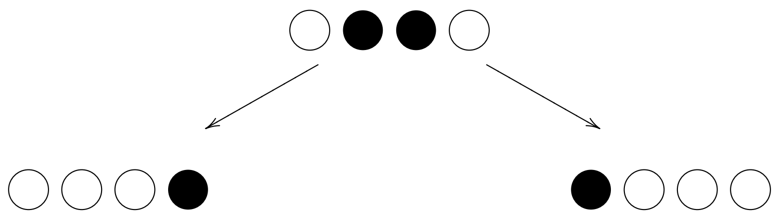
**Surprise:** In fact, whenever the search involves a relatively large number of nodes, the branching factor is very nearly 2 (see Figure 3). $\diamond$

The following reasoning, although rather incomplete, may in part justify this claim.

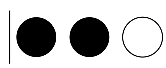
| **Order** | **Moves** | **Order** | **Moves** | **Order** | **Moves** |
|---|---|---|---|---|---|
| NESW | 20 278 | NWES | 20 278 | NSWE | 20 275 |
| ESWN | 7 667 769 | WESN | 7 667 769 | SWEN | 7 667 769 |
| SWEN | 7 667 769 | ESNW | 20 278 | WENS | 7 667 769 |
| WNES | 7 667 769 | SNWE | 20 275 | ENSW | 20 278 |
| NEWS | 20 278 | NWSE | 20 275 | NSEW | 20 275 |
| EWSN | 7 667 769 | WSEN | 7 667 769 | SEWN | 7 667 769 |
| WSNE | 7 667 769 | SENW | 20 275 | EWNS | 7 667 769 |
| SNEW | 20 275 | ENWS | 20 278 | WNSE | 7 667 769 |

Figure 5: Influence of order in the number of moves to find the first solution.

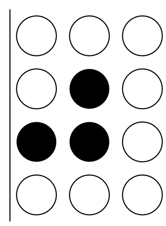
Let us call a node "next to leaves" if every move from it results in a terminal configuration. Very often for those nodes there are exactly 2 possibilities for the last move (before reaching a terminal node) and for the next to the last move; the corresponding configuration and transitions are something like (for the last move and where "dead" pegs are not represented).



For all these nodes the branching is 2. There are obviously other, less frequent, possibilities: the node may have for instance only one son:



It may also have 3 or more sons; in the following case, it has 3



| Problem | Nodes visited | Branching factor |
|---|---:|---:|
| Plus at first solution | 75 | 1.9737 |
| Plus, complete search | 580 | 2.0939 |
| Fireplace at first solution | 5 941 | 2.1706 |
| Fireplace, complete search | 15 827 | 2.2160 |
| Pyramid at first solution | 10 142 448 | 2.0843 |
| Pyramid, complete search | 735 033 270 | 2.1876 |
| Solitaire at first solution | 20 278 | 2.0205 |

Figure 6: Branching factor for some problems.

But what about higher (farther from the leaves) nodes? [1] The configurations of these nodes seem to have usually at least two sons. If they have exactly 2 sons they obviously contribute to the validity of the "2 branching factor claim". If they have 3 or more sons, it is easy to see that they are much less in number than the "next to the leaves" nodes so that the branching factor is essentially determined by the "next to the leaves" nodes.

## Number of nodes at a specific level

**Question 5 (Number of nodes as a function of depth, correct answer: 2)**
*The number of configurations (or nodes) as a function of depth is*

| Depth | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Nodes | 1 | 5 | 20 | 80 | 350 | 1272 | 2532 | 4860 | 5854 | 846 | 8 |
| Growth | | 5.00 | 4.00 | 4.00 | 4.37 | 3.63 | 1.99 | 1.92 | 1.20 | 0.14 | 0.0095 |

In the previous table we have also included the "growth" factor defined as

$$g(d) = \frac{\text{N. of nodes at depth } d}{\text{N. of nodes at depth } d-1}$$

The function $g(d)$ is relatively smooth, having a value between 4 and 5 up to depth 4 and then quickly decreasing until $d = 10$.

## Number of branches at a specific level

**Question 6 (Number of nodes/number of sons, correct answer: 2)**
*The number of configurations with 5 pegs as a function of the number of sons is*

[1] Notice that we not talking about nodes depth (distance from the root) but about their minimum distance to a leave.

| | Number of sons | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **Pegs** | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 11 | | | | | | 1 | | | |
| 10 | | | | 2 | 1 | 2 | | | |
| 9 | | | 2 | 8 | 6 | | 2 | | 2 |
| 8 | | | 20 | 4 | 26 | 6 | 10 | 8 | 6 |
| 7 | | 10 | 56 | 112 | 68 | 88 | 10 | 6 | |
| 6 | 126 | 540 | 122 | 300 | 114 | 44 | 18 | | 8 |
| 5 | 110 | 990 | 894 | 146 | 356 | | 32 | 4 | |
| 4 | 2204 | 424 | 1734 | 30 | 468 | | | | |
| 3 | 5396 | 70 | 388 | | | | | | |
| 2 | 842 | | 4 | | | | | | |
| 1 | 8 | | | | | | | | |

Figure 7: Fireplace problem: number of nodes having a given number of sons at each depth (depth=11-pegs). For clarity, when the number of nodes is 0, it is not represented. For pegs=1 there are 8 terminal nodes corresponding to the 8 solutions. Only the initial configuration has 11 pegs, having – as can be easily verified – 5 sons.

| *Number of sons* | *0* | *1* | *2* | *3* | *4* | *5* | *6* | *7* |
|---|---|---|---|---|---|---|---|---|
| *Number of nodes* | *110* | *990* | *894* | *146* | *356* | *0* | *32* | *4* |

**Surprise:** The number of sons as a function of the number of nodes and depth seems to be quite “chaotic”. Figure 3 shows this dependence for the fireplace problem. ⋄

# 4 Measure your perfomance

Read the correct answers below and check the your answers, counting the total number of correct ones. As there are 4 possible answers for each question, a random selection corresponds to an expected number of correct answers of $6/4 = 1.5$ so that 2 or less correct answers corresponds to very poor perfomance.

**2 or less correct answers:** Very poor.

**3 correct answers:** Poor.

**4 correct answers:** Fair.

**5 correct answers:** Good.

**6 correct answers:** Very good.

As explained in more detail in Section 5, we think that the questions proposed to the reader (or at least some of them) are intrinsically difficult so that your possibly bad score is quite comprehensible.

## 5 Conclusions and further research

Several competent people have tested their predictions by answering the questions described in this article. Their results were in general quite poor – and they often got very surprised when they saw the answers. But there are good reasons for this

– Trivial modifications of the program (like changing the ordering of the directions considered for the move of a peg) can have dramatic and unexpected effects on its running time.

– The search tree seems to be rather irregular; see for instance the table in Figure 3 where, for each depth, the number of nodes in function of the number of sons is represented.

We conjecture that the detailed behaviour of search algorithms for hard problems is very difficult to predict. One should be suspicious about quick predictions – even when made by experienced people – for they often turn out to be wrong. Apparently, only exhaustive and careful analysis can lead to reliable predictions. However, due to the seemingly "chaotic" behaviour of the search, we think that in many cases such analysis might be impossible.

We have been considering specific search trees of specific problems. In a more general setting we may consider the search trees associated with problems in certain *classes*; in particular it may be interesting to think about difficult instances of NP-complete problems. While many NP-complete problems seem to have a relatively small number of hard instances (an interesting example is the random 3-SAT problem discussed in [HHW96, CA96, GW96]), there are others such that every algorithm takes super-polynomial time almost everywhere; each problem which is not in P has such a "complexity core" (a hard sub-problem), see [Lynch75, BD87, ESY85]. In such cases the search trees are, by definition, large (except for a finite number of instances). We conjecture that, in this case, they are also complex, not in the sense of having a large Kolmogorov complexity ([LV94]) – the algorithm together with a particular instance are a short description of the tree – but in the computational sense; in other words we think that, for every algorithm that searches the solution of an NP-complete problem, no polynomial time algorithm can answer many of the questions

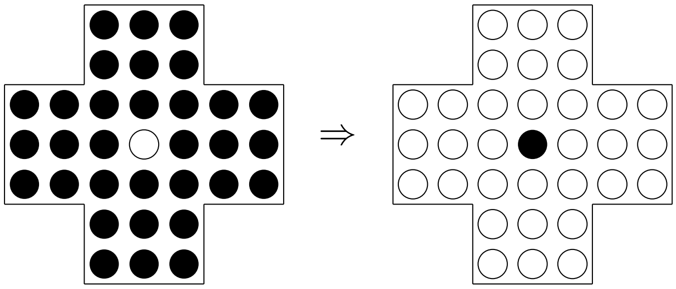


Figure 8: The Solitaire game. In the normal version, "solitaire", the final configuration must contain only one peg. In the challenging version, "solitaire$\star$", the final peg must be at the center of the board.

related with the corresponding search tree; one such difficult question (and this is admitly a trivial observation) is clearly the following: does the tree have a leaf which is a solution?

Must the structure of search trees associated with any algorithm that decides an NP-problem, be in some sense (and for difficult instances) computationally "chaotic"? The formalization of this question and its proof or disproof, corresponds to a clarification of the relationship between the complexity of a problem and the complexity of the corresponding search tree [2] (or graph); this is clearly an area deserving further work.

## Appendix 1: The rules of Peg Solitaire

*Peg Solitaire* is played by jumping a peg across any adjacent peg and placing it in an open space on the other side. Only horizontal and vertical moves are legal. A winning configuration is a board with only one peg. In a more challenging version of the game, the last peg must be at the center of the board.

In the "solitaire" version, the initial board has a peg in every hole except at the center; see Figure 8. Other common initial board configurations are shown in Figures 9 and 10.

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[2] More generally, instead of talking about the complexity of the search tree – which only makes sense for search algorithms – we could talk about the complexity of the corresponding *computation*.

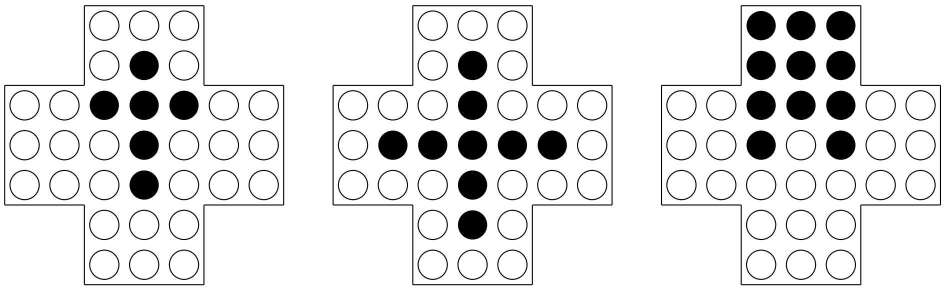

Figure 9: Left to right: initial configurations for versions “cross”, “plus” and “fireplace”. In the challenging versions, denoted by “cross⋆”, “plus⋆” and “fireplace⋆”, the final peg must be at the center of the board.

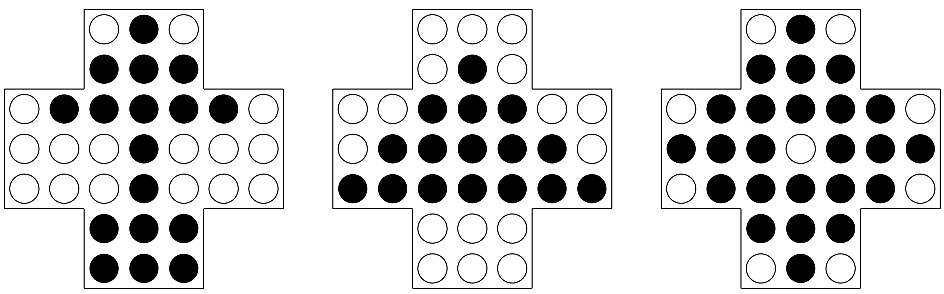

Figure 10: Left to right: initial configurations for versions “up”, “pyramid” and “diamond”. In the challenging versions, denoted by “up⋆”, “pyramid⋆” and “diamond⋆”, the final peg must be at the center of the board.

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