Constraint Satisfaction Problems - CSP

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Examples of Applications

- 8-puzzle
- blocks world (e.g.: Hanoi tower)
- n-queens
- cryptarithmetic
- missionaires and cannibals
- solitaire
- many others...

n-queens

- Problem: place n queens in a board $n \times n$ so that no queens attack each other in the rows, columns or diagonals
- 2 ways of solving the problem: incremental and complete
- Possible states (a) and moves (b):
  1. Incremental 1
     a) any arrangement of 0 to n queens in the board
     b) add 1 queen to any position in the board
  2. Incremental 2
     a) arrangements of 0 to n queens with no attacks
     b) add 1 queen to the next leftmost empty column such that this does not attack the others already placed
  3. Completo
     a) arrangements of n queens, 1 em cada coluna
     b) mover qq rainha atacada para outra posição na mesma coluna

Cryptarithmetic

```
              SOLUTION
  FORTY       29786    com: F = 2
+   TEN         850         O = 9
+   TEN         850         R = 7 etc
--------      ------
  SIXTY       31486
```

Cryptarithmetic

- states: set of letters that are replaced by digits
- operators (moves): replace all occurrences of a letter by a digit that was never used before
- possible rules to select digits: numerical order to avoid repetitions, more strict obeying the mathematical properties of the problem
- final state: solution contains only digits and represent a correct sum
- cost of the solution: zero

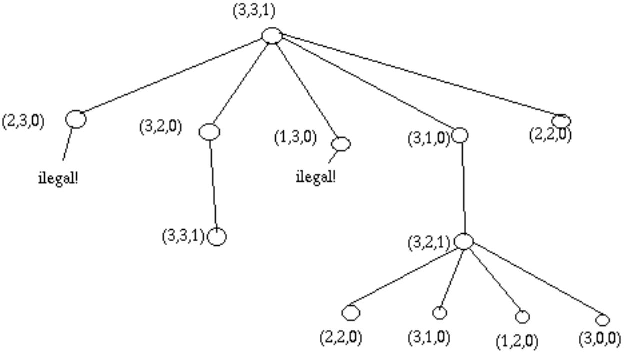
Missionaires and Cannibals

- 3 missionaires and 3 cannibals are by in a river bank with a boat that fits only 2 people
- Problem: find a way of moving all people from one river bank to the other
- : Constraint: we can never leave a number of cannibals larger than missionaires in a river bank :)

Missionaires and Cannibals

- 3 missionaires and 3 cannibals are by in a river bank with a boat that fits only 2 people
- Problem: find a way of moving all people from one river bank to the other
- Constraint: we can never leave a number of cannibals larger than missionaires in a river bank :)
- states: any ordered sequence of 3 numbers representing the missionaires, the cannibals and the boat
- initial state: (3,3,1)
- final state: (0,0,0)
- cost: number of river traversals
- rules (moves): take a missionaire, take a cannibal, take 2 cannibals etc...

Missionaires and Cannibals

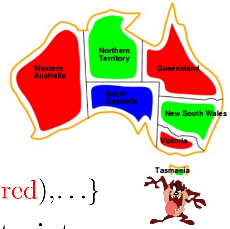


Other Approach and Representation

- Constraint Satisfaction
  - special type of problem that satisfies structural properties besides the basic requirements for general search problems
  - states: set of variables
  - variables can take values from a domain: set of possible values a variable can take (discrete/continuous, finite/infinite)
  - initial state: all variables with possible initial values from the domain
  - final state: final assignment of variable-value that does not violate the problem constraints
  - maximum search tree depth: number of variables
- Explicit representation for constraints (Portuguese translation: restrições)

Example CSP: Map-Coloring

- Variables: WA, NT, Q, NSW, V, SA, T
- Domain: D=red,green,blue
- Constraints: adjacent regions must have different colors
  - WA $\neq$ NT
  - (WA,NT) $\in$ { (red,green),(red,blue),(green,red),...}



- Solutions are assignments satisfying all constraints, e.g: {WA=red, NT=green, Q=red, NSW=green, V=red, SA=blue, T=green}

Example CSP: N-queens

- Formulation 1:
  - Variables: X_{ij}
  - Domains: {0,1}
  - Constraints:

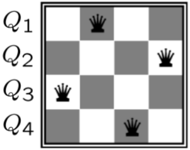
$$\forall i,j,k (X_{ij}, X_{ik}) \in \{(0,0),(0,1),(1,0)\}$$
$$\forall i,j,k (X_{ij}, X_{kj}) \in \{(0,0),(0,1),(1,0)\}$$
$$\forall i,j,k (X_{ij}, X_{i+k,j+k}) \in \{(0,0),(0,1),(1,0)\}$$
$$\forall i,j,k (X_{ij}, X_{i+k,j-k}) \in \{(0,0),(0,1),(1,0)\}$$

$$\sum_{i,j} X_{ij} = N$$

Example CSP: N-queens

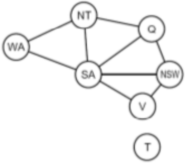
- Formulation 2:
  - Variables: Q_k
  - Domains: $\{1,2,3,\ldots\}$
  - Constraints:

$$\forall i,j \text{ non-threatening}(Q_i,Q_j) or$$
$$(Q1,Q2) \in \{(1,3),(1,4),\ldots\}$$
$$\ldots$$



Constraint Graphs

- Binary CSP: each constraint relates (at most) two variables
- Binary constraint graph: nodes are variables, arcs show constraints
- General-purpose CSP algorithms use the graph structure to speed up search. E.g., Tasmania is an independent subproblem



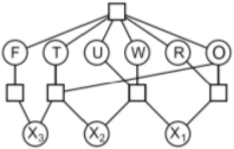
Example CSP: Cryptarithmetic

- Variables (circles): F T U W R O X1 X2 X3

```
  T W O
+ T W O
--------
F O U R
```

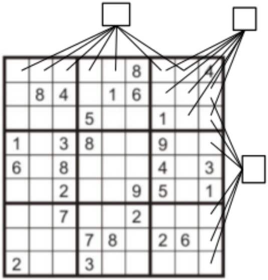
- Domains: {0,1,2,3,4,5,6,7,8,9}
- Constraints (boxes):

```
alldiff(F,T,U,W,R,O)
O + O = R + 10 . X1
...
```



Example CSP: Sudoku

- Variables: Each (open) square
- Domains: {1,2,...,9}
- Constraints:
  - 9-way alldiff for each column
  - 9-way alldiff for each row
  - 9-way alldiff for each region



Varieties of CSPs

- Discrete variables
  - Finite domains
    - Size d means $O(d^n)$ complete assignments
    - E.g., Boolean CSPs, including Boolean satisfiability (NP-complete)
  - Infinite domains (integers, strings, etc.)
    - E.g., job scheduling, variables are start/end times for each job
    - Linear constraints solvable, nonlinear undecidable
- Continuous variables
  - E.g., start-end state of a robot
  - Linear constraints solvable in polynomial time by LP methods

Varieties of Constraints

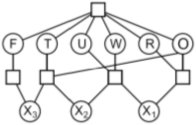
- Variables of Constraints
  - Unary constraints involve a single variable (equiv. to shrinking domains): SA $\neq$ green
  - Binary constraints involve pairs of variables: SA $\neq$ WA
  - Higher-order constraint involve 3 or more variables: e.g., cryptarithmetic column constraints
- Preferences (soft constraints):
  - E.g., red is better than green
  - Often representable by a cost for each variable assignment
  - Gives constrained optimization problems

Standard Search Formulation

- Standard search formulation of CSPs (incremental)
- Let's start with the straightforward, dumb approach, then fix it
- States are defined by the values assigned so far
  - Initial state: the empty assignment, {}
  - Successor function: assign a value to an unassigned variable
  - Goal test: the current assignment is complete and satisfies all constraints

Search Methods

- What does BFS do?
- What does DFS do?



Constraint Satisfaction: improving search

- n-queens
  - variables: possible positions in the board
  - constraints: no queen can attack each other
  - initial variable values: V_1 *in* $1..n$, V_2 *in* $1..n$ etc, with n the width of the board

Constraint Satisfaction: possible algorithms

- aloca uma nova rainha para uma nova coluna a cada nível (solução incremental)
- complexidade: $n^n = \approx \pi D_i = D_1 \times D_2 \times ... \times D_n$
- fator de ramificação: n
- fator máximo de ramificação: $n \times n = \sum D_i = D_1 + D_2 + ... + D_n$
  - se todas as variáveis fossem instanciadas com todos os valores possíveis no primeiro nível da árvore

Constraint Satisfaction: Possible algorithms

```
n = 8
                    (0,0,0,0,0,0,0,0)

N  (1,0,0,0,0,0,0,0)  (0,1,0,0,0,0,0,0)  (0,0,1,0,0,0,0,0) ....
i  (2,0,0,0,0,0,0,0)  (0,2,0,0,0,0,0,0)  (0,0,2,0,0,0,0,0) ....
v   ..........
1  (8,0,0,0,0,0,0,0)  (0,8,0,0,0,0,0,0)  (0,0,8,0,0,0,0,0) ....

N  (1,1,0,0,0,0,0,0)  (0,1,1,0,0,0,0,0)  (1,0,1,0,0,0,0,0) ....
i  (2,2,0,0,0,0,0,0)  (0,2,2,0,0,0,0,0)  (2,0,2,0,0,0,0,0) ....
v   ..........
2  (8,8,0,0,0,0,0,0)  (0,8,8,0,0,0,0,0)  (8,0,8,0,0,0,0,0) ....
```

Constraint Satisfaction: Possible algorithms

- using DFS, the tree that contains

  `(0,0,0,0,0,0,0,0) -> (1,0,0,0,0,0,0,0) ->`

  `(1,1,0,0,0,0,0,0) ->`

  `(1,1,1,0,0,0,0,0) -> (1,1,1,1,0,0,0,0) -> ...`

- will be explored even knowing that (1,1,1,1,1,1,1,1) is not a solution
- Class of algorithms "generate-and-test"

Constraint Satisfaction: Possible algorithms

- solution: test the constraint whenever a new queen is placed in the board
- it is not the best solution either
- Suppose that we managed to place 6 queens that are safe. This placement may threat the 8th queen and the algorithm does not know!
- DFS will end up testing ALL possibilities for placing the 7th queen in the board!
- Class of algorithms: “constrain-and-generate”

Constraint Satisfaction: Possible algorithms

- solution: Look ahead!
- *Forward Checking*: checks if the other unassigned variables domains are consistent with the new partial solution, removing from the domain of other variables all values that violate the constraints
- General Lookahead (arc-consistency): besides executing forward checking, checks if the new set of domains conflict with each other

Forward Checking: Example

```
n = 8
(1) V1 = 1  ==> V2 = {3,4,5,6,7,8}
                V3 = {2,4,5,6,7,8}
                V4 = {2,3,5,6,7,8}
                V5 = {2,3,4,6,7,8}
                V6 = {2,3,4,5,7,8}
                V7 = {2,3,4,5,6,8}
                V8 = {2,3,4,5,6,7}

(2) V2 = 3  ==> V3 = {5,6,7,8}
                V4 = {2,6,7,8}
                V5 = {2,4,7,8}
                V6 = {2,4,5,8}
                V7 = {2,4,5,6}
                V8 = {2,4,5,6,7}
```

Forward Checking: Example

n-queens
n=8

after V1=1
      V2=3

```
+---+---+---+---+---+---+---+---+
| X |   |   |   |   |   |   |   |
+---+---+---+---+---+---+---+---+
|   |   |   |   |   |   |   |   |
+---+---+---+---+---+---+---+---+
|   | X |   |   |   |   |   |   |
+---+---+---+---+---+---+---+---+
|   |   |   |   |   |   |   |   |
+---+---+---+---+---+---+---+---+
|   |   |   |   |   |   |   |   |
+---+---+---+---+---+---+---+---+
|   |   |   |   |   |   |   |   |
+---+---+---+---+---+---+---+---+
|   |   |   |   |   |   |   |   |
+---+---+---+---+---+---+---+---+
|   |   |   |   |   |   |   |   |
+---+---+---+---+---+---+---+---+
```

Forward Checking: Example

- Constraints:
  - $V_i = k \rightarrow V_j \neq k \quad \forall \quad j = 1, \ldots, 8; j \neq i$
  - $V_i = k_i, V_j = k_j \rightarrow \mid i - j \mid \neq \mid k_i - k_j \mid \quad \forall \quad j = 1, \ldots, 8; j \neq i$

Solutions to n-queens WITHOUT searching

(Explicit Solutions to the N-Queens Problem for all N, by Bo Bernhardsson)

- for n even and not of the form $6k + 2$
  - $(j, 2j)$
  - $\frac{n}{2} + j, 2j - 1$, for $j = 1, 2, \ldots, \frac{n}{2}$
- for n even, but not in the form 6k
  - $(j, 1 + [2(j-1) + \frac{n}{2} - 1 \mod n])$
  - $(n + 1 - j, n - [2(j-1) + \frac{n}{2} - 1 \mod n])$, for $j = 1, 2, \ldots, \frac{n}{2}$
- for n odd
  - Use any case above for n-1 and extend with 1 queen (n, n)

Solutions for n-queens WITHOUT searching

- Other references:
  - The n-Queens Problem, by Igor Rivin, Ilan Vardi, and Paul Zimmermann
  - A simplified solution of the N queens' problem, by Matthias Reichling
- Observation: find **all** solutions for the n-queens problem is not trivial :(

Constraint Satisfaction: other algorithms

- for inifinite domains: linear programming, simplex, revised simplex, convex hull, Gauss elimination
- for finite domains: forward checking, lookahead, arc-consistency in general
- for finite domains, two problems to solve:
  - choice of the *variable*:
    - *most-constrained*: smaller domain
    - *most constraining*: constrains the domain of other variables as much as possible
  - choice of the *value* for a variable:
    - *first-fail* principle.
    - *least constraining*: value that leaves more choices open to other variables

Constraint Satisfaction

- most-constrained: allows to solve n-queens with n equals to 100.
- pure forward checking: solves at most 30
- least-constraining value: allows to solve n-queens with n equals to 1000.

Algorithms

- Backtracking (systematic search)
- Constraint propagation: k-consistency
- All that use heuristics to order variables and their values
- Backjumping and dependency-directed backtracking

Basic Algorithm

```
CSP-BACKTRACKING(PartialAssignment a)
    If a is complete then return a
    X <- select an unassigned variable
    D <- select an ordering for the domain of X
    For each value v in D do
        If v is consistent with a then
            Add (X = v) to a
            result <- CSP-BACKTRACKING(a)
            If result <> failure then return result
            Remove (X = v) from a
    Return failure
```

Initial call: CSP-BACKTRACKING({})
Solves n-queens till $n \approx 25$
Combination of constraint propagation and heuristics can solve instances of size 1000

Random algorithms with min-conflicts plus parallelization can solve 3M queens in less than a minute! (see recommended reading in the discipline page)