Group B

3 Show that if ties in the ratio test are broken by favoring row 1 over row 2, then cycling occurs when the following LP is solved by the simplex:

$$
\begin{aligned}
\max z &= 2x_1 + 3x_2 - x_3 - 12x_4 \\
\text{s.t} \quad -2x_1 - 9x_2 + x_3 + 9x_4 &\le 0 \\
\tfrac{x_1}{3} + x_2 - \tfrac{x_3}{3} - 2x_4 &\le 0 \\
x_i \ge 0 \quad (i = 1, 2, 3, 4)
\end{aligned}
$$

4 Show that if ties are broken in favor of lower-numbered rows, then cycling occurs when the simplex method is used to solve the following LP:

$$
\begin{aligned}
\max z &= -3x_1 + x_2 - 6x_3 \\
9x_1 + x_2 - 9x_3 - 2x_4 &\le 0 \\
x_1 + \tfrac{x_2}{3} - 2x_3 - \tfrac{x_4}{3} &\le 0 \\
-9x_1 - x_2 + 9x_3 + 2x_4 &\le 1 \\
x_i \ge 0 \quad (i = 1, 2, 3, 4)
\end{aligned}
$$

5 Show that if Bland's Rule to prevent cycling is applied to Problem 4, then cycling does not occur.

6 Consider an LP (maximization problem) in which each basic feasible solution is nondegenerate. Suppose that x_i is the only variable in our current tableau having a negative coefficient in row 0. Show that any optimal solution to the LP must have $x_i > 0$.

4.12 The Big M Method

Recall that the simplex algorithm requires a starting bfs. In all the problems we have solved so far, we found a starting bfs by using the slack variables as our basic variables. If an LP has any $\ge$ or equality constraints, however, a starting bfs may not be readily apparent. Example 4 will illustrate that a bfs may be hard to find. When a bfs is not readily apparent, the Big M method (or the two-phase simplex method of Section 4.13) may be used to solve the problem. In this section, we discuss the **Big M method,** a version of the simplex algorithm that first finds a bfs by adding "artificial" variables to the problem. The objective function of the original LP must, of course, be modified to ensure that the artificial variables are all equal to 0 at the conclusion of the simplex algorithm. The following example illustrates the Big M method.

EXAMPLE 4 Bevco

Bevco manufactures an orange-flavored soft drink called Oranj by combining orange soda and orange juice. Each ounce of orange soda contains 0.5 oz of sugar and 1 mg of vitamin C. Each ounce of orange juice contains 0.25 oz of sugar and 3 mg of vitamin C. It costs Bevco 2¢ to produce an ounce of orange soda and 3¢ to produce an ounce of orange juice. Bevco's marketing department has decided that each 10-oz bottle of Oranj must contain at least 20 mg of vitamin C and at most 4 oz of sugar. Use linear programming to determine how Bevco can meet the marketing department's requirements at minimum cost.

Solution Let

x_1 = number of ounces of orange soda in a bottle of Oranj

x_2 = number of ounces of orange juice in a bottle of Oranj

Then the appropriate LP is

$$
\begin{aligned}
\min z &= 2x_1 + 3x_2 \\
\text{s.t.} \quad \tfrac{1}{2}x_1 + \tfrac{1}{4}x_2 &\le 4 \qquad \text{(Sugar constraint)} \\
x_1 + 3x_2 &\ge 20 \qquad \text{(Vitamin C constraint)}
\end{aligned}
\tag{17}
$$

$$x_1 + x_2 = 10 \qquad \text{(10 oz in bottle of Oranj)}$$
$$x_1, x_2 \geq 0$$

(The solution will be continued later in this section.)

To put (17) into standard form, we add a slack variable s_1 to the sugar constraint and subtract an excess variable e_2 from the vitamin C constraint. After writing the objective function as $z - 2x_1 - 3x_2 = 0$, we obtain the following standard form:

$$\begin{array}{ll} \text{Row 0:} & z - 2x_1 - 3x_2 = 0 \\ \text{Row 1:} & \tfrac{1}{2}x_1 + \tfrac{1}{4}x_2 + s_1 = 4 \\ \text{Row 2:} & x_1 + 3x_2 - e_2 = 20 \\ \text{Row 3:} & x_1 + x_2 = 10 \\ & \text{All variables nonnegative} \end{array} \tag{18}$$

In searching for a bfs, we see that $s_1 = 4$ could be used as a basic (and feasible) variable for row 1. If we multiply row 2 by -1, we see that $e_2 = -20$ could be used as a basic variable for row 2. Unfortunately, $e_2 = -20$ violates the sign restriction $e_2 \geq 0$. Finally, in row 3 there is no readily apparent basic variable. Thus, in order to use the simplex to solve (17), rows 2 and 3 each need a basic (and feasible) variable. To remedy this problem, we simply "invent" a basic feasible variable for each constraint that needs one. Because these variables are created by us and are not real variables, we call them **artificial variables.** If an artificial variable is added to row i, we label it a_i. In the current problem, we need to add an artificial variable a_2 to row 2 and an artificial variable a_3 to row 3. The resulting set of equations is

$$\begin{array}{l} z - 2x_1 - 3x_2 = 0 \\ \tfrac{1}{2}x_1 + \tfrac{1}{4}x_2 + s_1 = 4 \\ x_1 + 3x_2 - e_2 + a_2 = 20 \\ x_1 + x_2 + a_3 = 10 \end{array} \tag{18}$$

We now have a bfs: $z = 0$, $s_1 = 4$, $a_2 = 20$, $a_3 = 10$. Unfortunately, there is no guarantee that the optimal solution to (18) will be the same as the optimal solution to (17). In solving (18), we might obtain an optimal solution in which one or more artificial variables are positive. Such a solution may not be feasible in the original problem (17). For example, in solving (18), the optimal solution may easily be shown to be $z = 0$, $s_1 = 4$, $a_2 = 20$, $a_3 = 10$, $x_1 = x_2 = 0$. This "solution" contains no vitamin C and puts 0 ounces of soda in a bottle, so it cannot possibly solve our original problem! If the optimal solution to (18) is to solve (17), then we must make sure that the optimal solution to (18) sets all artificial variables equal to zero. In a min problem, we can ensure that all the artificial variables will be zero by adding a term Ma_i to the objective function for each artificial variable a_i. (In a max problem, add a term $-Ma_i$ to the objective function.) Here M represents a "very large" positive number. Thus, in (18), we would change our objective function to

$$\min z = 2x_1 + 3x_2 + Ma_2 + Ma_3$$

Then row 0 will change to

$$z - 2x_1 - 3x_2 - Ma_2 - Ma_3 = 0$$

Modifying the objective function in this way makes it extremely costly for an artificial variable to be positive. With this modified objective function, it seems reasonable that the optimal solution to (18) will have $a_2 = a_3 = 0$. In this case, the optimal solution to (18) will solve the original problem (17). It sometimes happens, however, that in solving the analog of (18), some of the artificial variables may assume positive values in the optimal solution. If this occurs, the original problem has no feasible solution.

For obvious reasons, the method we have just outlined is often called the Big M method. We now give a formal description of the Big M method.

Description of Big M Method

Step 1 Modify the constraints so that the right-hand side of each constraint is nonnegative. This requires that each constraint with a negative right-hand side be multiplied through by -1. Remember that if you multiply an inequality by any negative number, the direction of the inequality is reversed. For example, our method would transform the inequality $x_1 + x_2 \geq -1$ into $-x_1 - x_2 \leq 1$. It would also transform $x_1 - x_2 \leq -2$ into $-x_1 + x_2 \geq 2$.

Step 1′ Identify each constraint that is now (after step 1) an $=$ or $\geq$ constraint. In step 3, we will add an artificial variable to each of these constraints.

Step 2 Convert each inequality constraint to standard form. This means that if constraint i is a $\leq$ constraint, we add a slack variable s_i, and if constraint i is a $\geq$ constraint, we subtract an excess variable e_i.

Step 3 If (after step 1 has been completed) constraint i is a $\geq$ or $=$ constraint, add an artificial variable a_i. Also add the sign restriction $a_i \geq 0$.

Step 4 Let M denote a very large positive number. If the LP is a min problem, add (for each artificial variable) Ma_i to the objective function. If the LP is a max problem, add (for each artificial variable) $-Ma_i$ to the objective function.

Step 5 Because each artificial variable will be in the starting basis, all artificial variables must be eliminated from row 0 before beginning the simplex. This ensures that we begin with a canonical form. In choosing the entering variable, remember that M is a very large positive number. For example, $4M - 2$ is more positive than $3M + 900$, and $-6M - 5$ is more negative than $-5M - 40$. Now solve the transformed problem by the simplex. If all artificial variables are equal to zero in the optimal solution, then we have found the optimal solution to the original problem. If any artificial variables are positive in the optimal solution, then the original problem is infeasible.†

When an artificial variable leaves the basis, its column may be dropped from future tableaus because the purpose of an artificial variable is only to get a starting basic feasible solution. Once an artificial variable leaves the basis, we no longer need it. Despite this fact, we often maintain the artificial variables in all tableaus. The reason for this will become apparent in Section 6.7.

Solution **Example 4 (Continued)**

Step 1 Because none of the constraints has a negative right-hand side, we don't have to multiply any constraint through by -1.

†We have ignored the possibility that when the LP (with the artificial variables) is solved, the final tableau may indicate that the LP is unbounded. If the final tableau indicates the LP is unbounded and all artificial variables in this tableau equal zero, then the original LP is unbounded. If the final tableau indicates that the LP is unbounded and at least one artificial variable is positive, then the original LP is infeasible. See Bazaraa and Jarvis (1990) for details.

Step 1′ Constraints 2 and 3 will require artificial variables.

Step 2 Add a slack variable s_1 to row 1 and subtract an excess variable e_2 from row 2. The result is

$$\begin{aligned} &\min z = 2x_1 + 3x_2 \\ &\text{Row 1:} \quad \tfrac{1}{2}x_1 + \tfrac{1}{4}x_2 + s_1 = 4 \\ &\text{Row 2:} \quad x_1 + 3x_2 - e_2 = 20 \\ &\text{Row 3:} \quad x_1 + x_2 = 10 \end{aligned}$$

Step 3 Add an artificial variable a_2 to row 2 and an artificial variable a_3 to row 3. The result is

$$\begin{aligned} &\min z = 2x_1 + 3x_2 \\ &\text{Row 1:} \quad \tfrac{1}{2}x_1 + \tfrac{1}{4}x_2 + s_1 = 4 \\ &\text{Row 2:} \quad x_1 + 3x_2 - e_2 + a_2 = 20 \\ &\text{Row 3:} \quad x_1 + x_2 + a_3 = 10 \end{aligned}$$

From this tableau, we see that our initial bfs will be $s_1 = 4$, $a_2 = 20$, and $a_3 = 10$.

Step 4 Because we are solving a min problem, we add $Ma_2 + Ma_3$ to the objective function (if we were solving a max problem, we would add $-Ma_2 - Ma_3$). This makes a_2 and a_3 very unattractive, and the act of minimizing z will cause a_2 and a_3 to be zero. The objective function is now

$$\min z = 2x_1 + 3x_2 + Ma_2 + Ma_3$$

Step 5 Row 0 is now

$$z - 2x_1 - 3x_2 - Ma_2 - Ma_3 = 0$$

Because a_2 and a_3 are in our starting bfs (that's why we introduced them), they must be eliminated from row 0. To eliminate a_2 and a_3 from row 0, simply replace row 0 by row 0 + M(row 2) + M(row 3). This yields

$$\begin{aligned} &\text{Row 0:} && z - 2x_1 - 3x_2 - Ma_2 - Ma_3 = 0 \\ &M(\text{row 2}): && Mx_1 + 3Mx_2 - Me_2 + Ma_2 = 20M \\ &M(\text{row 3}): && Mx_1 + Mx_2 + Ma_3 = 10M \\ &\text{New row 0:} && z + (2M - 2)x_1 + (4M - 3)x_2 - Me_2 = 30M \end{aligned}$$

Combining the new row 0 with rows 1–3 yields the initial tableau shown in Table 33.

We are solving a min problem, so the variable with the most positive coefficient in row 0 should enter the basis. Because $4M - 3 > 2M - 2$, variable x_2 should enter the basis. The ratio test indicates that x_2 should enter the basis in row 2, which means the artificial variable a_2 will leave the basis. The most difficult part of doing the pivot is eliminating

TABLE 33
Initial Tableau for Bevco

z	x_1	x_2	s_1	e_2	a_2	a_3	rhs	Basic Variable	Ratio
1	$2M - 2$	$4M - 3$	0	$-M$	0	0	$30M$	$z = 30M$	
0	$\frac{1}{2}$	$\frac{1}{4}$	1	0	0	0	4	$s_1 = 4$	16
0	1	3	0	-1	1	0	20	$a_2 = 20$	$\frac{20}{3}$*
0	1	1	0	0	0	1	10	$a_3 = 10$	10

TABLE 34
First Tableau for Bevco

z	x_1	x_2	s_1	e_2	a_2	a_3	rhs	Basic Variable	Ratio
1	$\frac{2M-3}{3}$	0	0	$\frac{M-3}{3}$	$\frac{3-4M}{3}$	0	$\frac{60+10M}{3}$	$z = \frac{60+10M}{3}$	
0	$\frac{5}{12}$	0	1	$\frac{1}{12}$	$-\frac{1}{12}$	0	$\frac{7}{3}$	$s_1 = \frac{7}{3}$	$\frac{28}{5}$
0	$\frac{1}{3}$	1	0	$-\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{20}{3}$	$x_2 = \frac{20}{3}$	20
0	$\frac{2}{3}$	0	0	$\frac{1}{3}$	$-\frac{1}{3}$	1	$\frac{10}{3}$	$a_3 = \frac{10}{3}$	5*

x_2 from row 0. First, replace row 2 by $\frac{1}{3}$(row 2). Thus, the new row 2 is

$$\tfrac{1}{3}x_1 + x_2 - \tfrac{1}{3}e_2 + \tfrac{1}{3}a_2 = \tfrac{20}{3}$$

We can now eliminate x_2 from row 0 by adding $-(4M - 3)$(new row 2) to row 0 or $(3 - 4M)$(new row 2) + row 0. Now

$$(3 - 4M)(\text{new row 2}) \quad =$$

$$\frac{(3 - 4M)x_1}{3} + (3 - 4M)x_2 - \frac{(3 - 4M)e_2}{3} + \frac{(3 - 4M)a_2}{3} = \frac{20(3 - 4M)}{3}$$

$$\text{Row 0:} \qquad z + (2M - 2)x_1 + (4M - 3)x_2 - Me_2 = 30M$$

$$\text{New row 0: } z + \frac{(2M - 3)x_1}{3} + \frac{(M - 3)e_2}{3} + \frac{(3 - 4M)a_2}{3} = \frac{60 + 10M}{3}$$

After using EROs to eliminate x_2 from row 1 and row 3, we obtain the tableau in Table 34. Because $\frac{2M-3}{3} > \frac{M-3}{3}$, we next enter x_1 into the basis. The ratio test indicates that x_1 should enter the basis in the third row of the current tableau. Then a_3 will leave the basis, and our next tableau will have $a_2 = a_3 = 0$. To enter x_1 into the basis in row 3, we first replace row 3 by $\frac{3}{2}$(row 3). Thus, new row 3 will be

$$x_1 + \frac{e_2}{2} - \frac{a_2}{2} + \frac{3a_3}{2} = 5$$

To eliminate x_1 from row 0, we replace row 0 by row 0 + $(3 - 2M)$(new row 3)/3.

$$\text{Row 0:} \qquad z + \frac{(2M - 3)x_1}{3} + \frac{(M - 3)e_2}{3} + \frac{(3 - 4M)a_2}{3} = \frac{60 + 10M}{3}$$

$$\frac{(3 - 2M)(\text{new row 3})}{3}: \frac{(3 - 2M)x_1}{3} + \frac{(3 - 2M)e_2}{6} + \frac{(2M - 3)a_2}{6}$$

$$+ \frac{(3 - 2M)a_3}{2} = \frac{15 - 10M}{3}$$

$$\text{New row 0:} \qquad z - \frac{e_2}{2} + \frac{(1 - 2M)a_2}{2} + \frac{(3 - 2M)a_3}{2} = 25$$

New row 1 and new row 2 are computed as usual, yielding the tableau in Table 35. Because all variables in row 0 have nonpositive coefficients, this is an optimal tableau; all artificial variables are equal to zero in this tableau, so we have found the optimal solution to the Bevco problem: $z = 25$, $x_1 = x_2 = 5$, $s_1 = \frac{1}{4}$, $e_2 = 0$. This means that Bevco can hold the cost of producing a 10-oz bottle of Oranj to 25¢ by mixing 5 oz of orange soda and 5 oz of orange juice. Note that the a_2 column could have been dropped after a_2 left the basis (at the conclusion of the first pivot), and the a_3 column could have been dropped after a_3 left the basis (at the conclusion of the second pivot).

TABLE 35
Optimal Tableau for Bevco

z	x_1	x_2	s_1	e_2	a_2	a_3	rhs	Basic Variable
1	0	0	0	$-\frac{1}{2}$	$\frac{1-2M}{2}$	$\frac{3-2M}{2}$	25	$z = 25$
0	0	0	1	$-\frac{1}{8}$	$\frac{1}{8}$	$-\frac{5}{8}$	$\frac{1}{4}$	$s_1 = \frac{1}{4}$
0	0	1	0	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	5	$x_2 = 5$
0	1	0	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{3}{2}$	5	$x_1 = 5$

How to Spot an Infeasible LP

We now modify the Bevco problem by requiring that a 10-oz bottle of Oranj contain at least 36 mg of vitamin C. Even 10 oz of orange juice contain only 3(10) = 30 mg of vitamin C, so we know that Bevco cannot possibly meet the new vitamin C requirement. This means that Bevco's LP should now have no feasible solution. Let's see how the Big M method reveals the LP's infeasibility. We have changed Bevco's LP to

$$\begin{aligned} \min z &= 2x_1 + 3x_2 \\ \text{s.t.}\quad \tfrac{1}{2}x_1 + \tfrac{1}{4}x_2 &\le 4 &&\text{(Sugar constraint)} \\ x_1 + 3x_2 &\ge 36 &&\text{(Vitamin C constraint)} \\ x_1 + x_2 &= 10 &&\text{(10 oz constraint)} \\ x_1, x_2 &\ge 0 \end{aligned} \tag{19}$$

After going through Steps 1–5 of the Big M method, we obtain the initial tableau in Table 36. Because $4M - 3 > 2M - 2$, we enter x_2 into the basis. The ratio test indicates that x_2 should be entered in row 3, causing a_3 to leave the basis. After entering x_2 into the basis, we obtain the tableau in Table 37. Because each variable has a nonpositive coefficient in row 0, this is an optimal tableau. The optimal solution indicated by this tableau is $z = 30 + 6M$, $s_1 = \frac{3}{2}$, $a_2 = 6$, $x_2 = 10$, $a_3 = e_2 = x_1 = 0$. An artificial variable (a_2) is positive in the optimal tableau, so Step 5 shows that the original LP has no feasible solution.† In summary, *if any artificial variable is positive in the optimal Big M tableau, then the original LP has no feasible solution.*

TABLE 36
Initial Tableau for Bevco (Infeasible)

z	x_1	x_2	s_1	e_2	a_2	a_3	rhs	Basic Variable	Ratio
1	$2M - 2$	$4M - 3$	0	$-M$	0	0	$46M$	$z = 46M$	
0	$\frac{1}{2}$	$\frac{1}{4}$	1	0	0	0	4	$s_1 = 4$	16
0	1	3	0	−1	1	0	36	$a_2 = 36$	12
0	1	1	0	0	0	1	10	$a_3 = 10$	10*

†To explain why (19) can have no feasible solution, suppose that it does $(\bar{x}_1, \bar{x}_2)$. Clearly, if we set $a_3 = a_2 = 0$, $(\bar{x}_1, \bar{x}_2)$ will be feasible for our modified LP (the LP with artificial variables). If we substitute $(\bar{x}_1, \bar{x}_2)$ into the modified objective function ($z = 2\bar{x}_1 + 3\bar{x}_2 + Ma_2 + Ma_3$), we obtain $z = 2\bar{x}_1 + 3\bar{x}_2$ (this follows because $a_3 = a_2 = 0$). Because M is large, this z-value is certainly less than $6M + 30$. This contradicts the fact that the best z-value for our modified objective function is $6M + 30$. This means that our original LP (19) must have no feasible solution.

TABLE 37
Tableau Indicating Infeasibility for Bevco (Infeasible)

z	x_1	s_2	s_1	e_2	a_2	a_3	rhs	Basic Variable
1	$1 - 2M$	0	0	$-M$	0	$3 - 4M$	$30 + 6M$	$z = 6M + 30$
0	$\frac{1}{4}$	0	1	0	0	$-\frac{1}{4}$	$\frac{3}{2}$	$s_1 = \frac{3}{2}$
0	-2	0	0	-1	1	-3	6	$a_2 = 6$
0	1	1	0	0	0	1	10	$x_2 = 10$

Note that when the Big M method is used, it is difficult to determine how large M should be. Generally, M is chosen to be at least 100 times larger than the largest coefficient in the original objective function. The introduction of such large numbers into the problem can cause roundoff errors and other computational difficulties. For this reason, most computer codes solve LPs by using the two-phase simplex method (described in Section 4.13).

PROBLEMS

Group A

Use the Big M method to solve the following LPs:

1 min $z = 4x_1 + 4x_2 + x_3$

s.t. $x_1 + x_2 + x_3 \le 2$
$2x_1 + x_2 \le 3$
$2x_1 + x_2 + 3x_3 \ge 3$
$x_1, x_2, x_3 \ge 0$

2 min $z = 2x_1 + 3x_2$

s.t. $2x_1 + x_2 \ge 4$
$x_1 - x_2 \ge -1$
$x_1, x_2 \ge 0$

3 max $z = 3x_1 + x_2$

s.t. $x_1 + x_2 \ge 3$
$2x_1 + x_2 \le 4$
$x_1 + x_2 = 3$
$x_1, x_2 \ge 0$

4 min $z = 3x_1$

s.t. $2x_1 + x_2 \ge 6$
$3x_1 + 2x_2 = 4$
$x_1, x_2 \ge 0$

5 min $z = x_1 + x_2$

s.t. $2x_1 + x_2 + x_3 = 4$
$x_1 + x_2 + 2x_3 = 2$
$x_1, x_2, x_3 \ge 0$

6 min $z = x_1 + x_2$

s.t. $x_1 + x_2 = 2$
$2x_1 + 2x_2 = 4$
$x_1, x_2 \ge 0$

4.13 The Two-Phase Simplex Method†

When a basic feasible solution is not readily available, the two-phase simplex method may be used as an alternative to the Big M method. In the two-phase simplex method, we add artificial variables to the same constraints as we did in the Big M method. Then we find a bfs to the original LP by solving the Phase I LP. In the Phase I LP, the objective function is to minimize the sum of all artificial variables. At the completion of Phase I, we reintroduce the original LP's objective function and determine the optimal solution to the original LP.

The following steps describe the two-phase simplex method. Note that steps 1–3 for the two-phase simplex are identical to steps 1–3 for the Big M method.

†This section covers topics that may be omitted with no loss of continuity.