

Exact Generation of Acyclic Deterministic Finite Automata

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Abstract.

We give a canonical representation for trim acyclic deterministic finite automata (ADFA) with n states over an alphabet of k symbols. Using this normal form, we present a backtracking algorithm for the exact generation of ADFAs. This algorithm is a non trivial adaptation of the algorithm for the exact generation of minimal acyclic deterministic finite automata (MADFA), presented by Almeida *et al.*

Keywords: acyclic deterministic finite automata, finite languages, generation, enumeration

1 Introduction

Recently, Liskovets [10] obtained a formula for the enumeration of unlabelled (non-isomorphic) initially connected acyclic deterministic finite automata with n states over an alphabet of k symbols. Callan [4] presented a canonical form for those automata and showed that a certain determinant of Stirling cycle numbers can also count them. That canonical form is obtained by observing that if we mark the visited states, starting with the initial state, it is always possible to find a state whose only incident states are already marked. This induces a unique labelling of states, but it is not clear how these representations can be used in automata generation. Almeida *et al.* [2] obtained a canonical form for (non-isomorphic) minimal acyclic deterministic finite automata (MADFA) and an exact generation algorithm. Unfortunately the canonical form did not provide directly an enumeration formula for MADFAs. One of the applications of such an enumeration formula would be in the development of uniform random generators of automata, useful for the average case analysis of algorithms for that class of automata. The enumeration of different kinds of finite automata was considered by several authors since late 1950s. For more complete surveys we

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refer the reader to Domaratzki *et al.* [7] and to Domaratzki [6]. Liskovets [9] and Robinson [14] counted non-isomorphic initially connected deterministic finite automata (ICDFA). More recently, several authors examined related problems. Domaratzki *et al.* [7] studied the (exact and asymptotic) enumeration of distinct languages accepted by finite automata with n states. Nicaud [12], Champarnaud and Paranthoën [5] presented a method for randomly generating complete ICDFA's. Bassino and Nicaud [3] showed that the number of complete ICDFA's is $\Theta(n2^n S(kn, n))$, where $S(kn, n)$ is a Stirling number of the second kind. Based on a canonical string representation for ICDFA's, Almeida *et al.* [1] obtained a new formula for the number of non-isomorphic ICDFA's, and provided exact and uniform random generators for them.

In this paper, we give a canonical representation for trim (complete) acyclic deterministic finite automata (ADFA). By trim we mean that from the initial state all other states are reachable (initially connected) and that from all states (but the dead) at least one final state is reachable (useful).

This canonical form extends the one for MADFAs by taking into consideration equivalent states. The backtracking algorithm for the exact generation of ADFAs is a non-trivial adaptation of the one for MADFAs, because we must properly consider the equivalence classes but still avoid the multiple generation of isomorphic automata. It is easy to order equivalent states according to the words that reach them (i.e., their left languages) but to obtain a feasible generator algorithm we must find an ordering such that:

- the canonical representation for ADFAs is a natural extension of the canonical representation for MADFAs (i.e., preserves its characteristics);
- it allows the detection of an ill-formed automata representation as soon as possible (as the algorithm proceeds backwards, towards the initial state);
- it allows the exact generation algorithm to output the automata canonical representations in increasing order.

ADFAs, as defined here, are a proper subset of the class of acyclic automata enumerated by Liskovets and Callan because we only consider automata where all the states are useful. Once more, their formulae can not be used directly, but in this paper we hope to contribute to a better understanding of the internal structure of ADFAs.

The paper is organized as follows. In the next section some basic concepts and notation are introduced. In Section 3 we review some concepts about acyclic deterministic finite automata and the canonical form for MADFAs. In Section 4 we show how to extend that canonical form to ADFAs. In Section 5 we describe an algorithm to efficiently generate equivalent states as an extension to the exact generator for MADFAs. Some experimental results are also summarized in that section. In Section 6 we consider ADFAs enumeration formulas for small values of n and k . Finally Section 7 concludes.

2 Basic concepts

We review some basic concepts we need in this paper. For more details we refer the reader to Hopcroft *et al.* [8], Yu [15] or Lothaire [11].

Let $[n, m]$ denote the set $\{i \in \mathbb{Z} \mid n \leq i \leq m\}$. In a similar way, we consider the variants $]n, m]$, $[n, m[$ and $]n, m[$. Whenever we have a finite ordered set A , and a function f on A , the expression $(f(a))_{a \in A}$ denote the values of f for increasing values of A .

Let Σ be an *alphabet* and Σ^* be the set of all words over Σ . The empty word is denoted by ε . The length of a word $x = \sigma_1 \sigma_2 \cdots \sigma_n$, denoted by $|x|$, is n . A language L is a subset of Σ^* . A language is finite if its cardinality is finite.

The alphabet Σ can be equipped with a total order $<$ that allows the definition of total orders on Σ^* . A *lexicographical* order on Σ^* is defined as follows. Let $x = x_1 \cdots x_m$, $y = y_1 \cdots y_n \in \Sigma^*$. Then $x < y$ if:

- (i) there exists an integer $j \in [1, \min\{m, n\}]$ such that $(\forall i \in [1, j]) x_i = y_i$ and $x_j < y_j$;
- (ii) $m < n$ and $(\forall i \in [1, m]) x_i = y_i$.

A *deterministic finite automaton* (DFA) \mathcal{A} is a tuple $(S, \Sigma, \delta, s_0, F)$ where S is a finite set of states, Σ is the alphabet, $\delta : S \times \Sigma \rightarrow S$ is the transition function, s_0 the initial state and $F \subseteq S$ the set of final states.

We assume that the transition function is total, so we consider only *complete* DFAs. The transition function δ is inductively extended to Σ^* , by $(\forall s \in S) \delta(s, \varepsilon) = s$ and $\delta(s, x\sigma) = \delta(\delta(s, x), \sigma)$.

A DFA is *initially connected* (or *accessible*) (ICDFA) if for each state $s \in S$ there exists a word $x \in \Sigma^*$ such that $\delta(s_0, x) = s$. A DFA is *trim* if it is an ICDFA and every state is *useful*, i.e., $(\forall s \in S)(\exists x \in \Sigma^*) \delta(s, x) \in F$.

Two DFAs $(S, \Sigma, \delta, s_0, F)$ and $(S', \Sigma', \delta', s'_0, F')$ are called *isomorphic* if $|\Sigma| = |\Sigma'| = k$, there exist bijections $\Pi_1 : \Sigma \rightarrow [0, k-1]$, $\Pi_2 : \Sigma' \rightarrow [0, k-1]$ and a bijection $\iota : S \rightarrow S'$ such that $\iota(s_0) = s'_0$, $\iota(F) = F'$, and for all $\sigma \in \Sigma$ and $s \in S$, $\iota(\delta(s, \sigma)) = \delta'(\iota(s), \Pi_2^{-1}(\Pi_1(\sigma)))$.

The *language* accepted by a DFA \mathcal{A} is $\mathcal{L}(\mathcal{A}) = \{x \in \Sigma^* \mid \delta(s_0, x) \in F\}$. For a state $s \in S$ we denote

$$\begin{aligned} \mathcal{L}_L(\mathcal{A}, s) &= \{x \in \Sigma^* \mid \delta(s_0, x) = s\}, \\ \mathcal{L}_R(\mathcal{A}, s) &= \{x \in \Sigma^* \mid \delta(s, x) \in F\}, \end{aligned}$$

the *left* and the *right language* of state s , respectively. We omit \mathcal{A} whenever no confusion arises. All states of a DFA have distinct *left* languages.

Two DFAs are *equivalent* if they accept the same language. We say that two states s and s' are *equivalent*, $s \sim s'$, if and only if $\mathcal{L}_R(\mathcal{A}, s) = \mathcal{L}_R(\mathcal{A}, s')$. A DFA is *minimal* if it has no equivalent states and it is initially-connected. Minimal DFAs are unique up to isomorphism.

3 Acyclic finite automata

An *acyclic deterministic finite automaton* is a DFA $\mathcal{A} = (S \cup \{\Omega\}, \Sigma, \delta, s_0, F)$ with $F \subseteq S$ and $s_0 \neq \Omega$ such that $(\forall \sigma \in \Sigma) \delta(\Omega, \sigma) = \Omega$ and $(\forall x \in \Sigma^* \setminus \{\varepsilon\})(\forall s \in S) \delta(s, x) \neq s$. The state Ω is called the *dead state*, and is the only *cyclic* state of \mathcal{A} . The *size* of \mathcal{A} is

$|S|$. We are going to consider only trim complete acyclic deterministic finite automata (ADFA), where all states but Ω are useful. It is obvious that the language of an ADFA is finite.

A state $s \in S$ is called *pre-dead* if $(\forall \sigma \in \Sigma) \delta(s, \sigma) = \Omega$. Every ADFA has at least a pre-dead state and all pre-dead states are final.

Given an ADFA, $\mathcal{A} = (S \cup \{\Omega\}, \Sigma, \delta, s_0, F)$, the *rank* of a state $s \in S$, denoted $\text{rk}(s)$, is the length of the longest word $x \in \Sigma^*$ such that $\delta(s, x) \in F$ (i.e., $x \in \mathcal{L}_R(\mathcal{A}, s)$). The *rank*¹ of an ADFA \mathcal{A} , $\text{rk}(\mathcal{A})$, is $\max\{\text{rk}(s) \mid s \in S\}$. Trivially, we have that $\text{rk}(s_0) = \text{rk}(\mathcal{A})$ and $\text{rk}(s) = 0$, for all pre-dead states s .

For every state $s \in S$, with $\text{rk}(s) > 0$ there exists a transition to a state with rank immediately lower than s 's.

Lemma 1. *Let $\mathcal{A} = (S \cup \{\Omega\}, \Sigma, \delta, s_0, F)$ be an ADFA, then*

$$(\forall s \in S)(\text{rk}(s) \neq 0 \Rightarrow (\exists \sigma \in \Sigma) \text{rk}(\delta(s, \sigma)) = \text{rk}(s) - 1).$$

Two states s and s' are *mergeable* if they are both either final or not final, and the transition function is identical, i.e.,

$$(s \in F \leftrightarrow s' \in F) \wedge (\forall \sigma \in \Sigma) \delta(s, \sigma) = \delta(s', \sigma).$$

For instance, in the ADFA of Figure 1 the states s_2 and s_3 are mergeable, and s_7 and s_8 are mergeable too.

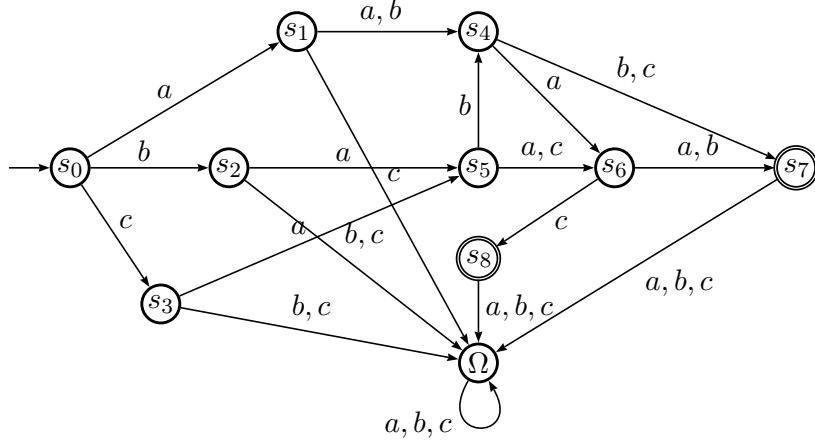


Figure 1. An ADFA.

An ADFA can be minimized by merging mergeable states, thus, a minimal ADFA (MADFA) can be characterized by:

Lemma 2 [13, 11]. *An ADFA $\mathcal{A} = (S \cup \{\Omega\}, \Sigma, \delta, s_0, F)$ is minimal if and only if it has no mergeable states.*

¹Also called the *diameter* of \mathcal{A} .

It is a direct consequence of Lemma 2 that every MADFA has a unique pre-dead state, $\pi \in S$, and that mergeable states have the same rank. This implies that to minimize an ADFFA it is only necessary to merge states by increasing rank order (see Revuz [13] or Lothaire [11]).

3.1 A normal form for MADFAs

Based upon the above considerations, Almeida *et al.* [2] presented a canonical representation for MADFAs.

Let $\mathcal{A} = (S \cup \{\Omega\}, \Sigma, \delta, s_0, F)$ be a MADFA with $k = |\Sigma|$ and $n = |S| \geq 2$. Consider a total order over Σ and let $\Pi : \Sigma \rightarrow [0, k[$ be the bijection induced by that order. Let $R_l = \{s \in S \mid \text{rk}(s) = l\}$. It is possible to obtain a canonical numbering of the states $\varphi : S \cup \{\Omega\} \rightarrow [0, n]$ proceeding by increasing rank order and considering an ordering over the $(k + 1)$ -tuples that represent the transition function and the finality of each state. For each state $s \in S$, let its *representation* be a $(k + 1)$ -tuple $\Delta(s) = (\varphi(\delta(s, \Pi^{-1}(0))), \dots, \varphi(\delta(s, \Pi^{-1}(k - 1))), f)$, where the first k values represent the transitions from state s and the last value, f , is 1 if $s \in F$ or 0, otherwise. Let $\varphi(\Omega) = 0$ and $\varphi(\pi) = 1$. Thus, the representations of Ω and π are $(0^k, 0)$, and $(0^k, 1)$, respectively. We can continue this process considering the states by increasing rank order, and in each rank we number the states by lexicographic order over their transition representations. It is important to note that transitions from a given state can only refer to states of a lower rank, and thus already numbered. The sequence of tuples $(\Delta(i))_{i \in [0, n]}$ is the *canonical string representation* of \mathcal{A} . Formally, the assignment of state numbers, φ , can be described by the following simple algorithm:

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 $\varphi(\Omega) \leftarrow 0; \varphi(\pi) \leftarrow 1; i \leftarrow 2$ 
for  $l$  in  $]0, \text{rk}(\mathcal{A})]$ 
  for  $s \in R_l$  by lexicographic order over  $\Delta(s)$ 
     $\varphi(s) \leftarrow i$ 
     $i \leftarrow i + 1$ 

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In Figure 2, we present a MADFA ($n = 7$ and $k = 3$), the φ function and its canonical representation.

The characterization of these strings and that they constitute a canonical representation for MADFAs is given by the following theorem:

Theorem 3 [2, Thms.3-5]. *There exists a bijection between non-isomorphic MADFAs with n states and k symbols and the set of strings $(s_i)_{i \in [0, (k+1)(n+1)[}$, with $s_i \in [0, n[$ that satisfy the following conditions. Let $(f_i)_{i \in [1, n[}$ be the sequence of the positions in $(s_i)_i$ of the first occurrence of each $i \in [1, n[$. Let $d \leq n$ and let $(r_l)_{l \in [0, d]} \in [1, n]$*

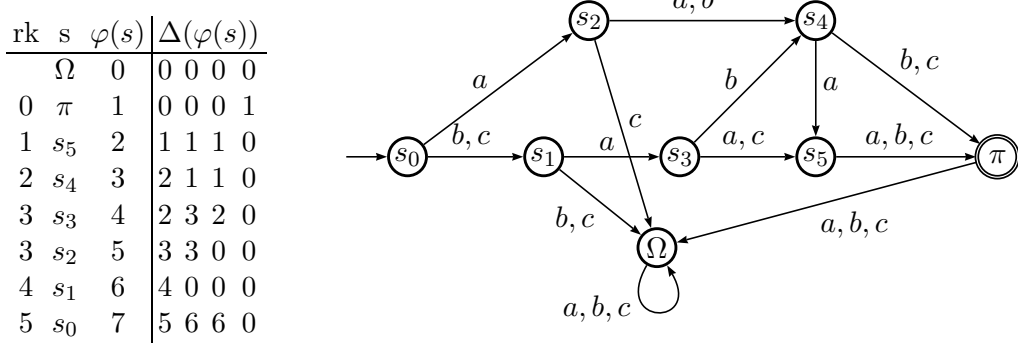


Figure 2. An example of a MADFA that can be described by the canonical representation $[[0, 0, 0, 0], [0, 0, 0, 1], [1, 1, 1, 0], [2, 1, 1, 0], [2, 3, 2, 0], [3, 3, 0, 0], [4, 0, 0, 0], [5, 6, 6, 0]]$.

be the sequence of the first states of each rank in $(s_i)_i$. Then:

$$s_0 = \dots = s_k = \dots = s_{2k} = 0 \wedge s_{2k+1} = 1 \quad (\text{N0})$$

$$(\forall i \in [0, n]) s_{(k+1)i+k} \in \{0, 1\} \quad (\text{N1})$$

$$r_0 = 1 \wedge r_1 = 2 \wedge r_d = n \wedge (\forall l \in [0, d]) r_l < r_{l+1} \quad (\text{N2})$$

$$((\forall i \in [1, n]) s_{f_i} = i \wedge \quad (\text{N3})$$

$$(\forall j \in [0, n])(\forall m \in [0, k]) ((k+1)j + m < f_i \Rightarrow s_{(k+1)j+m} \neq i))$$

$$(\forall l \in [0, d])(\forall i \in [r_l, r_{l+1}[) kr_{l+1} + 1 \leq f_i \quad (\text{N4})$$

$$(\forall l \in [0, d])(\forall i \in [r_l, r_{l+1}[) (\exists m \in [0, k]) s_{(k+1)i+m} \in [r_{l-1}, r_l[\quad (\text{N5})$$

$$(\forall l \in [0, d])(\forall i \in [r_l, r_{l+1} - 1]) (s_{(k+1)i+m})_{m \in [0, k]} < (s_{(k+1)(i+1)+m})_{m \in [0, k]} \quad (\text{N6})$$

The condition N0 gives the representation of the dead (Ω) and the pre-dead state (π). The condition N1 states that the last symbol of each state representation indicates if the state is final or not. The condition N2 ensures that states are numbered by increasing rank order. The condition N3 defines the sequence $(f_i)_{i \in [1, n]}$, and ensures that \mathcal{A} is initially connected. The condition N4 is a direct consequence of the rank definition, i.e., a state can only refer to a state of a lower rank. The condition N5 states that every state has a transition to a state with rank immediately lower than its own. The condition N6 ensures that within a rank the state representations are lexicographically ordered.

4 A normal form for ADFAs

If an ADFA is not minimal, then it has at least two mergeable states, but not all equivalent states need to be mergeable. The two following lemmas give characterizations of equivalent states in an ADFA that will be used to obtain a canonical representation.

Lemma 4. *In an ADFA every two equivalent states must belong to the same rank.*

This follows directly from the definitions.

Lemma 5. *Let $\mathcal{A} = (S \cup \{\Omega\}, \Sigma, \delta, s_0, F)$ be an ADFA. For all $s, s' \in S$, if $s \sim s'$ then there exists $w \in \Sigma^*$, such that $\delta(s, w)$ and $\delta(s', w)$ are mergeable states.*

Proof: If $s \sim s'$ then $(\forall \sigma \in \Sigma), \delta(s, \sigma) \sim \delta(s', \sigma)$. Suppose that there exists $\sigma_1 \in \Sigma$ such that $s_1 = \delta(s, \sigma_1) \neq \delta(s', \sigma_1) = s'_1$. Because $s_1 \sim s'_1$ we can proceed as before, but because \mathcal{A} is acyclic and $|S|$ is finite this process must stop, and two mergeable states, s_j and s'_j for $j \leq |S|$, must be reached. The concatenation of the $\sigma_1 \dots \sigma_j$ provides the word w . \square

In order to have a canonical representation for ADFAs we must provide an ordering for the equivalent states. Because they must appear in the same rank we may restrict the state ordering by rank and consider a proper extension of the function φ (assignment of state numbers), and so a proper extension of the canonical representation for MADFAs. In particular we take $\varphi(\Omega) = 0$. Because ADFAs are deterministic, we have

Lemma 6. *Let $\mathcal{A} = (S \cup \{\Omega\}, \Sigma, \delta, s_0, F)$ be an ADFA. Then*

$$(\forall s \neq s' \in S \cup \{\Omega\}), \mathcal{L}_L(s) \cap \mathcal{L}_L(s') = \emptyset.$$

Any two different states can be distinguished, if we define any injective function $\Psi : S \rightarrow O$, where O must be a total ordered set.

For instance, given an order over Σ we could have $\Psi : S \rightarrow \Sigma^*$ given by $\Psi(s) = \min\{w \mid w \in \mathcal{L}_L(s)\}$, for $s \in S$, where \min is taken considering the lexicographical order on Σ^* . Then, whenever two mergeable states s and s' were found, we could take $s < s'$ if and only if $\Psi(s) < \Psi(s')$ (lexicographically).

In the general case, given an injective function Ψ , let \prec_Ψ be an ordering such that $(\forall s, s' \in S), s \prec_\Psi s'$ if:

- (i) $\Delta(s) < \Delta(s')$, where $<$ is the lexicographical order;
- (ii) if $\Delta(s) = \Delta(s')$ then $\Psi(s) < \Psi(s')$.

The algorithm of page 5, that computes the function φ can be adapted for ADFAs by not considering the state π , initializing i with 1 and considering the order \prec_Ψ . Consider the ADFA of Figure 1. Its state ranks are the following: $R_0 = \{s_7, s_8\}$, $R_1 = \{s_6\}$, $R_2 = \{s_4\}$, $R_3 = \{s_1, s_5\}$, $R_4 = \{s_2, s_3\}$ and $R_5 = \{s_0\}$. Regarding the function Ψ above, the function φ is defined by the following tuples: $(s_7, 1), (s_8, 2), (s_6, 3), (s_4, 4), (s_5, 5), (s_1, 6), (s_3, 7), (s_2, 8), (s_0, 9)$. And, its string representation is

$$[[0, 0, 0, 0][0, 0, 0, 1][0, 0, 0, 1][1, 1, 2, 0][3, 1, 1, 0][0, 1, 0, 0][4, 4, 0, 0][5, 0, 0, 0][5, 0, 0, 0][6, 7, 8, 0]];$$

which is lexicographically ordered within a rank (i.e., respects condition N6, considering \leq instead of $<$).

As we aim to obtain an exact generator that will proceed by increasing rank order, it is convenient that $\Psi(s)$ is related to a maximal word of $\mathcal{L}_L(s)$. To assure that in a rank the state representations are lexicographically ordered we also take into consideration the ranks and the finalities of the states.

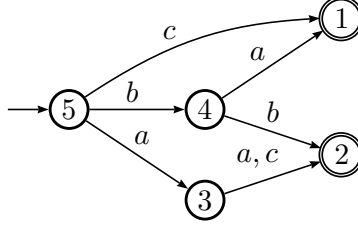


Figure 3. An ADFA which canonical string representation considering Ψ_c is: $[[0, 0, 0, 0][0, 0, 0, 1][0, 0, 0, 1][1, 0, 1, 0][1, 2, 0, 0][3, 4, 2, 0]]$.

Let $\mathcal{A} = (S \cup \{\Omega\}, \Sigma, \delta, s_0, F)$ be an ADFA, with Σ ordered. For each state $s \in S$, let $\delta^{-1}(s) = \{(s', \sigma) \mid \delta(s', \sigma) = s\}$, and for $(s', \sigma) \in \delta^{-1}(s)$ let consider the tuple $\tau = (rk(s'), \sigma, f_{s'})$ with $f_{s'} = 1$, if $s' \in F$ or 0, otherwise. We define $\mathcal{L}_L^{rk}(s)$ to be the set of sequences of these tuples $\tau_0 \dots \tau_l$ such that $\sigma_l \dots \sigma_0 \in \mathcal{L}_L(s)$. The *characteristic word* of s , is

$$\Psi_c(s) = \min\{\tau_0 \dots \tau_l \mid \tau_0 \dots \tau_l \in \mathcal{L}_L^{rk}(s)\},$$

where min is taken lexicographically.

In the example of Figure 3, we have $\Psi_c(1) = 1a02b0$ and $\Psi_c(2) = 1a02a0$ which shows that the numbers assigned to these states must be reversed, i.e., $\varphi(1) = 2$ and $\varphi(2) = 1$.

The following three theorems guarantee that this representation is indeed a canonical representation for ADFAs.

Theorem 7. *Let $\mathcal{A} = (S \cup \{\Omega\}, \Sigma, \delta, s_0, F)$ be an ADFA with $rk(\mathcal{A}) = d$, $n = |S|$ and $k = |\Sigma|$. Let $(s_i)_{i \in [0, (k+1)(n+1)[}$, with $s_i \in [0, n[$, be the string representation of \mathcal{A} obtained using the the function Ψ_c . Then the conditions N0–N5 of Theorem 3 are satisfied, together with the following condition N6':*

$$(\forall l \in [0, d[)(\forall i \in R_l) i \prec_{\Psi_c} i + 1. \quad (\text{N6}')$$

Proof: Follows from the above considerations. \square

Theorem 8. *Let $(s_i)_{i \in [0, (k+1)(n+1)[}$ with $s_i \in [0, n[$ be a string that satisfies conditions N0–N5 and condition N6', then the corresponding automaton is an ADFA with n states and an alphabet of k symbols.*

Proof: From conditions N0–N5, we knew that we could obtain a trim complete acyclic deterministic finite automaton. The relaxation of condition N6 to condition N6' allows some states to be mergeable. \square

Theorem 9. *Let $(s_i)_{i \in [0, (k+1)(n+1)[}$ and $(s'_i)_{i \in [0, (k+1)(n+1)[}$ be two distinct strings satisfying conditions N0–N5 and condition N6'. Then they correspond to distinct ADFAs.*

Proof: The proof follows exactly the lines of Theorem 5 in Almeida *et al.*[2], because of condition N6'. \square

5 Exact generation of ADFAs

To generate all the string representations of the ADFAs with n states and k symbols, we will use the same approach described by Almeida *et al* [2], traversing the search tree, backtracking on its way, to generate all possible representations. The representations will appear lexicographically. The conditions to generation are the same but with N6 replaced by its relaxed form N6'. The satisfaction of the conditions on the order of equivalent states is too complex to be included in the generation. When a pair of equivalent states is generated, instead of renumbering them according to the first word (for some order) that reaches each state, we proceed with the generation of all the states in lexicographical order of their Δ values, and discard the automata for which the previously stated order is contradicted.

The problem with this strategy is that, with the “natural” lexicographical order, the contradiction to the order of two states in rank 0 may appear only when generating the last state, i.e., the *initial state* of the automaton. This is very inconvenient, because a lot of generating work is going to be discarded and because of the backtracking strategy, the corresponding search tree is not pruned as it should. On the other hand, using the order described in Section 4 we can evaluate the possible contradictions after the complete generation of each rank of states.

The algorithm goes as follows:

- At the beginning of the generation of each rank, there are two data structures:
 - Probl** a set of lists of states that are equivalent and for which we want to ensure that the characteristic words that reach them are in accordance with that order;
 - Refs** an empty set of lists of states that, in that rank, have transitions to states in some list in **Probl**.
- Every time two or more states with the same Δ are generated, they are added as a new list to **Probl**.

$$\begin{aligned} (\Delta(s_1) = \Delta(s_2) = \dots = \Delta(s_l)) \wedge (\varphi(s_1) < \varphi(s_2) < \dots < \varphi(s_l)) \Rightarrow \\ \Rightarrow \mathbf{Probl} \leftarrow \mathbf{Probl} \cup \{[s_1, s_2, \dots, s_l]\}. \end{aligned}$$

- Every time a newly generated state has a transition to a state present in a list of **Probl**, it is added to **Refs** with information about the state it has a transition to.
- When the state generation of a given rank is finished (because no more states in that rank can be generated according to rules N0–N6'), each list R in **Refs** of states with transitions to states in a list L in **Probl** is examined.
 - * For $x \in L$, let $m(x) = \min\{(\sigma, f_s) \mid (s, \sigma) \in \delta^{-1}(x) \wedge s \in R\}$, where f_s represents the finality of s . If for some pair of states of L a contradiction is found, i.e.,

$$\begin{aligned} (\exists x_1, x_2 \in L)(\exists s_1, s_2 \in R)(\exists \sigma_1, \sigma_2 \in \Sigma) \\ (\delta(s_1, \sigma_1) = x_1 \wedge \delta(s_2, \sigma_2) = x_2 \wedge m(x_1) < m(x_2) \wedge \varphi(x_1) > \varphi(x_2)); \end{aligned}$$

then the generation of this automaton is aborted and the process is continued by backtracking.

- * For all the non-singleton sublists $M_{(\sigma,f)}$ of states in L such that

$$(\forall x \in M_{(\sigma,f)})m(x) = (\sigma, f);$$

its elements are removed from L , and the list of the states s of R such that $(\delta(s, \sigma) \in M_{(\sigma,f)} \wedge f_s = f)$, with the order induced by L , is added to **Probl**.

- * Finally, if

$$\begin{aligned} & (\forall x_1, x_2 \in L)(\exists s_1, s_2 \in R)(\exists \sigma_1, \sigma_2 \in \Sigma) \\ & ((\delta(s_1, \sigma_1) = x_1 \wedge \delta(s_2, \sigma_2) = x_2) \Rightarrow (m(x_1) < m(x_2) \Rightarrow \varphi(x_1) < \varphi(x_2))); \end{aligned}$$

then all the states in L that are the image of a transition from a state in R are removed from L , and R is removed from **Refs**.

- * All empty or singleton lists are removed from **Probl**.
- * Before the generation of a new rank is started, **Refs** is emptied.

The correctness of this algorithm follows from the considerations in Section 4.

5.1 Some experimental results

In Table 1 the number of MADFAs and ADFAs for some small values of n and k are summarized. We observe that almost all ADFAs are MADFAs. Several performance times are also presented. For the enumeration of ADFAs and MADFAs instead of the exact generators, we also generate initially-connected deterministic automata (ICDFAs), using the method presented in Almeida *et al.* [1], and then test for acyclicity, trimness and possibly, for minimality. But the number of IDFAs grows much faster than the number of ADFAs (or MADFAs), so the generate-test-reject method is not feasible. In column **Time B** of Table 1 we present the running times obtained by this method (for small values of n and k). In column **Time A** of Table 1 we present the running times obtained by the exact generation methods.

Considering only the performance times for MADFAs and $k = 2$, we obtained a curve fitting for both methods: for the exact generation a function $f(n) = e^{3.66n-20.76}$ and for the test-reject a function $g(n) = e^{4.21n-23.0}$, which gives $g(n)/f(n) = e^{0.55n-2.24}$.

As for the performance values we should only consider their order of magnitude as they were obtained using different CPUs and programs implemented in different programming languages. Both performance times **B**, were obtained using a **C** implementation and running on a AMD Athlon 64 at 2.5GHz. Performance times **A** were obtained using a **C++** implementation and running on a Intel[®] Xeon[®] 5140 at 2.33GHz, and a **Python** implementation running on a AMD Athlon 64 at 2.5GHz, respectively for MADFAs and for ADFAs (in general the **C++** implementation for MADFA is two times faster than the correspondent **Python** implementation).

It is reasonable that for (very) small values of n the test-reject method is faster, as the pruning of non legal ADFAs is a relatively costly operation. But because of the

$k = 2$							
n	MADFA	Time A (s)	Time B (s)	A/B	ADFA	Time A (s)	Time B (s)
3	60	0	0	1	62	0	0
4	900	0	0	1	964	0.14	0
5	18480	0.1	0.1	1	20424	3.65	0.2
6	487560	3	16	0.18	553472	110.92	13.92
7	15824880	116	687	0.16	18384552		736.89
8	612504240	4742	35774	0.13	726133776		20284.14
9	27619664640	224243	2345124	0.09			
$k = 3$							
n	MADFA	Time A (s)	Time B (s)	A/B	ADFA	Time A (s)	Time B (s)
2	14	0	0	1	14	0	0
3	532	0.1	0.1	1	544	0.07	0
4	42644	0.22	0.58	0.38	44290	7.59	6.16
5	6011320	43	3340	0.01	6306672		3142
6	1330452032	11501	2431307	0.005			
$k = 4$							
n	MADFA	Time A (s)	Time B (s)	A/B	ADFA	Time A (s)	Time B (s)
2	30	0	0	1	30	0	0
3	3900	0.2	1.6	0.13	3950	0.51	1.55
4	1460700	7.7	5549	0.001	1488120	236	5326
$k = 5$							
n	MADFA	Time A (s)	Time B (s)	A/B	ADFA	Time A (s)	Time B (s)
2	62	0	0	1	62	0	0
3	26164	0.121	2.396	0.05	26344	3.4	116
4	43023908	213			43411218	6805	4872111

Table 1. Number of MADFAs and ADFAs for small values of n and k . Performance times for its generation: exact (A) and with a test-rejection pass (B).

much faster growing of the number of ICDFAs (when compared with the number of ADFAs), that will not happen for larger ns .

6 Counting ADFAs for n and k

Let $A_k(n)$ be the number of ADFAs with n states over an alphabet of k symbols and let $M_k(n)$ be the corresponding number of MADFAs. In Almeida *et al.* [2], the values of $M_k(n)$ were determined for $n \in [1, 5]$. The same kind of results can be obtained for $A_k(n)$. The values of $A_k(n)$ for small values of n can be determined by considering the possible distribution of states by ranks and the number of *dangling* states that are targets of transitions from a state of a previous rank, for the first time. Using the Principle of Inclusion and Exclusion we have:

$$\begin{aligned} A_k(2) &= M_k(2) = 2(2^k - 1). \\ A_k(3) &= M_k(3) + (3^k - 2^{k+1} + 1) = 2^2(3^k - 2^k)(2^k - 1) + (3^k - 2^{k+1} + 1). \\ A_k(4) &= 2^3(4^k - 3^k)(3^k - 2^k)(2^k - 1) + 2^2(4^k - 3^k 2 + 2^k)(2^k - 1)^2 \\ &\quad + 2(4^k - 3^k)(3^k - 2^k 2 + 1) + (4^k - 3^k 3 + 2^k 3 - 1)/3. \end{aligned}$$

For $n = 5$ there are already 12 configurations to be considered. For values of $n \in [2, 5]$, $\lim_{k \rightarrow \infty} M_k(n)/A_k(n) = 1$. We note that this behaviour is also observed (experimentally) in the case of arbitrary IC DFA's.

7 Conclusions

A canonical representation for minimal acyclic deterministic finite automata was extended to allow equivalent states, and thus uniquely represent trim acyclic deterministic finite automata. A method for the exact generation of MADFAs was extended to allow the generation of equivalent states, while still avoiding the multiple generation of non-isomorphic automata. More experimental tests must be carried on in order to see what really is the overhead of pruning non-legal equivalent states.

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