The recursive circle packing problem

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Origin of the problem: industrial setting

- A company produces tubes
- Orders are sent to customers in containers
- The company wants to know how to load the containers
  - previously: solution constructed by hand
  - very tedious and error prone
  - using an expensive resource: production engineer's time
  - the cost of sending one container to distant customers is very high

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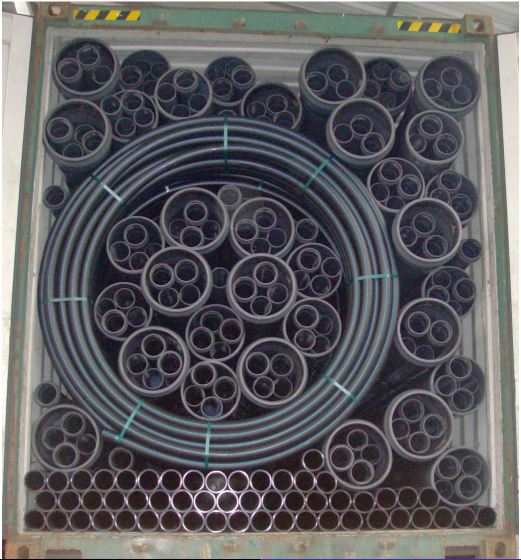


Problem statement: real world

Given a set of tubes characterized by internal and external diameters, all of them cut to the length of a container:

- determine the number of containers required to send them
- decide in which, from the set of containers to insert each tube
- determine each tube's position

minimizing:

- the number of containers required (first goal)
- the center of gravity of each container (second goal)

maximizing:

- the number/value of additional tubes that can be inserted

being allowed to:

- insert tubes inside other tubes (telescoping)

respecting:

- maximum container weight.

Problem statement: abstraction

Simplified version:

- given a set of *annuli* (*i.e.*, concentric circles, or rings), each characterized by
  - external and internal radii
  - value
- given a rectangle;
- determine the maximum value that can be inserted, such that:
  - each ring may be inside other rings, but they cannot overlap
  - all rings must be completely inside the rectangle
- (solution: each ring's position)

Background: some algorithms

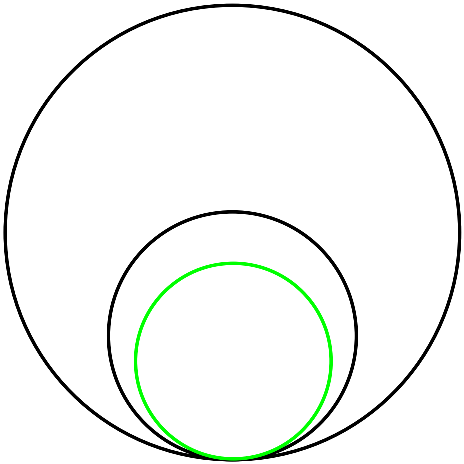
- Circle packing
  - in rectangles
  - in circles
  - (no previous work of our knowledge in *telescoping*)

Algorithms – practical solution

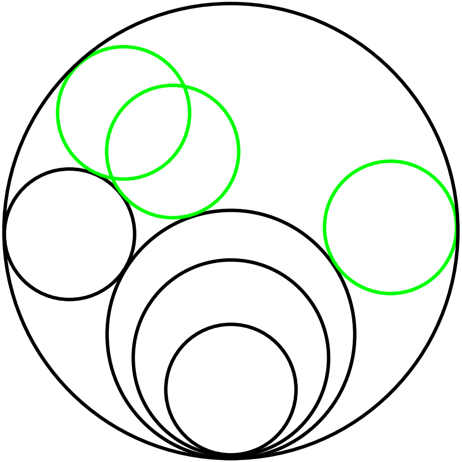
Solution construction in two interconnected phases:

- Telescoping: inserting thinner tubes inside thicker
- Packing in rectangle, using (possibly) previously telescoped tubes

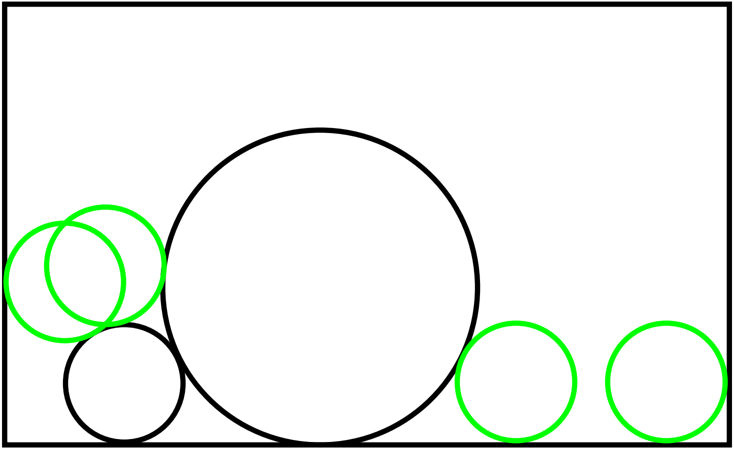
Practical solution: telescoping



Telescoping



Practical solution: packing in rectangle



Approximation algorithm

Practical solutions are obtained with a (semi-greedy) variant of:
For each ring, sorted from largest to thinnest

- Determine positions in the rectangle where it can be inserted (if some)
- Choose the lowest, leftmost position
- For each ring that can be inserted in this ring:
  - Determine positions where it can be inserted (if some)
  - Choose the lowest, leftmost position
  - For each ring that can be inserted in this ring:
    - Determine positions where it can be inserted (if some)
    - Choose the lowest, leftmost position
    - ...

A mathematical model: parameters, variables

Parameters for describing an instance:

- width W and height H of the rectangular container;
- set of tube indicess $\mathcal{A}$
- for each tube $i \in \mathcal{A}$:
  - external radius r_i^{ext}
  - internal radius r_i^{int}

Variables:

- position of the center of each ring i, (x_i, y_i);
- $w_i = 1$ if ring i is placed directly in the rectangle, zero otherwise;
- $u_{ki} = 1$ if ring i is placed directly inside ring k, zero otherwise;

A mathematical model: placement inside rectangle

Tube placement:

$$w_i + \sum_k u_{ki} \leq 1, \qquad \forall i,$$

$$\sum_{i:i \neq k} u_{ki} \leq w_k + \sum_i u_{ik}, \quad \forall k.$$

Placement inside rectangle with vertices $(0,0)$, $(W,0)$, $(0,H)$, (W,H).

$$\begin{aligned} x_i - r_i^{\text{ext}} &\geq 0, \\ y_i - r_i^{\text{ext}} &\geq 0, \\ x_i + r_i^{\text{ext}} &\leq W, \\ y_i + r_i^{\text{ext}} &\leq H, \quad \forall i, \end{aligned}$$

A mathematical model: placement among circles

Distance between circles in rectangle must be larger than the sum of their radii:

$$\sqrt{(x_i - x_j)^2 + (y_i - y_j)^2} \geq (r_i^{\text{ext}} + r_j^{\text{ext}})(w_i + w_j - 1), \quad \forall i, j$$
$$\Leftrightarrow (x_i - x_j)^2 + (y_i - y_j)^2 \geq (r_i^{\text{ext}} + r_j^{\text{ext}})^2(w_i + w_j - 1), \quad \forall i, j$$

The same for each pair of tubes i, j directly placed inside the same tube k:

$$(x_i - x_j)^2 + (y_i - y_j)^2 \geq (r_i^{\text{ext}} + r_j^{\text{ext}})^2(2 - u_{ki} - u_{kj}),$$

If tube i is placed directly in tube k, their circles cannot overlap:

$$(x_k - x_i)^2 + (y_k - y_i)^2 \leq (r_k^{\text{int}} - r_i^{\text{ext}})^2 + M(1 - u_{ki}).$$

(M makes constraints redundant if one of u_{ki} is zero)

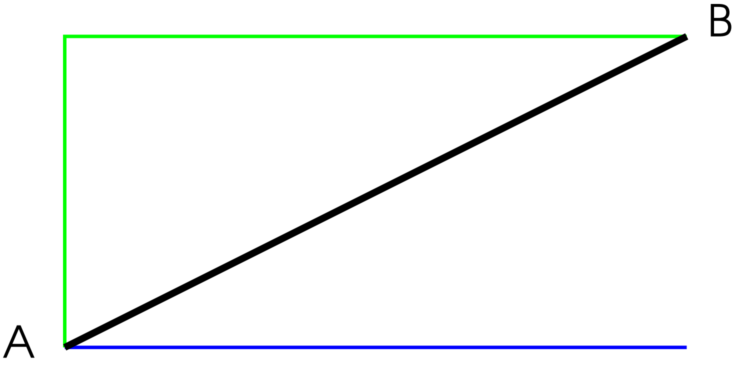
A mathematical model: objective

$$\text{maximize } V = \sum_i v_i \left(w_i + \sum_k u_{ki} \right)$$

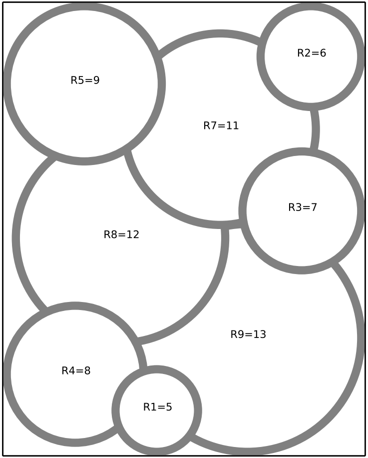
Bounds

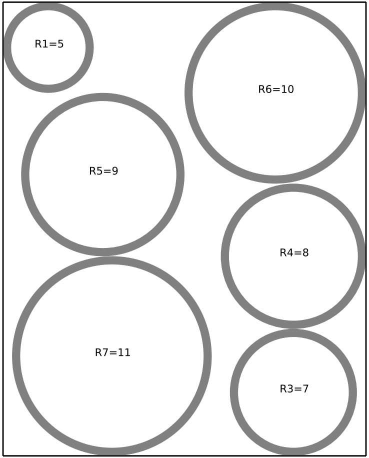
- The previous model is exact, but is it nonlinear and very difficult to solve
- Challenges: linear model for obtaining
  - **lower** bounds (feasible solutions)
  - **upper** bounds (how much they can be improved)
- some ideas:
  - linear version with L_1-norm distance: lower bound?
  - linear version with L_∞-norm distance: upper bound?
  - L_2-norm with piecewise linear approximation?
  - what are the limits for a MINLP solver (*e.g.*, couenne)?

Distance with different norms



Distance with different norms





A mathematical model: linear version (lower bound)

$$||x-y||_\infty = \max(|x_i - x_j|, |y_i - y_j|) \geq r_i^{\text{ext}} + r_j^{\text{ext}} - M(2 - w_i - w_j)$$
$$||x-y||_\infty = \max(|x_i - x_j|, |y_i - y_j|) \geq r_i^{\text{ext}} + r_j^{\text{ext}} - M(2 - u_{ki} - u_{kj})$$
$$||x-y||_1 = |x_i - x_j| + |y_i - y_j| \leq r_k^{\text{int}} - r_i^{\text{ext}} + M(1 - u_{ki})$$

A mathematical model: linear version (upper bound)

$$||x - y||_1 = |x_i - x_j| + |y_i - y_j| \geq r_i^{\text{ext}} + r_j^{\text{ext}} - M(2 - w_i - w_j)$$

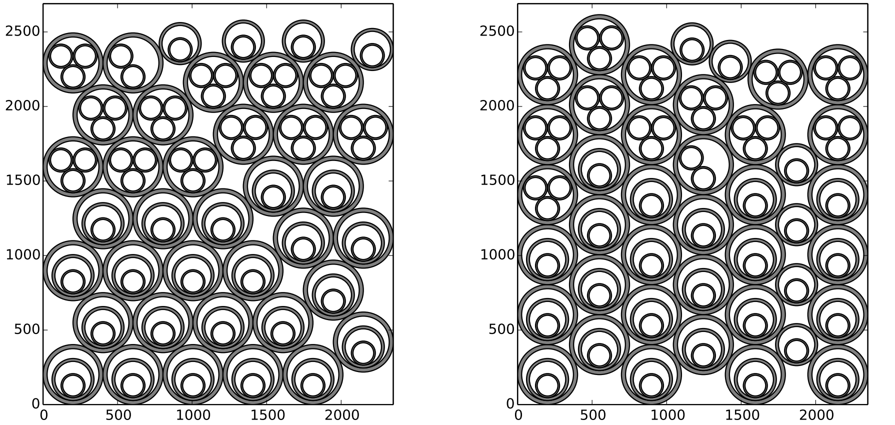
$$||x - y||_1 = |x_i - x_j| + |y_i - y_j| \geq r_i^{\text{ext}} + r_j^{\text{ext}} - M(2 - u_{ki} - u_{kj})$$

$$||x - y||_\infty = \max(|x_i - x_j|, |y_i - y_j|) \leq r_k^{\text{int}} - r_i^{\text{ext}} + M(1 - u_{ki})$$

Conclusions

- Main contributions:
  - definition of the recursive circle packing problem;
  - new method and model for cycle packing, including telescoping;
  - approximate methods: obtain *bounds* on the objective value;
  - upper bounds are important: firms want to assess how much could be improved.
- Further work:
  - exact methods for solving this problem.

Some solutions



Some solutions