The combinatorial properties of a small subset of the reversible language SRL

Relation with the modular group $\mathrm{SL}_2(\mathbb{Z})$

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Abstract

We study the $\mathbb{Z}^2$ transformations that can be implemented in $\mathsf{SRL\text{-}FOR}$, a reversible subset of the language SRL. The programs of $\mathsf{SRL\text{-}FOR}$ consist in sequences of zero or more instructions of one of the forms "$\mathsf{for}\, x(\mathsf{inc}\, y)$", "$\mathsf{for}\, x(\mathsf{dec}\, y)$", "$\mathsf{for}\, y(\mathsf{inc}\, x)$", and "$\mathsf{for}\, y(\mathsf{dec}\, x)$". Thus, at most two programming registers are used in a program. The $\mathsf{SRL\text{-}FOR}$ language is equivalent to the linear subset of SRL limited to two registers and having no outer inc nor dec instructions.

A closely related combinatoric problem is the following: define a partition of $\mathbb{Z}^2$ such that (x, y) and (x', y') belong to the same set of the partition iff it is possible to move from (x, y) to (x', y') using a finite number of steps of the form: add or subtract one coordinate from the other. For instance, the application of one step to $(2, 5)$ may result in $(2, 7)$, $(2, 3)$, $(7, 5)$, or $(-3, 5)$.

Due to the reversibility of $\mathsf{SRL\text{-}FOR}$, the sets of points reachable from a given initial point (x, y) form a partition of $\mathbb{Z}^2$ which is related to the value $\gcd(x, y)$.

The close relation between the language $\mathsf{SRL\text{-}FOR}$ and the modular group $\mathrm{SL}_2(\mathbb{Z})$ is discussed.

A brief presentation of the paper

A sub-language of SRL

Language SRL-FOR: a program is a sequence (concatenation) of zero or more instructions of the forms

$$\mathsf{for}\,x(\mathsf{inc}\,y) \qquad \mathsf{for}\,x(\mathsf{dec}\,y) \qquad \mathsf{for}\,y(\mathsf{inc}\,x) \qquad \mathsf{for}\,y(\mathsf{dec}\,x)$$

Thus, a program uses at most 2 registers.

Points of $\mathbb{Z}^2$ accessible from (0,1)

Definition: a point $(\mathsf{x}', \mathsf{y}') \in \mathbb{Z}^2$ is accessible (or "reachable") from the point (x, y) if there is a SRL-FOR program P such that $P((\mathsf{x}, \mathsf{y})) = (\mathsf{x}', \mathsf{y}')$.

Example: in the text we show that all the points of a certain "square" of $\mathbb{Z}^2$ are accessible from the point (0,1).

Notation. The set of all points accessible from the point (x, y) is denoted by

$$\mathsf{S}_{(x,y)} = \{P((x,y)) : P \in \mathsf{SRL\text{-}FOR}\}$$

These sets form a *partition* of $\mathbb{Z}^2$. The reversibility of the SRL-FOR language is essential for the symmetry of the "accessibility" binary relation.
Figures 3, 4, 6, and 7 illustrate parts of the sets $\mathsf{S}_{(0,1)}$.

The general result: points accessible from $(0, a)$

The sets $\mathsf{S}_{(x,y)}$ form a partition of $\mathbb{Z}^2$ because

— The empty program is in SRL-FOR
— SRL-FOR is closed for concatenation (symbol ";")
— SRL-FOR is reversible (very important)

Two pairs of points (x, y) and (w, z) belong to the same set of the partition iff $\gcd(\mathsf{x}, \mathsf{y}) = \gcd(\mathsf{w}, \mathsf{z})$, see Theorem 13. See also Figure 9.

Other topics:

The average distance to the origin.
SRL-FOR and the modular group: see Figure 1 and equation (9).

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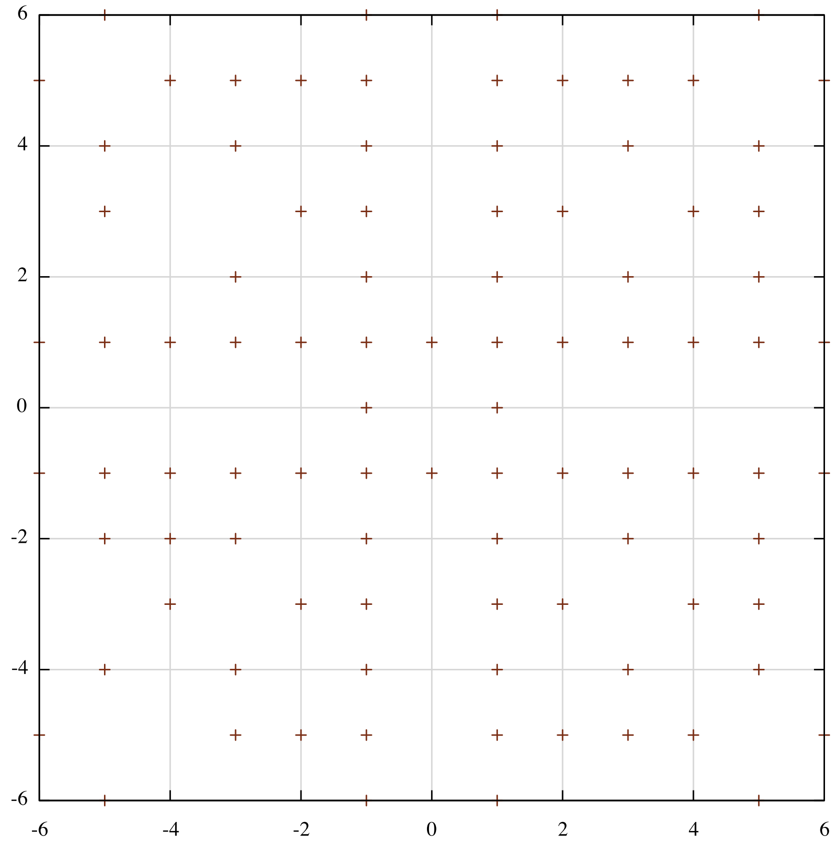

Figure 3: Let x and y be the contents of registers x and y, respectively. The factor n has been removed. Start in $\mathbb{Z}^2$ in the point $(0, 1)$, that is, the initial contents of x and y are 0 and 1, respectively. The points in the range $[-6, 6] \times [-6, 6]$ that are shown are accessible from the point $(0, 1)$. As explained in the text, some accessible points, like (-4,-5) may be not represented.

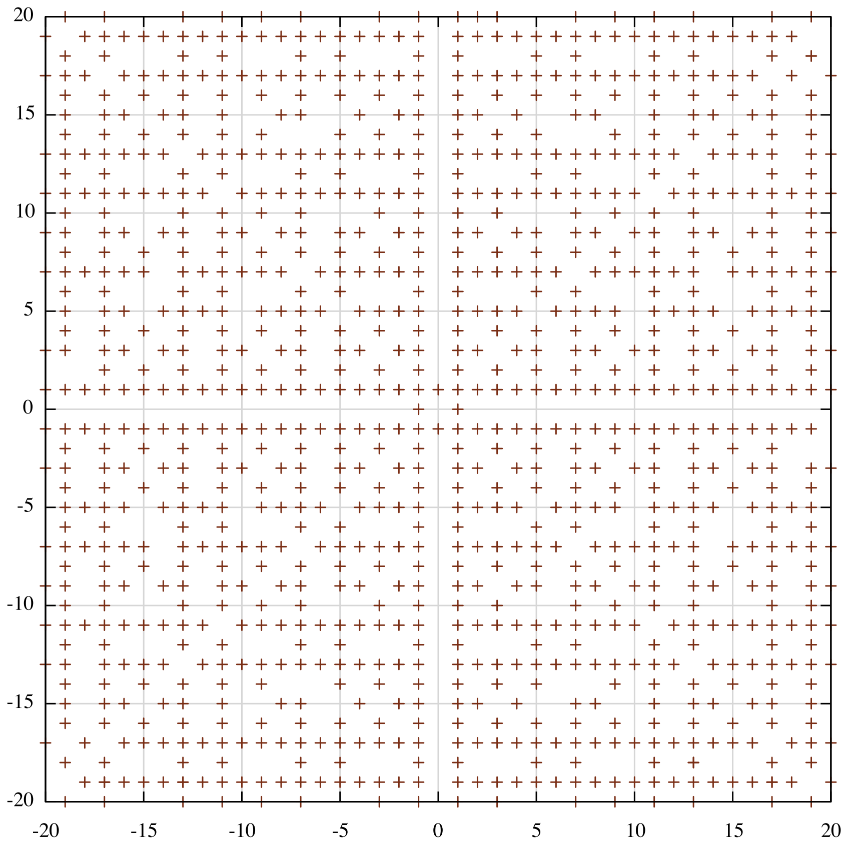

Figure 4: Similar to Figure 3, but with range $[-20, 20] \times [-20, 20]$.

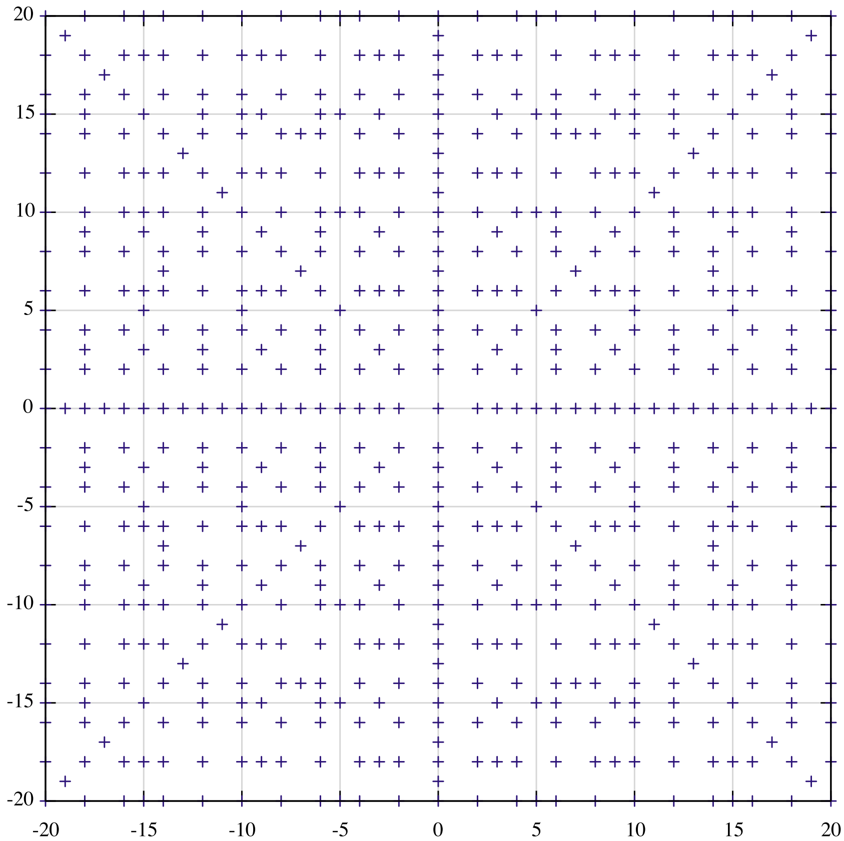

Figure 5: The complement of the set in Figure 4, that is, points in the range $[-20, 20] \times [-20, 20]$ that are *not* accessible from $(0, 1)$.

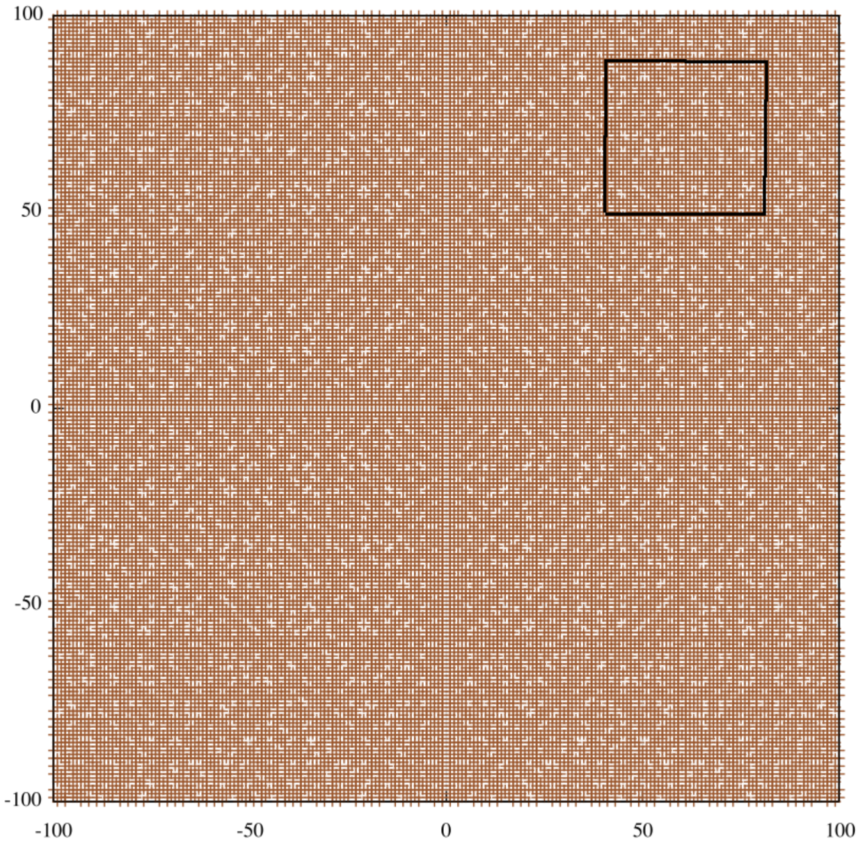

Figure 6: Similar to Figure 3, but with range $[-100, 100] \times [-100, 100]$. See also Figure 7, page 36, where the boxed area is shown in detail.

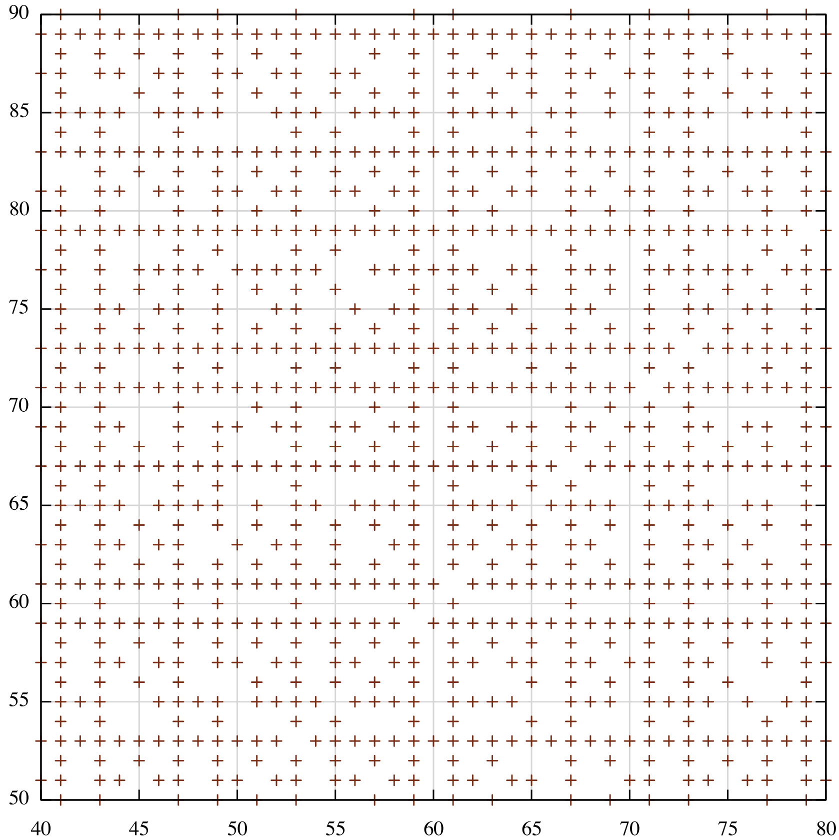

Figure 7: A close-up zoom over the diagram of Figure 6, page 35. The range displayed is $[40, 80] \times [50, 90]$. Apparently, the distribution of points accessible from the point $(0, 1)$ — or, equivalently, from any point of the diagram — is rather "homogeneous".

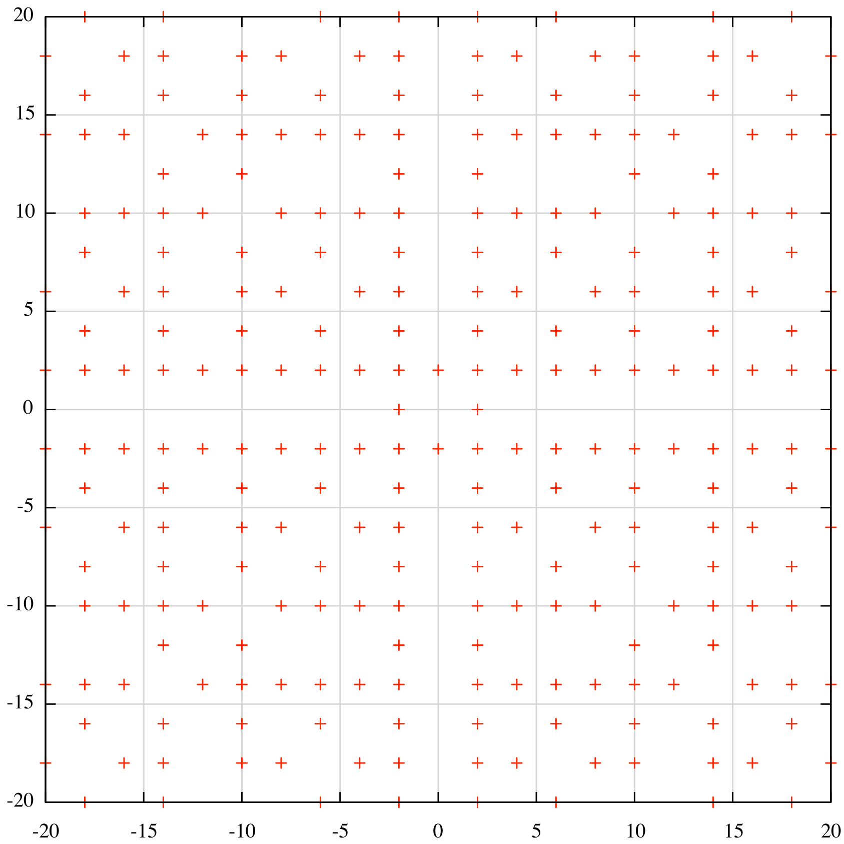

Figure 8: Points accessible from $(2,2)$ in range $[-20,20]\times[-20,20]$.

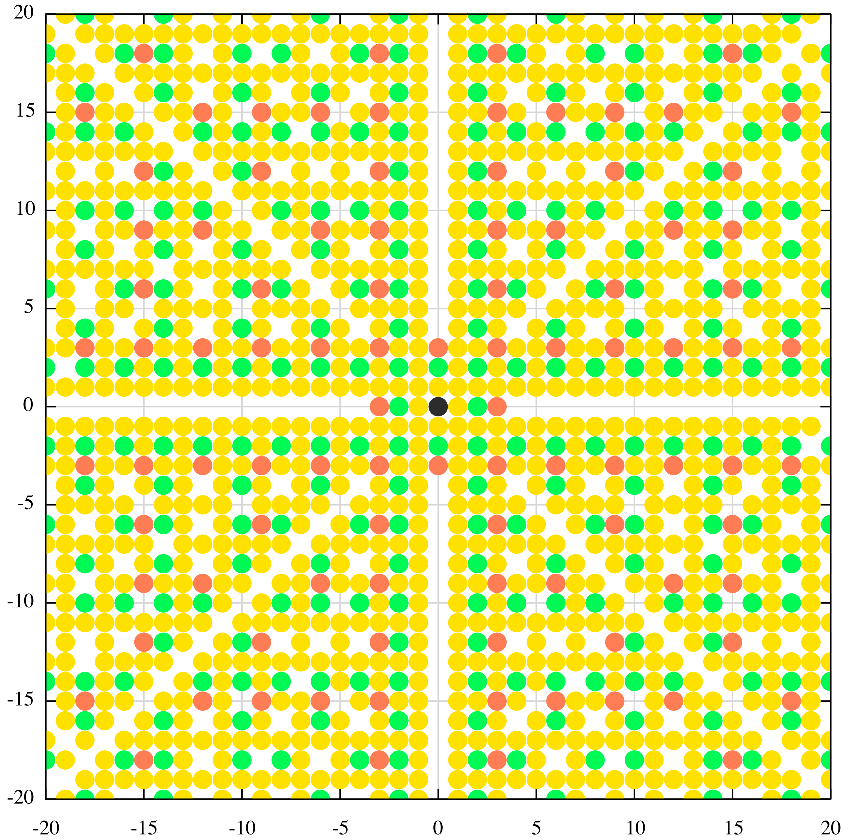

Figure 9: Four partitions of $\mathbb{Z}^2$: points accessible
from $(0,0)$: set $\{(0,0)\}$, coloured black,
from $(0,1)$: coloured yellow,
from $(0,2)$: coloured green,
from $(0,3)$: coloured red.
See also Figures 4 (page 33) and 5 (page 34).

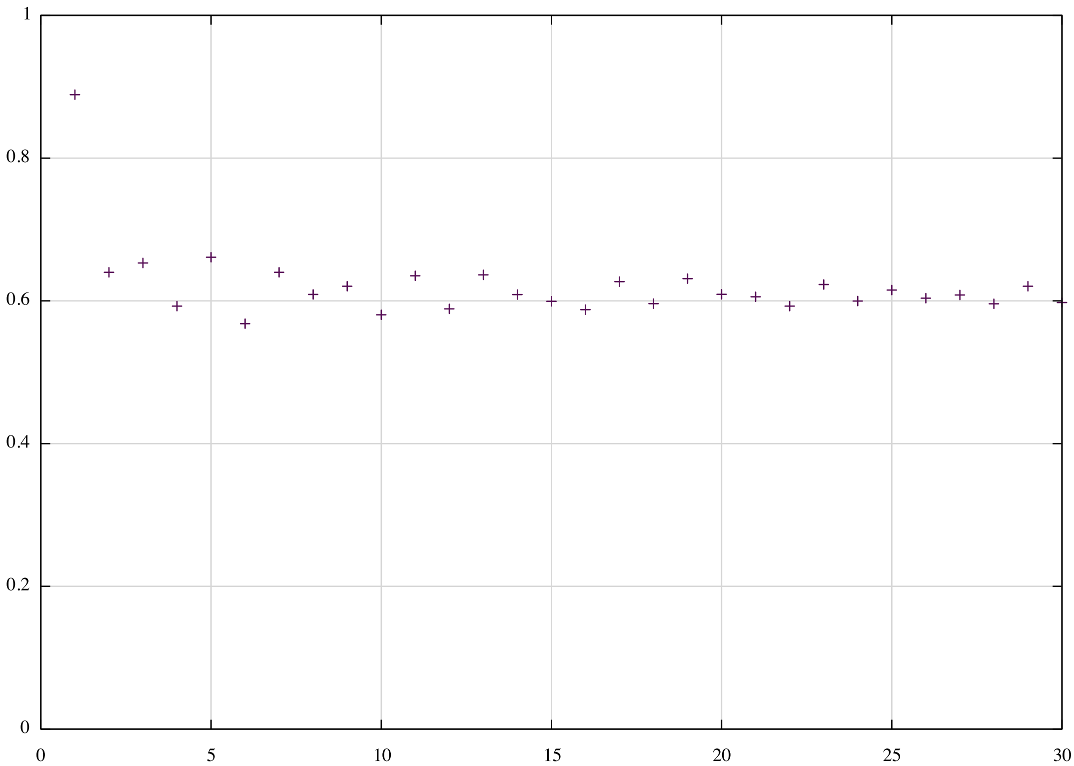

Figure 10: For $b = 1, \ldots, 30$ (horizontal axis), the fraction (vertical axis) of points of the range $[-b, b] \times [-b, b]$ that are accessible from $(0, 1)$.

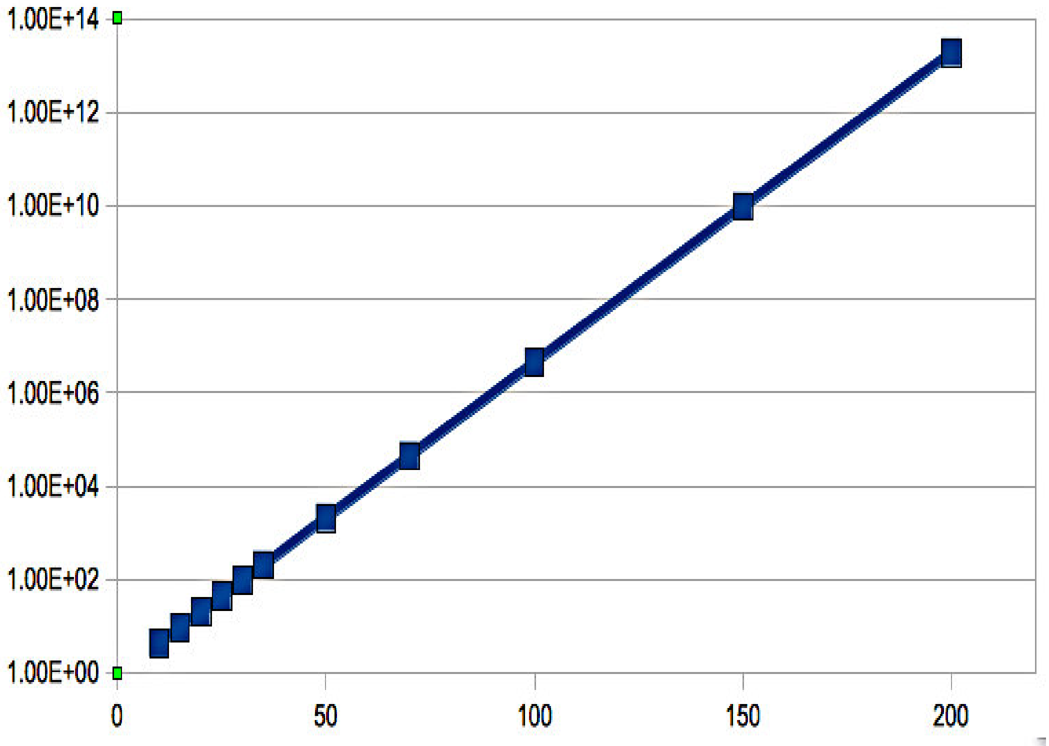

Figure 11: Average distance to the origin (vertical axis) as a function of the length of the path (horizontal axis). These results were obtained by Monte Carlo simulations. For each length represented (10, 15, 20, 35,. . . , 200) the number of statistical experiments is 1 million. The vertical scale is logarithmic.