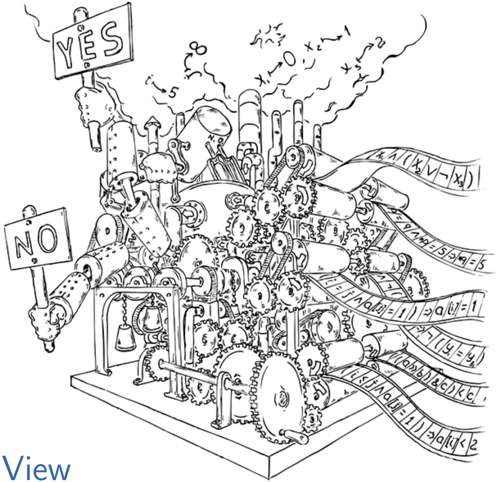
Bit-Vectors

Chapter 6



Decision Procedures

An Algorithmic Point of View

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Revision 1.2

Outline

Bit-Vector Logic: Syntax

$$
\begin{array}{rcl}
\mathit{formula} & : & \mathit{formula} \vee \mathit{formula} \mid \neg \mathit{formula} \mid \mathit{atom} \\
\mathit{atom} & : & \mathit{term}\ \mathit{rel}\ \mathit{term} \mid \mathit{Boolean\text{-}Identifier} \mid \mathit{term}[\,\mathit{constant}\,] \\
\mathit{rel} & : & = \ \mid < \\
\mathit{term} & : & \mathit{term}\ \mathit{op}\ \mathit{term} \mid \mathit{identifier} \mid \ \sim \mathit{term} \mid \mathit{constant} \mid \\
 & & \mathit{atom}\,\mathtt{?}\,\mathit{term}\,\mathtt{:}\,\mathit{term} \mid \\
 & & \mathit{term}[\,\mathit{constant} : \mathit{constant}\,] \mid \mathit{ext}(\,\mathit{term}\,) \\
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\end{array}
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- $\sim x$: bit-wise negation of x
- $ext(x)$: sign- or zero-extension of x
- $x \ll d$: left shift with distance d
- $x \circ y$: concatenation of x and y

Danger!

$$(x - y > 0) \iff (x > y)$$

Valid over $\mathbb{R}/\mathbb{N}$, but not over the bit-vectors.
(Many compilers have this sort of bug)

Width and Encoding

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- Typical encodings:
  - Binary encoding

$$\langle x \rangle_U := \sum_{i=0}^{l-1} a_i \cdot 2^i$$

  - Two's complement

$$\langle x \rangle_S := -2^{n-1} \cdot a_{n-1} + \sum_{i=0}^{l-2} a_i \cdot 2^i$$

  - But maybe also fixed-point, floating-point, ...

Examples

$$\langle 11001000\rangle_U = 200$$

$$\langle 11001000\rangle_S = -128 + 64 + 8 = -56$$

$$\langle 01100100\rangle_S = 100$$

Notation to clarify width and encoding:

$$x_{[32]S}$$

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Width in bits

U: unsigned binary
S: signed two's complement

Definition (Bit-Vector)

A *bit-vector* is a vector of Boolean values with a given length l:

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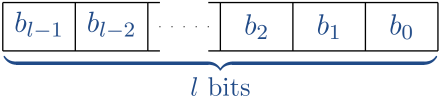
The value of bit number i of x is $x(i)$.



We also write x_i for $x(i)$.

Lambda-Notation for Bit-Vectors

λ expressions are functions without a name

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Examples:

- The vector of length l that consists of zeros:

$$\lambda i \in \{0, \ldots, l-1\}.0$$

- A function that inverts (flips all bits in) a bit-vector:

$$\textit{bv-invert}(x) := \lambda i \in \{0, \ldots, l-1\}.\neg x_i$$

- A bit-wise OR:

$$\textit{bv-or}(x, y) := \lambda i \in \{0, \ldots, l-1\}.(x_i \vee y_i)$$

$\Longrightarrow$ we now have semantics for the bit-wise operators.

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$$(x_{[10]} \circ y_{[5]})[14] \iff x[9]$$

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- Final result:

$$(\lambda i.(i < 5)?y_i : x_{i-5})(14) \iff x_9$$

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unsigned char number = 200;
number = number + 100;
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 & 11001000 & = 200 \\
+ & 01100100 & = 100 \\
\hline
= & 00101100 & = 44
\end{array}
$$

$\Longrightarrow$ Bit-vector arithmetic uses modular arithmetic!

Semantics for Arithmetic Expressions

Semantics for addition, subtraction:

$$a_{[l]} +_U b_{[l]} = c_{[l]} \quad \Longleftrightarrow \quad \langle a \rangle_U + \langle b \rangle_U = \langle c \rangle_U \mod 2^l$$
$$a_{[l]} -_U b_{[l]} = c_{[l]} \quad \Longleftrightarrow \quad \langle a \rangle_U - \langle b \rangle_U = \langle c \rangle_U \mod 2^l$$

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We can even mix the encodings:

$$a_{[l]U} +_U b_{[l]S} = c_{[l]U} \iff \langle a \rangle_U + \langle b \rangle_S = \langle c \rangle_U \mod 2^l$$

Semantics for $<$, $\leq$, $\geq$, and so on:

$$a_{[l]U} < b_{[l]U} \iff \langle a \rangle_U < \langle b \rangle_U$$
$$a_{[l]S} < b_{[l]S} \iff \langle a \rangle_S < \langle b \rangle_S$$

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Mixed encodings:

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$$a_{[l]S} < b_{[l]U} \iff \langle a \rangle_S < \langle b \rangle_U$$

Note that most compilers don't support comparisons with mixed encodings.

Complexity

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- It is NP-complete otherwise.

A Simple Decision Procedure

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- Most commonly used decision procedure
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Bit-Vector Flattening

1. Convert propositional part as before
2. Add a *Boolean variable for each bit* of each sub-expression (term)
3. Add *constraint* for each sub-expression

We denote the new Boolean variable for bit i of term t by $\mu(t)_i$.

Algorithm 6.2.1: BV-FLATTENING

Input: A formula φ in bit-vector arithmetic
Output: An equisatisfiable Boolean formula $\mathcal{B}$

function BV-FLATTENING
  $\mathcal{B} := e(\varphi)$; ▷ the propositional skeleton of φ
  for each $t_{[l]} \in T(\varphi)$ **do**
    for each $i \in \{0, \ldots, l-1\}$ **do**
      set $e(t)_i$ to a new Boolean variable;
  for each $a \in At(\varphi)$ **do**
    $\mathcal{B} := \mathcal{B} \wedge$ BV-CONSTRAINT(e, a);
  for each $t_{[l]} \in T(\varphi)$ **do**
    $\mathcal{B} := \mathcal{B} \wedge$ BV-CONSTRAINT(e, t);
  return $\mathcal{B}$;

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- This is easy for the bit-wise operators.

- Example for $a|_{[l]}b$:

$$\bigwedge_{i=0}^{l-1}(\mu(t)_i = (a_i \vee b_i))$$

(read $x = y$ over bits as $x \iff y$)

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- We can transform this into CNF using Tseitin's method.

Flattening Bit-Vector Arithmetic

How to flatten $a + b$?

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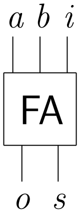
$\longrightarrow$ we can build a *circuit* that adds them!



Full Adder

$$s \equiv (a + b + i) \bmod 2 \equiv a \oplus b \oplus i$$

$$o \equiv (a + b + i) \operatorname{div} 2 \equiv a \cdot b + a \cdot i + b \cdot i$$

The full adder in CNF:

$$(a \lor b \lor \neg o) \land (a \lor \neg b \lor i \lor \neg o) \land (a \lor \neg b \lor \neg i \lor o) \land$$
$$(\neg a \lor b \lor i \lor \neg o) \land (\neg a \lor b \lor \neg i \lor o) \land (\neg a \lor \neg b \lor o)$$

Ok, this is good for one bit! How about more?

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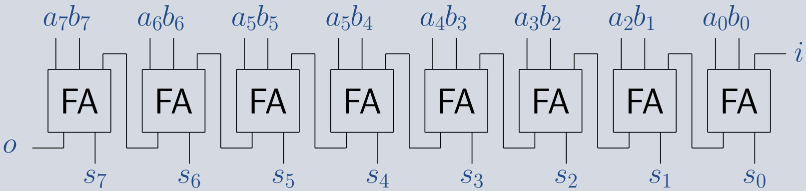
8-Bit ripple carry adder (RCA)



- Also called *carry chain adder*
- Adds l variables
- Adds $6 \cdot l$ clauses

- Multipliers result in very hard formulas
- Example:

$$a \cdot b = c \land b \cdot a \neq c \land x < y \land x > y$$

CNF: About 11000 variables, unsolvable for current SAT solvers

- Similar problems with division, modulo
- Q: Why is this hard?

- Multipliers result in very hard formulas
- Example:

$$a \cdot b = c \wedge b \cdot a \neq c \wedge x < y \wedge x > y$$

CNF: About 11000 variables, unsolvable for current SAT solvers

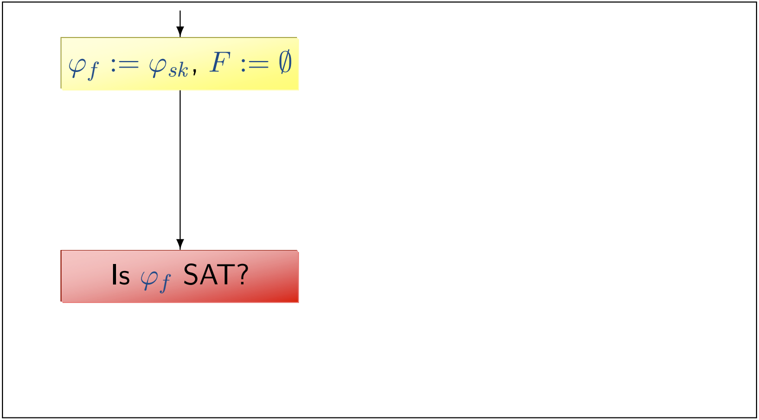
- Similar problems with division, modulo
- Q: Why is this hard?
- Q: How do we fix this?

Incremental Flattening

$\varphi_f := \varphi_{sk}$, $F := \emptyset$

φ_{sk}: Boolean part of φ
F: set of terms that are in the encoding

Incremental Flattening



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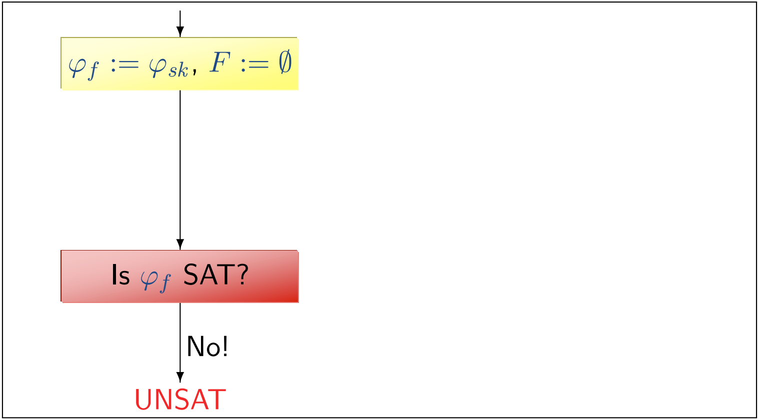

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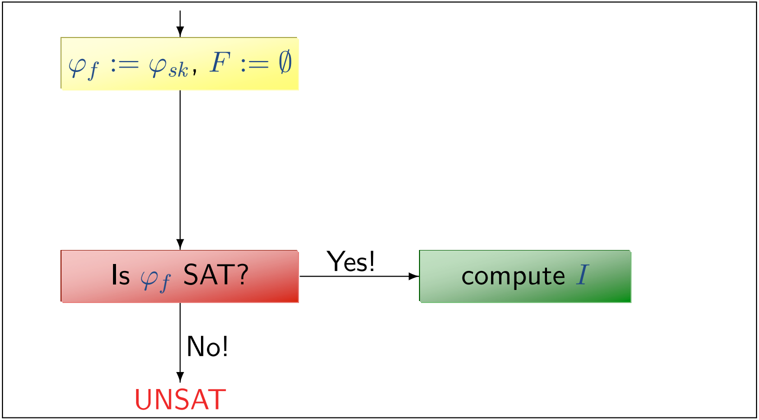
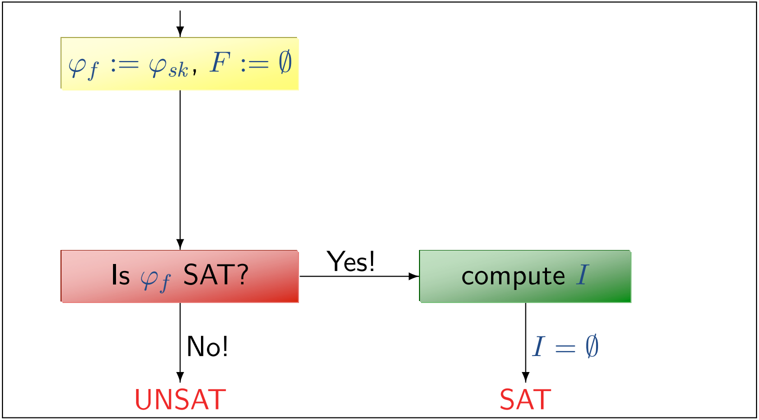


φ_{sk}: Boolean part of φ

F: set of terms that are in the encoding

I: set of terms that are inconsistent with the current assignment



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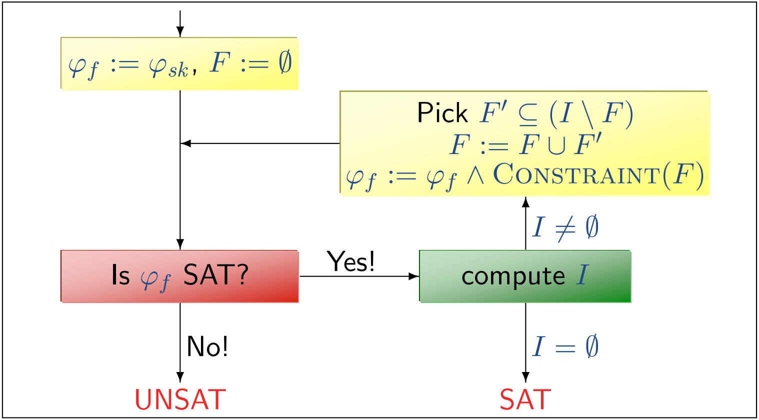

φ_{sk}: Boolean part of φ

F: set of terms that are in the encoding

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- Idea: add 'easy' parts of the formula first
- Only add hard parts when needed
- φ_f only gets stronger – use an incremental SAT solver

Incomplete Assignments

- Hey: initially, we only have the skeleton!
  How do we know what terms are inconsistent with the current assignment if the variables aren't even in φ_f?

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- Solution: guess some values for the missing variables.
  If you guess right, it's good.

Algorithm 6.3.1: INCREMENTAL-BV-FLATTENING

Input: A formula φ in bit-vector logic

Output: "Satisfiable" if the formula is satisfiable, and "Unsatisfiable" otherwise

```
function INCREMENTAL-BV-FLATTENING(φ)
    B := e(φ);                                   ▷ propositional skeleton of φ
    for each t_[l] ∈ T(φ) do
        for each i ∈ {0, ..., l − 1} do
            set e(t)_i to a new Boolean variable;
    while (TRUE) do
        α := SAT-SOLVER(B);
        if α="Unsatisfiable" then
            return "Unsatisfiable";
        else
            Let I ⊆ T(φ) be the set of terms that are inconsistent with the
                satisfying assignment;
            if I = ∅ then
                return "Satisfiable";
            else
                Select "easy" F′ ⊆ I;
                for each t_[l] ∈ F′ do
                    B:=B ∧ BV-CONSTRAINT(e, t);
```

Incomplete Assignments

- Hey: initially, we only have the skeleton!
  How do we know what terms are inconsistent with the current assignment if the variables aren't even in φ_f?

- Solution: guess some values for the missing variables.
  If you guess right, it's good.

- Ideas:
  - All zeros
  - Sign extension for signed bit-vectors
  - Try to propagate constants ($a = b + 1$)