Maximizing Expectation on Vertex-disjoint Cycle Packing

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International Conference on Computational Science and its Applications
Guimarães, July 2014

[1] This work is financed by the ERDF - European Regional Development Fund through the COMPETE Programme (operational programme for competitiveness) and by National Funds through the FCT - Fundação para a Ciência e a Tecnologia (Portuguese Foundation for Science and Technology) within project *Kidney Exchange Programme*, PTDC/EGE-GES/110940/2009, and includes contributions from its research team.

Background: kidney exchange programs

- in many countries, recent legislation allows patients needing a kidney transplant to receive it from a living donor
- what to do when the transplant from that donor is not possible?
  - blood type
  - other incompatibilities
- patient-donor pair may enter a kidney exchange program (KEP)

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Kidney exchanges

- idea: allow two (or more) patients in incompatible pairs to exchange their donors
- each recipient receives a compatible kidney from the donor of another pair

Incompatible pairs $P_1 - D_1$ and $P_2 - D_2$ exchange donors

- P_1 receives a transplant from D_2 and vice versa

Graph representation:

- vertices are patient-donor pairs
- arcs link a patient to compatible donors

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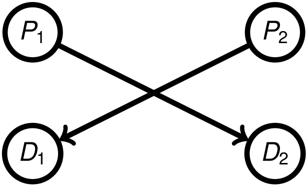

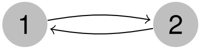

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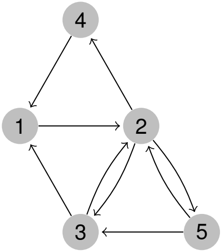
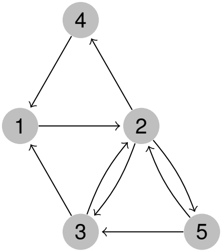
Kidney exchanges: example



- instance with five pairs
- what is the maximum number of transplants?
- what if the allowed number of simultaneous transplants is limited?
- how to optimize if there is some probability of vertex/arc failure?

Kidney exchanges: example



- feasible exchange: a set of vertex-disjoint cycles (*e.g.*, 1 – 2 – 3 – 1)
- size of an exchange: sum of the lengths of its cycles
- maximum exchange in this example: 4 (cycle 1 – 2 – 5 – 3 – 1)

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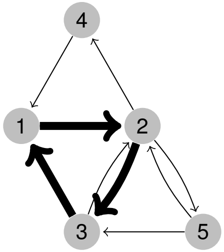

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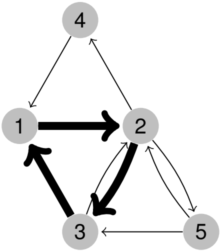

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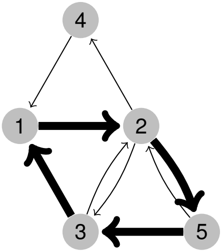

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  - limitations in the number of operation rooms
  - number of surgeons available
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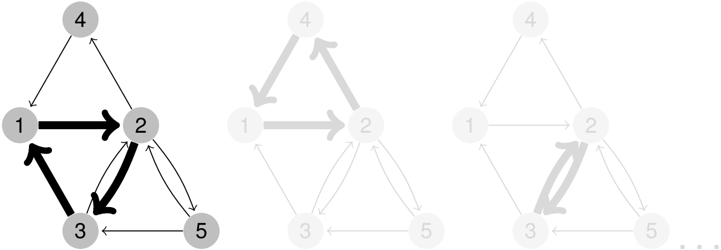

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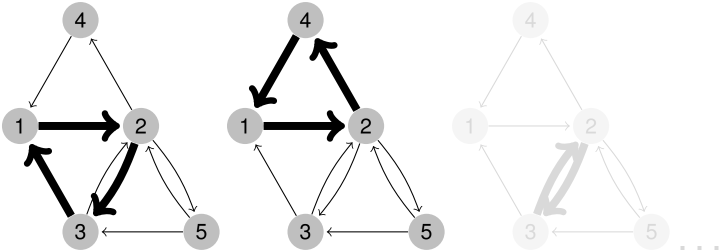

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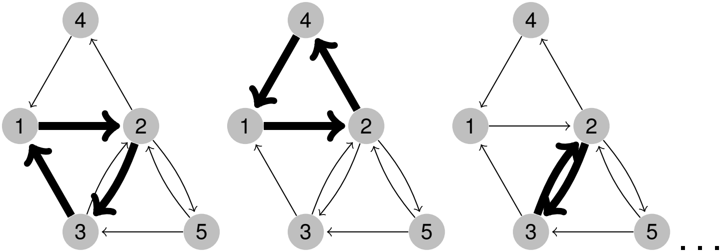
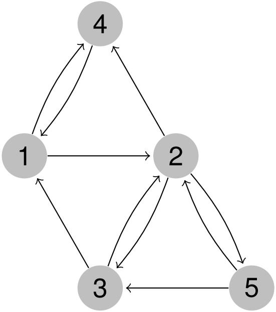


Another example



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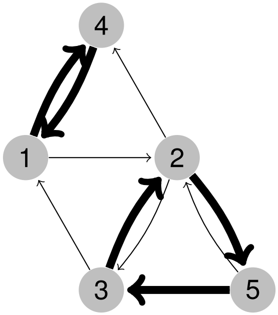

Maximum cycle size and NP-hardness

- In many situations the length of each cycle is limited
- If length is not limited $\longrightarrow$ ***assignment problem*** (polynomial algorithms are known, *e.g.*, hungarian algorithm).
- If length is limited to 2 $\longrightarrow$ ***matching problem*** (polynomial algorithms are known: Edmonds algorithm).
- If length is limited to 3, 4, ... $\longrightarrow$ ***problem is NP-hard*** (no polynomial algorithms are known).

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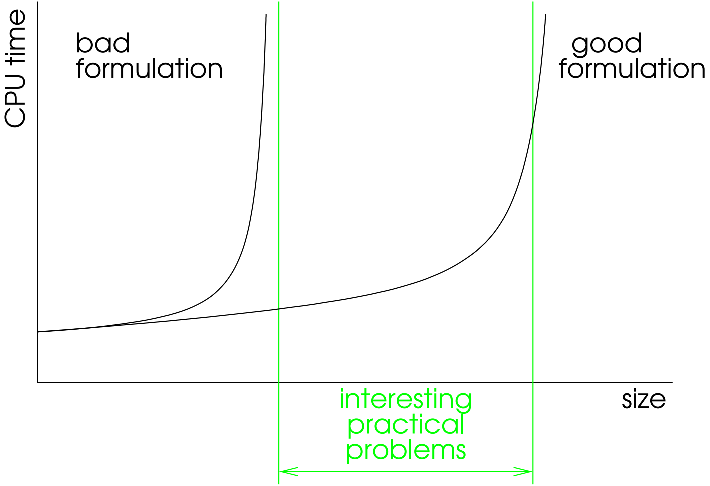
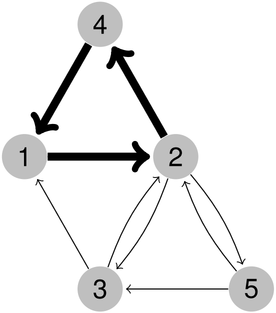
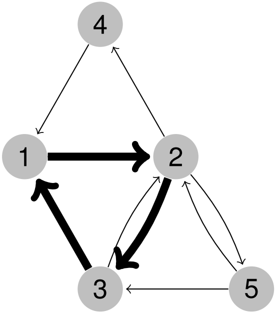
NP-hard problems



Mathematical programming formulations

- There are several possibilities for modeling the problem in mathematical programming
- One of the most successful is the *cycle formulation:*
  - enumerate all cycles in the graph with length at most K
  - for each cycle c, let variable x_c be 1 if c is chosen, 0 otherwise
  - every feasible solution corresponds to a set of vertex-disjoint cycles





Cycle formulation

$$\text{maximize} \quad \sum_{c} w_c x_c \tag{1a}$$

$$\text{subject to} \quad \sum_{c:i \in c} x_c \leq 1 \quad \forall i \tag{1b}$$

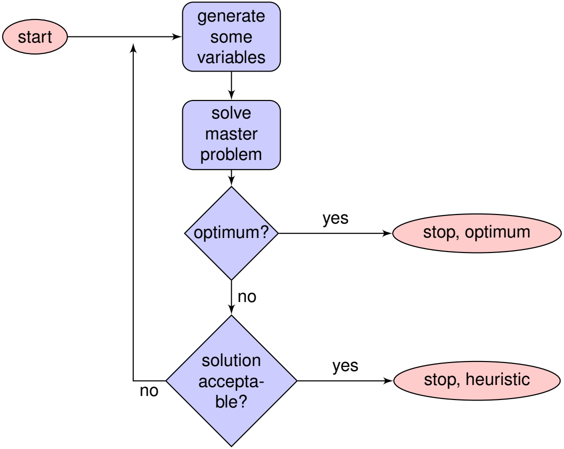
$$x_c \in \{0, 1\} \quad \forall c$$

- case of $0-1$ weights: $w_c = |c|$, (length of cycle c)
- objective: maximize the weight of the exchange
- constraints: every vertex is at most in one cycle (*i.e.*, donate/receive at most one kidney)
- difficulty: number of variables

Cycle formulation

- Exponential number of variables
- Not all are needed for solving the problem
- Use only those necessary $\longrightarrow$ column generation

Column generation



Previous results

- Cycle formulation seems to be more than able to process foreseen number of patient-donor pairs in the KEP in Portugal
- Besides, it may allow to treat slightly different objectives:
  - produce robust solutions
  - maximize expectation of the number of transplants
- What if the *“market”* becomes the European Union?

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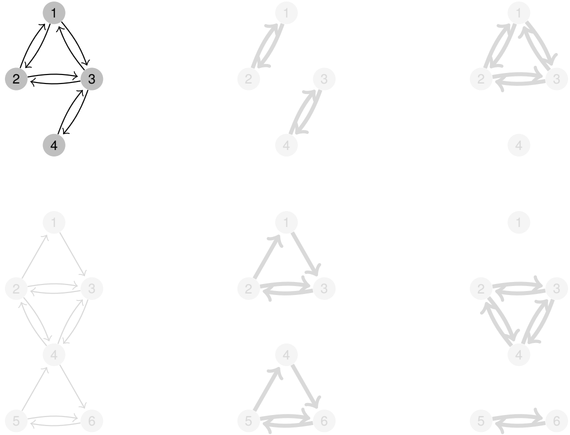
Maximizing expectation

- Basis: cycle formulation
- On standard approach: cycle evaluation is the number of arcs in the cycle (*i.e.*, the *number of transplants*)
- Our proposal: use the expectation of the number of transplants instead
- Problem: not straightforward to tackle. . .
  1. computation of the expectation is heavy, even for small cycles
  2. optimization is just a small part in the solution process. . .

Unreliable vertices



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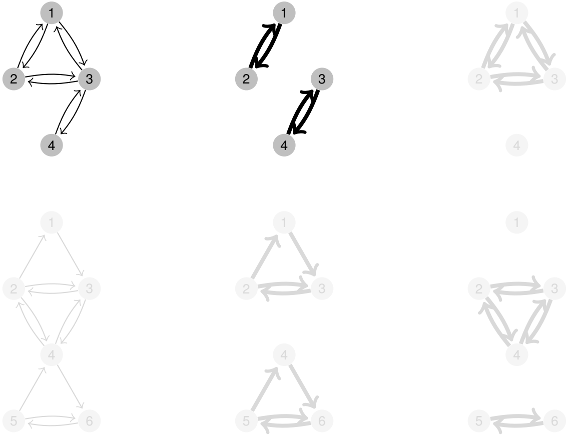

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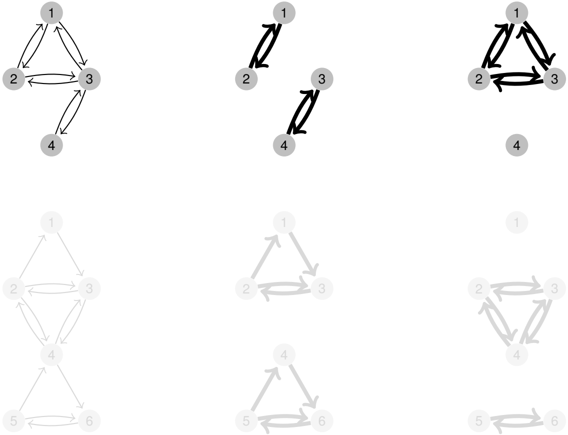

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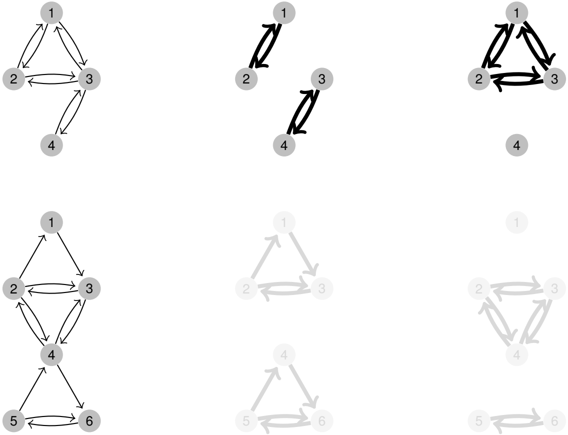

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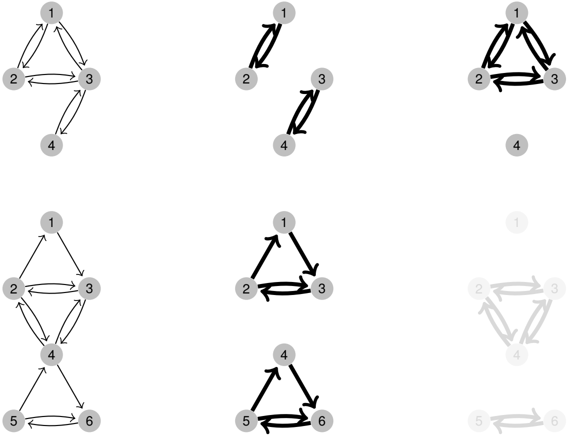

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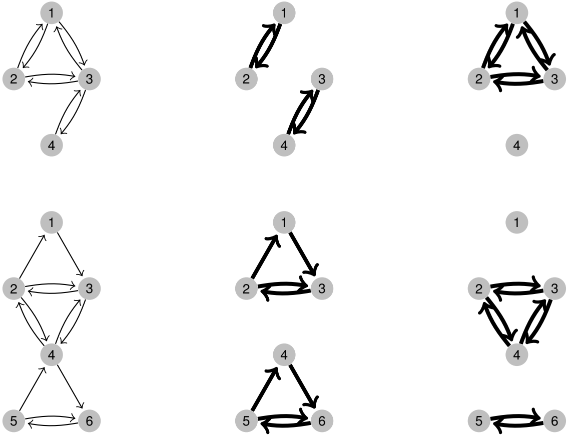
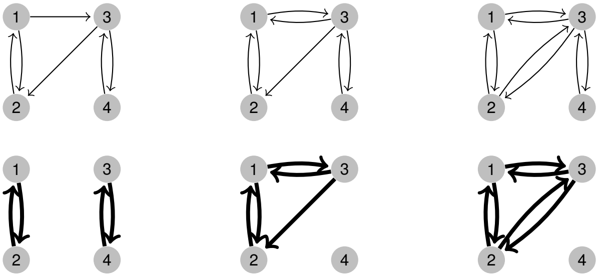
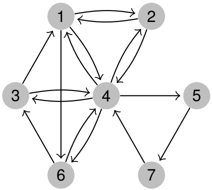


Unreliable arcs



Arc withdrawal



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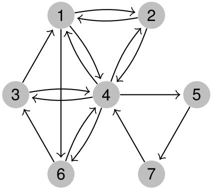

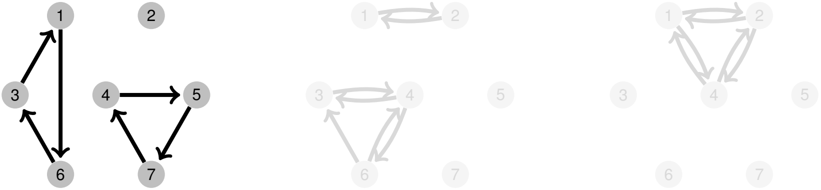

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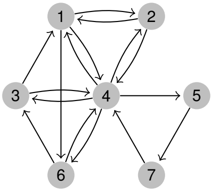

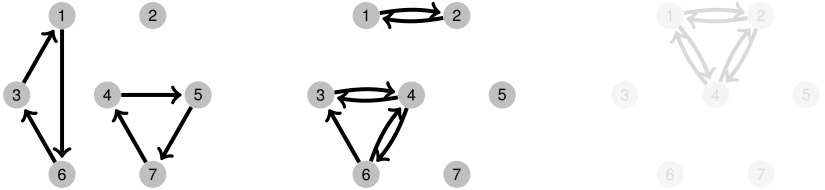

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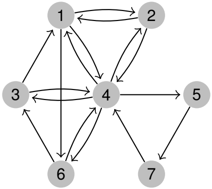

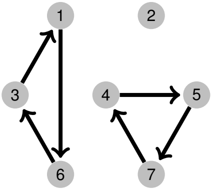

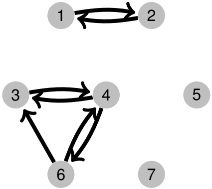

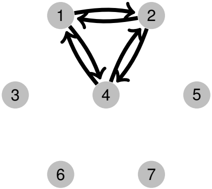

Solution procedure

- Preprocessing
- Solution optimization
- Implementation

Solution procedure: preprocessing

- Preprocessing
  1. prepare a database of cycle configurations for the relevant sizes
  2. precompute formulas for expectations for these configurations
     HARD
  3. store this information in a database
- Example:
- Use the expectation as objective coefficient for each cycle

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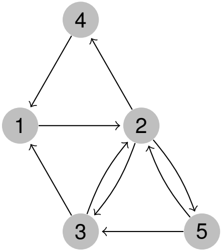

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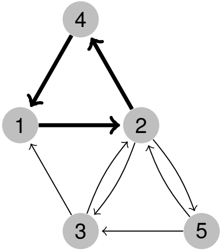

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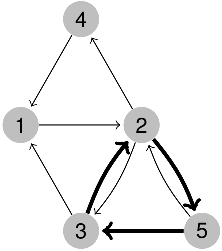

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Solution procedure: cycle configuration database

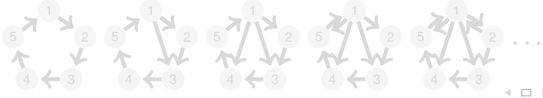
- Two-vertex graphs (1 graph)

vertices:	$2\,(1-p_1)\,(1-p_2)$
arcs:	$2\,(1-p_{12})\,(1-p_{21})$
both:	$2\,(1-p_1)\,(1-p_2)\,(1-p_{12})\,(1-p_{21})$

- Three-vertex graphs (4 graphs)

- Four-vertex graphs (61 graphs)

- Five-vertex graphs (3725 graphs)



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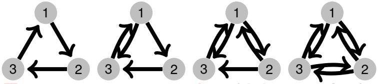

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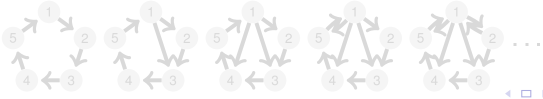

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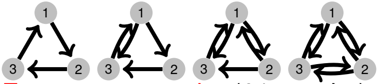

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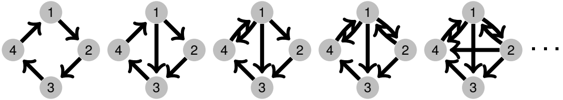

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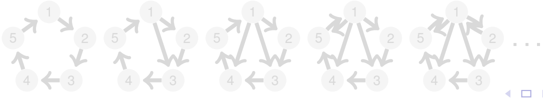

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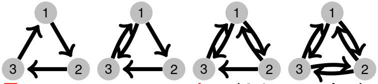

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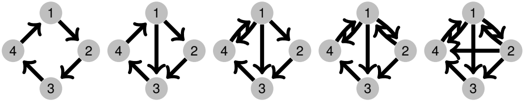

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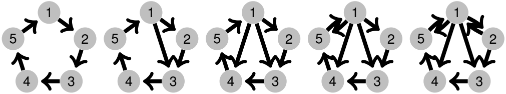

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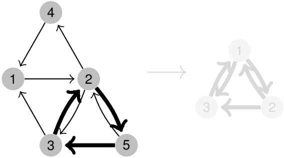
1. Match enumerated graph with one in the database



2. Extract expectation formula from the database

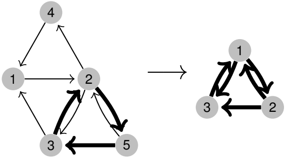
$3 - p_{21}p_{23} - p_{21}p_{31}p_{23} + p_{21}p_{32}p_{23} + p_{21}p_{31}p_{32}p_{23} + p_{31}p_{32}p_{23} - p_{32}p_{23} - p_{21}p_{31} - p_{21}p_{31}p_{32} - p_{31}p_{32} + p_{13}\left(p_{23}\left(p_{31} - 1\right)\left(-p_{32} + p_{21}\left(p_{32} + 1\right) + 1\right) - \left(p_{21} - 1\right)p_{31}\left(p_{32} - 1\right)\right) + p_{12}\left(-\left(p_{23}\left(p_{31} - 1\right) + p_{31} + 1\right)p_{32} + p_{21}\left(p_{23}\left(p_{31} - 1\right)\left(p_{32} - 1\right) + p_{32} + p_{31}\left(p_{32} + 1\right) - 1\right) + p_{13}\left(p_{23}\left(p_{31} + 1\right)\left(p_3 \right.\right.$

3. Map probabilities from original graph to the one stored

p_2	$\leftrightarrow$	p_1	p_{25}	$\leftrightarrow$	p_{12}
p_5	$\leftrightarrow$	p_2	p_{52}	$\leftrightarrow$	p_{21}
p_3	$\leftrightarrow$	p_3	p_{53}	$\leftrightarrow$	p_{23}
			p_{32}	$\leftrightarrow$	p_{31}
			p_{23}	$\leftrightarrow$	p_{13}

4. Compute expectation for the cycle $(2 - 5 - 3)$; it will be its coefficient at the objective

1. Match enumerated graph with one in the database



2. Extract expectation formula from the database

$3 - p_{21}p_{23} - p_{21}p_{31}p_{23} + p_{21}p_{32}p_{23} + p_{21}p_{31}p_{32}p_{23} + p_{31}p_{32}p_{23} - p_{32}p_{23} - p_{21}p_{31} - p_{21}p_{31}p_{32} -$
$p_{31}p_{32} + p_{13}\left(p_{23}\left(p_{31} - 1\right)\left(-p_{32} + p_{21}\left(p_{32} + 1\right) + 1\right) - \left(p_{21} - 1\right)p_{31}\left(p_{32} - 1\right)\right) +$
$p_{12}\left(-\left(p_{23}\left(p_{31} - 1\right) + p_{31} + 1\right)p_{32} + p_{21}\left(p_{23}\left(p_{31} - 1\right)\left(p_{32} - 1\right) + p_{32} + p_{31}\left(p_{32} + 1\right) - 1\right) + p_{13}\left(p_{23}\left(p_{31} + 1\right)\left(p_{3}\right.\right.\right.$

3. Map probabilities from original graph to the one stored

p_2	$\leftrightarrow$	p_1	p_{25}	$\leftrightarrow$	p_{12}
p_5	$\leftrightarrow$	p_2	p_{52}	$\leftrightarrow$	p_{21}
p_3	$\leftrightarrow$	p_3	p_{53}	$\leftrightarrow$	p_{23}
			p_{32}	$\leftrightarrow$	p_{31}
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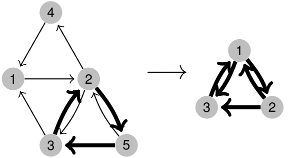

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$$p_{31}p_{32} + p_{13}\left(p_{23}\left(p_{31} - 1\right)\left(-p_{32} + p_{21}\left(p_{32} + 1\right) + 1\right) - \left(p_{21} - 1\right)p_{31}\left(p_{32} - 1\right)\right) +$$
$$p_{12}\left(-\left(p_{23}\left(p_{31} - 1\right) + p_{31} + 1\right)p_{32} + p_{21}\left(p_{23}\left(p_{31} - 1\right)\left(p_{32} - 1\right) + p_{32} + p_{31}\left(p_{32} + 1\right) - 1\right) + p_{13}\left(p_{23}\left(p_{31} + 1\right)\left(p_3\right.\right.$$

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p_2	$\leftrightarrow$	p_1	p_{25}	$\leftrightarrow$	p_{12}
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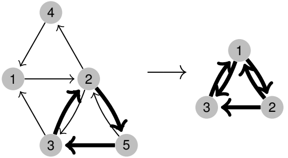

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$p_{31}p_{32} + p_{13}\left(p_{23}\left(p_{31}-1\right)\left(-p_{32}+p_{21}\left(p_{32}+1\right)+1\right)-\left(p_{21}-1\right)p_{31}\left(p_{32}-1\right)\right)+$

$p_{12}\left(-\left(p_{23}\left(p_{31}-1\right)+p_{31}+1\right)p_{32}+p_{21}\left(p_{23}\left(p_{31}-1\right)\left(p_{32}-1\right)+p_{32}+p_{31}\left(p_{32}+1\right)-1\right)+p_{13}\left(p_{23}\left(p_{31}+1\right)\left(p_3\right.\right.\right.$

3. Map probabilities from original graph to the one stored

p_2	$\leftrightarrow$	p_1
p_5	$\leftrightarrow$	p_2
p_3	$\leftrightarrow$	p_3

p_{25}	$\leftrightarrow$	p_{12}
p_{52}	$\leftrightarrow$	p_{21}
p_{53}	$\leftrightarrow$	p_{23}
p_{32}	$\leftrightarrow$	p_{31}
p_{23}	$\leftrightarrow$	p_{13}

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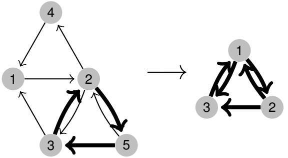

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$p_{31}p_{32} + p_{13}\left(p_{23}\left(p_{31} - 1\right)\left(-p_{32} + p_{21}\left(p_{32} + 1\right) + 1\right) - \left(p_{21} - 1\right)p_{31}\left(p_{32} - 1\right)\right) +$

$p_{12}\left(-\left(p_{23}\left(p_{31} - 1\right) + p_{31} + 1\right)p_{32} + p_{21}\left(p_{23}\left(p_{31} - 1\right)\left(p_{32} - 1\right) + p_{32} + p_{31}\left(p_{32} + 1\right) - 1\right) + p_{13}\left(p_{23}\left(p_{31} + 1\right)\left(p_{3}\right.\right.\right.$

3. Map probabilities from original graph to the one stored

p_2	$\leftrightarrow$	p_1		p_{25}	$\leftrightarrow$	p_{12}
				p_{52}	$\leftrightarrow$	p_{21}
p_5	$\leftrightarrow$	p_2		p_{53}	$\leftrightarrow$	p_{23}
				p_{32}	$\leftrightarrow$	p_{31}
p_3	$\leftrightarrow$	p_3		p_{23}	$\leftrightarrow$	p_{13}

4. Compute expectation for the cycle $(2 - 5 - 3)$; it will be its coefficient at the objective

Solution procedure: solution optimization

- Solution optimization
  1. read instance: compatibility between pairs, failure probability for vertices/arcs
  2. prepare compatibility graph
  3. enumerate cycles of relevant size
  4. setup optimization model
     1. one variable for each cycle
     2. constraints: each vertex in at most one cycle
     3. objective coefficient: expectation of number of transplants HARD
  5. solve optimization model easy?!!

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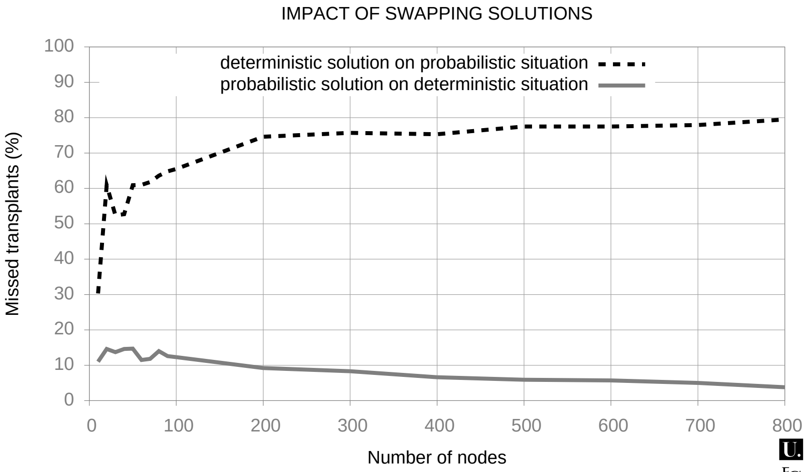
Solution procedure: implementation

- Implementation
  1. contact selected pairs
  2. verify solution (check back outs)
  3. make last-minute compatibility check
  4. make transplants

Results: cross-formulation performance



Conclusions

- There are many applications of information technologies in health care
- Applications involve many disciplines in computer science and informatics
- KEP: case where welfare of patients can be maximized
  - number of transplants
  - robustness of the solution
  - quality of the solution (maximize patient-donor compatibility)
- Careful implementation of operations research program leads to significant social benefits