

SMT Solvers

- SAT can codify operations and relations between integers with bounded precision
 - using representations as bit vectors
 - representing addition, etc as Boolean circuits
- As well as other finite datatypes and structures
- But, cannot represent unbounded types (e.g., reals) or infinity data structures (stacks, lists)
- Bounded arithmetic is not very efficient for large values
- There are efficient procedures for these FOL theories for conjunctions of atomic formulas
- Use search strategies based on SAT solvers
- Are called SMT (*Satisfiability Modulo Theories*) solvers.

(Classic) First Order Logic

- Infinity set of variables x_1, x_2, \dots ($Vars$)
- Logic symbols: Boolean connectives ($\wedge, \vee, \implies, \neg, \dots$), quantifiers (\forall, \exists) and parentheses '(', ')'
- Non-logic symbols: alphabet Σ for functional and predicate symbols
- Syntax: for terms and for formulae

$$t ::= x \mid a \mid f(t, \dots, t)$$

$$\varphi ::= R(t_1, \dots, t_n) \mid t_1 = t_2 \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid (\varphi \rightarrow \varphi) \mid \forall x\varphi \mid \exists x\varphi$$

- A formula is *closed* if does have free variables.
- Semantics: a Σ structure A with domain A , interpretation of non logic symbols (\cdot^A) and a assignment of domain elements to the free variables, α .
- A formula φ is satisfiable if there exists a structure A of Σ and assignment α such that φ is true ($A \models_{\alpha} \varphi$)
- A formula is valid if for all structures A of Σ , φ is true ($\models \varphi$)

Satisfiability

- (i) $A \models_{\alpha} t_1 = t_2$ if $\alpha(t_1) = \alpha(t_2)$
- (ii) $A \models_{\alpha} R(t_1, \dots, t_n)$ if $(\alpha(t_1), \dots, \alpha(t_n)) \in R^A$
- (iii) $A \models_{\alpha} \neg\varphi$ if $A \not\models_{\alpha} \varphi$
- (iv) $A \models_{\alpha} \varphi \wedge \psi$ if $A \models_{\alpha} \varphi$ and $A \models_{\alpha} \psi$
- (v) $A \models_{\alpha} \varphi \vee \psi$ if $A \models_{\alpha} \varphi$ or $A \models_{\alpha} \psi$
- (vi) $A \models_{\alpha} \varphi \rightarrow \psi$ if $A \not\models_{\alpha} \varphi$ or $A \models_{\alpha} \psi$
- (vii) $A \models_{\alpha} \forall x \varphi$ if for all $a \in A$ if $A \models_{\alpha[a/x]} \varphi$ where:

$$\alpha[a/x](y) = \begin{cases} \alpha(y) & \text{if } y \neq x \\ a & \text{if } y = x \end{cases}$$

- (viii) $A \models_{\alpha} \exists x \varphi$ if exists $a \in A$ such that $A \models_{\alpha[a/x]} \varphi$

Decidability

Given formulae φ and ψ , we have the following decision problems

Validity problem: Is φ valid?

Satisfiability problem: Is φ satisfiable?

Consequence problem: Is ψ a consequence of φ ?

Equivalence problem: Are ψ and φ equivalent?

These are, in some sense, variations of the same problem :

$\models \varphi \iff \neg\varphi$ is unsatisfiable $\psi \models \varphi \iff \neg(\psi \implies \varphi)$ is unsatisfiable
 $\psi \equiv \varphi \iff (\psi \models \varphi \wedge \varphi \models \psi)$ φ is satisfiable $\iff \neg\varphi$ is not valid

Undecidability of FOL

A *solution* to a decision problem is a program that takes problem instances as input and always terminates, producing a correct *yes* or *no* output.

A decision problem is *decidable* if it has a solution. A decision problem is *undecidable* if it is not decidable.

Theorem 1 (Church & Turing). • *The decision problem of validity in first-order logic is undecidable: no program exists which, given any φ , decides whether $\models \varphi$.*

- *The decision problem of satisfiability in first-order logic is undecidable: no program exists which, given any φ , decides whether φ is satisfiable.*

Semi-decidability

However, there is a procedure that halts and says yes if is valid.

A decision problem is *semi-decidable* if exists a procedure that, given an input,

- halts and answers “yes” \iff “yes” is the correct answer,
- halts and answers “no” if “no” is the correct answer,
- or does not halt if “no” is the correct answer

Unlike a decidable problem, the procedure is only guaranteed to halt if the correct answer is “yes”.

The decision problem of *validity* in first-order logic is *semi-decidable*.

FOL Theories

Let Σ be an alphabet of a first-order language.

- A *theory* \mathcal{T} is a set of closed formulae such that $\mathcal{T} \models \varphi$ implies $\varphi \in \mathcal{T}$ (closed under derivability).
- A \mathcal{T} -structure is a Σ that validates all formulae of \mathcal{T}
- A formula φ is \mathcal{T} -satisfiable if it is satisfiable in a \mathcal{T} -structure; in the same way we define \mathcal{T} -valid
- A theory \mathcal{T} is *finitely (recursively) axiomatizable* if there exists a finite (recursive) set $\mathcal{A} \subseteq \mathcal{T}$ (axioms) such that $\forall \varphi, \varphi \in \mathcal{T} \iff \mathcal{A} \models \varphi$
- A theory \mathcal{T} is *complete* if for all closed formulae φ , $\mathcal{T} \models \varphi$ or $\mathcal{T} \models \neg\varphi$.
- A theory \mathcal{T} is *decidable* if for all closed formulae it is possible to decide if $\mathcal{T} \models \varphi$.
- A *axiomatizable* and *complete* theory is *decidable*.
- Given a Σ structure A ,

$$Th(A) = \{\varphi \mid A \models \varphi\}$$

is complete.

- Semantically defined theories are important as they allow to reason about mathematical domains (naturals, integers, algebraic structures, etc.), but they must be axiomatizable.
- A *fragment* of a theory \mathcal{T} is a subset of formulae of \mathcal{T} with a syntactic restriction, e.g.:
 - only conjunctions of literals;
 - quantifier-free;
 - etc.

Some Theories

1. Equality Theory and Uninterpreted Functions \mathcal{T}_E
2. Theory of Arithmetic - Peano Axioms \mathcal{T}_{PA}
3. Theory of Arithmetic of Presburger (additive fragment) $\mathcal{T}_{\mathbb{N}}$
4. Linear Integer Arithmetic $\mathcal{T}_{\mathbb{Z}}$ (same expressiveness of $\mathcal{T}_{\mathbb{N}}$)
5. Real Arithmetic and Linear Rational Arithmetic ($\mathcal{T}_{\mathbb{R}}, \mathcal{T}_{\mathbb{Q}}$)
6. Set Theory of Zermelo-Frankle
7. Geometry Theory (Euclides, non-standard, etc.)
8. Group Theory
9. Theory of Regular Languages (expressions) (Kleene Algebras)

Equality and Uninterpreted Functions \mathcal{T}_E

$$\begin{aligned} & \forall x. x = x \\ & \forall x, y. x = y \rightarrow y = x \\ & \forall x, y, z. x = y \wedge y = z \rightarrow x = z \\ & \forall \bar{x}, \bar{y}. x_1 = y_1 \wedge \dots \wedge x_n = y_n \rightarrow f(x_1, \dots, x_n) = f(y_1, \dots, y_n) \\ & \forall \bar{x}, \bar{y}. x_1 = y_1 \wedge \dots \wedge x_n = y_n \rightarrow P(x_1, \dots, x_n) \rightarrow P(y_1, \dots, y_n) \end{aligned}$$

The last two axioms are congruences: functional and predicates.

\mathcal{T}_E -validity is undecidable. Quantifier-free fragment is decidable. Fragment with only functions where the conjunctive fragment is \mathcal{EUF} , is decidable.

Arithmetic (natural and integer numbers)

First Incompleteness Theorem Kurt Gödel (1931)

Any effectively generated (i.e., recursively enumerable) theory capable of expressing elementary arithmetic cannot be both consistent and complete. In particular, for any consistent, effectively generated formal theory that proves certain basic arithmetic truths, there is an arithmetical statement that is true, but not provable in the theory.

A semantic theory $\text{Th}(M)$, where M interprets each symbol with its standard mathematical meaning in the interpretation domain, is always a complete theory. Therefore, the semantic theories of natural numbers and integers cannot be axiomatizable, not even by an infinite recursive set of axioms.

Peano Axioms, $T_{\mathcal{PA}}$

Let $\Sigma = \{0, 1, +, x, =, <\}$. The axioms define basic facts of naturals and $+$ and x ($\mathbb{N} \models \mathcal{PA}$):

1. $\forall x(x + 1 \neq 0)$
2. $\forall x \forall y(x + 1 = y + 1 \rightarrow x = y)$
3. $0 + 1 = 1$
4. $\forall x x + 0 = x$
5. $\forall x \forall y x + (y + 1) = (x + y) + 1$
6. $\forall x x \times 0 = 0$
7. $\forall x \forall y x \times (y + 1) = (x \times y) + x$
8. (induction principle) $(Q(0) \wedge (\forall x(Q(x) \rightarrow Q(x + 1))) \rightarrow \forall x Q(x))$

$T_{\mathcal{PA}}$ -validity is undecidable (Gödel's Incompleteness). Even the quantifier-free fragment of \mathcal{PA} is undecidable. (Matiyasevich, 1970).

Presburger Arithmetic $\mathcal{T}_{\mathbb{N}}$

- If we exclude axioms 6 and 7 we obtain Presburger Arithmetics $\mathcal{T}_{\mathbb{N}}$ which is complete and decidable.
- This theory has many applications in formal verification and is related with automata theory.
- Linear integer arithmetic, $\mathcal{T}_{\mathbb{Z}}$, ($\Sigma = \{\dots, -1, 1, 0, 1, 2, \dots, +, =, <\}$) reduces to Presburger theory.
- Suppose

$$\forall w, x. \exists y, z. x + 2y - z - 13 > -3w + 5.$$

we can introduce new variables v_p and v_n (in \mathbb{N}) for each variable v

$$\forall w_p, w_n, x_p, x_n. \exists y_p, y_n, z_p, z_n. (x_p - x_n) + 2(y_p - y_n) - (z_p - z_n) - 13 > -3(w_p - w_n) + 5,$$

change the side of the $-$ to

$$\forall w_p, w_n, x_p, x_n. \exists y_p, y_n, z_p, z_n. x_p + 2y_p + z_p + 3w_p > x_n + 2y_n + z_n + 13 + 3w_n + 5$$

and code in unary.

- Of course $\mathcal{T}_{\mathbb{N}}$ reduces to $\mathcal{T}_{\mathbb{Z}}$:
- The $\mathcal{T}_{\mathbb{N}}$ -formula

$$\forall x \exists y. x = y + 1$$

is equisatisfiable to the $\mathcal{T}_{\mathbb{Z}}$ -formula

$$\forall x. (x > -1 \implies \exists y. y > -1 \wedge x = y + 1)$$

Real Arithmetic $\mathcal{T}_{\mathbb{R}}$

Let $\Sigma = \{0, 1, +, x, =, \geq\}$. The Real Arithmetic (or elementary algebra) is :

- with addition an abelian group $(\mathbb{R}, +, 0)$, $+$ associative and commutative, 0 identity and all elements have inverse $(-)$.:
- with multiplication a ring $(\mathbb{R}, +, \times, 1, 0)$: \times associative and distributes over addition, 1 identity.
- and a field: \times commutative, $1 \neq 0$, non 0 elements have multiplicative inverse.
- closed: \geq total order

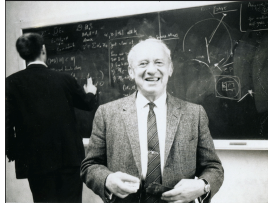
1. $\forall x, y, z. x \geq y \implies x + z \geq y + z$

2. $\forall x, y. x \geq 0 \wedge y \geq 0 \implies xy \geq 0$

3. $\forall x. \exists y. x = y^2 \vee x = -y^2$

4. for each odd integer n , polynomials of odd degree have at least one root.

$$\forall \vec{x}. \exists y. y^n + x_1 y^{n-1} + \dots + x_{n-1} y + x_n = 0$$



Tarski proved that $\mathcal{T}_{\mathbb{R}}$ was decidable in the 1930s, although the Second World War prevented his publishing the result until 1956. Collins (1975) proposed a more efficient technique of cylindrical algebraic decomposition (CAD) AD runs in time proportionate to $2^{2^k |F|}$, for some constant k and for $|F|$ the length of F ..

Linear theory of rationals $\mathcal{T}_{\mathbb{Q}}$

The full theory of rational numbers (with addition and multiplication) is undecidable, since the property of being a natural number can be encoded in it. For the linear theory of rationals the alphabet is $\Sigma = \{0, 1, +, =, \geq\}$ and corresponds to $\mathcal{T}_{\mathbb{R}}$ without multiplication.

1. $\forall x, y. x \geq y \wedge y \geq x \implies x = y$

2. $\forall x, y, z. x \geq y \wedge y \geq z \implies x \geq z$

3. $\forall x, y. x \geq y \vee y \geq x$

4. $\forall x, y, z. (x + y) + z = x + (y + z)$

5. $\forall x. x + 0 = x$
6. $\forall x. x + (-x) = 0$
7. $\forall x, y. x + y = y + x$
8. $\forall x, y, z. x \geq y \implies x + z \geq y + z$
9. for each positive integer n , $\forall x. nx = 0 \implies x = 0$
10. for each positive integer n , $\forall x. \exists y. x = ny$

Models are divisible torsion-free abelian groups. (Axiom 9).

Difference Arithmetic

- Difference logic is a fragment (a sub-theory) of linear arithmetic.
- Atomic formulas have the form $x - y \leq c$, for variables x and y and constant c .
- Conjunctions of difference arithmetic inequalities can be checked very efficiently for satisfiability by searching for negative cycles in weighted directed graphs.
- Graph representation: each variable corresponds to a node, and an inequality of the form $x - y \leq c$ corresponds to an edge from y to x with weight c .
- The quantifier-free satisfiability problem is solvable in $O(|V||E|)$.

Theory of Lists, \mathcal{T}_L

The alphabet is $\Sigma_L = \{cons, head, tail, atom, =\}$.

$$\begin{aligned}
 & \mathcal{T}_E \\
 & \forall x, y. head(cons(x, y)) = x \\
 & \forall x, y. tail(cons(x, y)) = y \\
 & \forall y. \neg atom(y) \implies cons(head(y), tail(y)) = y \\
 & \forall x, y. \neg atom(cons(x, y))
 \end{aligned}$$

- $atom(x)$ is a predicate that is true if the argument x is a singleton.
- In Lisp $head$ is car (contents of address register) and cdr (contents of decrement register).
- The axioms of \mathcal{T}_E ensure that $head$ and $tail$ are functional congruences and $atom$ a predicate congruence.
- Satisfiability of the quantifier-free fragment is decidable.
- Can be extended to other recursive data structures \mathcal{T}_{RDS}

Theory of Arrays, \mathcal{T}_A^-

The alphabet is $\Sigma_A = \{read, write, =\}$. Arrays are functions that can be modified. The term $read(a, i)$ corresponds to $a[i]$, and $write(a, i, v)$ corresponds to $a[i \leftarrow v]$.

$$\begin{aligned} & \mathcal{T}_E \\ & \forall a, i, j. i = j \rightarrow read(a, i) = read(a, j) \\ & \forall a, i, j, v. i = j \rightarrow read(write(a, i, v), j) = v \\ & \forall a, i, j, v. \neg(i = j) \rightarrow read(write(a, i, v), j) = read(a, j) \\ & \forall a, b. (\forall i. read(a, i) = read(b, i)) \rightarrow a = b \text{ (extensionality)} \end{aligned}$$

\mathcal{T}_A^- -validity is undecidable. Quantifier-free fragment is decidable. Without the last axiom (extensionality) (\mathcal{T}_A) even that fragment was not decidable.

Other theories

- *Fixed-size bit-vectors*
 - Model bit-level operations of machine words, including 2^n -modular operations (where n is the word size), shift operations, etc.
 - Decision procedures for the theory of fixed-size bit vectors often rely on appropriate encodings in propositional logic.
- *Pointer logic*, allows to reasoning variables that refer to some other program construct, such as a variable, a procedure or an address. A pointer corresponds to the unique address of a memory cell. The way the memory cells are addresses is given by the memory model. It is characterised by a *memory valuation* $M : A \rightarrow D$ where A is a set of addresses and D the set of data words stored in each memory cell; and A *memory layout* $L : V \rightarrow A$ that associates to each program variable an address.

Decision procedures

- Are specific to a given theory.
- Determine if a formula is inconsistent, satisfiable, or valid.
- Can work on conjunctions of atomic formulae or decide if a formula is consequence of other formulae.
- Can use heuristics to improve performance, but have to give the correct answer and terminate (sound and complete).

Decidable theories

- As we saw there are many useful decidable theories (or at least fragments):

- Equality with uninterpreted functional symbols \mathcal{EUF}

$$x = y \wedge f(f(f(x))) = f(x) \rightarrow f(f(f(f(y)))) = f(x)$$

- Updates of functions, registers and tuples
- Linear integer arithmetics and rational (\mathcal{LIA} , \mathcal{LRA} , linear programming, Simplex, etc.)

$$x \leq y \wedge x \leq 1 - y \wedge 2x \geq 1 \rightarrow 4x = 2$$

- Difference logic

$$x - y < c$$

- Bit vectors: modular arithmetics
- Lists and other RDS.
- Pointer Logics

- Combinations of decidable theories are also decidable, in general

Complexity of decidability

Theory	Complexity
PL	NP-complete
$T_{\mathbb{N}}, T_{\mathbb{Z}}$	$\Omega(2^{2^n}), O(2^{2^{2^{kn}}})$
$T_{\mathbb{R}}$	$O(2^{2^{kn}})$
$T_{\mathbb{Q}}$	$\Omega(2^n), O(2^{2^{kn}})$
T_{RDS}^+	not elementary recursive

Decidable theories

Theory	Complexity	Theory	Complexity
PL	$\Theta(n)$	$T_{\mathbb{E}}$	$O(n \log n)$
$T_{\mathbb{N}}, T_{\mathbb{Z}}$	NP-complete	$T_{\mathbb{R}}$	$O(2^{2^{kn}})$
$T_{\mathbb{Q}}$	PTIME	T_{RDS}^+	$\Theta(n)$
T_{RDS}	$O(n \log n)$	$T_{\mathbb{A}}$	NP-complete

for quantifier-free, conjunctive fragments of theories

Combining theories

Let

$$x + 2 = y \implies f(\text{read}(\text{write}(a, x, 3), y - 2) = f(y - x + 1)$$

What theories are involved here?

- equality and uninterpreted functions, \mathcal{T}_E
- arrays, $\mathcal{T}_A^=$
- arithmetic, \mathcal{T}_Z

Combining theories

Let \mathcal{T}_1 and \mathcal{T}_2 two theories with alphabets Σ_1 and Σ_2 ; and axioms A_1 and A_2 .

The combined theory $\mathcal{T}_1 \cup \mathcal{T}_2$ such that $\Sigma_1 \cap \Sigma_2 = \{=\}$ is given by:

- alphabet $\Sigma_1 \cup \Sigma_2$
- axioms: $A_1 \cup A_2$

Nelson & Oppen, 1979

Satisfiability of the quantifier-free fragment of $\mathcal{T}_1 \cup \mathcal{T}_2$ is decidable if

- satisfiability of the quantifier-free fragment of \mathcal{T}_1 is decidable
- satisfiability of the quantifier-free fragment of \mathcal{T}_2 is decidable
- and certain technical requirements are met

SMT Solvers

- Extend SAT solvers to FOL
- Use decision procedures alone or combined to decide conjunctions of atomic formulae
- SMT use the propositional backbone of the formulae
 - Use search strategies of modern SAT solvers
 - Terms are substituted by propositional variables
 - Find a solution with a SAT solver
 - If found, consider the interpretation of the variables and evaluates the FOL formula with the appropriate solver.

SMT Solvers

Let $\text{prop}(\varphi)$ be a function that maps $\varphi \in \mathcal{T}$ (in CNF and quantifier-free) into a propositional formula (substituting atomic formulae by propositional variables) and unprop the inverse function. Given an assignment ρ for $\text{prop}(\varphi)$ let

$$\varphi(\rho) = \{\text{unprop}(p_i) \mid \rho(p_i) = \top\} \cup \{\neg\text{unprop}(p_i) \mid \rho(p_i) = \perp\}$$

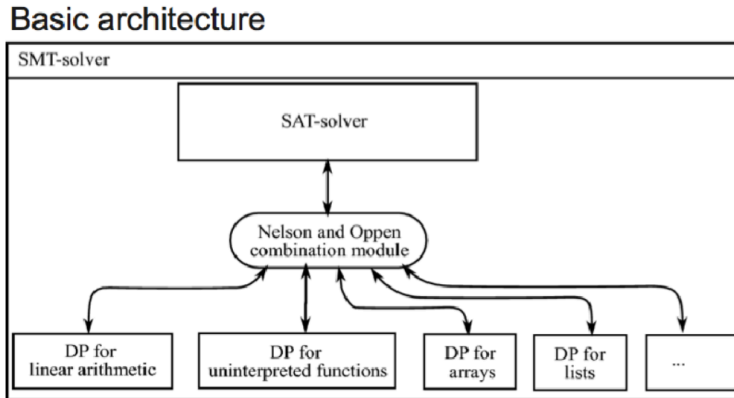
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SMT-Solver( $\varphi$ ) {
  A := prop( $\varphi$ )
  loop
    ( $r, \rho$ ) := SAT(A)
    if  $r = \text{unsat}$  then return unsat
    ( $\theta, r$ ) :=  $DP_{\mathcal{T}}$ ( $\varphi(\rho)$ )
    if  $r = \text{sat}$  then return sat
    C :=  $\bigvee_{B \in \theta} \neg \text{prop}(B)$ 
    A := A  $\wedge$  C
  }

```

where $\theta \subseteq \varphi(\rho)$ corresponds to unsatisfiable formulae. Then we add to A the propositional equivalent of θ (C) to ensure that the assignment ρ is not used again in SAT . $DP_{\mathcal{T}}$ is the decision procedure for \mathcal{T} .

SMT-solvers basic architecture



Efficient SMT Solvers

- In the last two decades, SMT procedures have undergone dramatic progress. There has been enormous improvements in efficiency and expressiveness of SMT procedures for the more commonly occurring theories.
- The annual competition for SMT procedures plays an important role in driving progress in this area.

- A key ingredient is SMT-LIB, an online resource that proposes, as a standard, a unified notation and a collection of benchmarks for performance evaluation and comparison of tools.
- Some SMT solvers: Z3, CVC4, Alt-Ergo, Yices 2, MathSAT, Boolector, etc.
- Usually, SMT solvers accept input either in a proprietary format or in SMT-LIB format.

SMT Solvers- Links

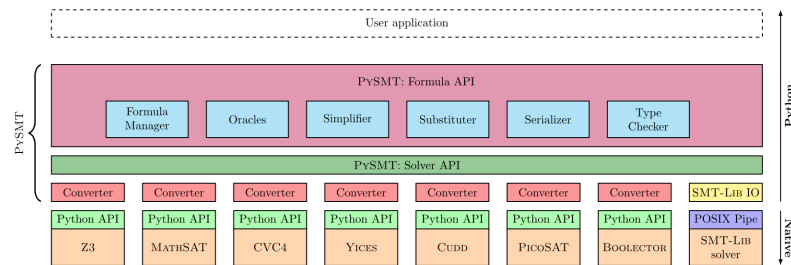
- SMT-LIB: The Satisfiability Modulo Theories Library
<http://smtlib.cs.uiowa.edu>
- SMT-COMP: The Satisfiability Modulo Theories Competition
<https://github.com/SMT-COMP>
<http://www.smtcomp.org>
- Decision procedures - an algorithmic point of view
<https://www.decision-procedures.org/>
- SAT Association <http://satassociation.org/sat-smt-school.html>
- SAT/SMT Examples <https://sat-smt.codes>

SMT LIB

- Catalog of theory declarations - semi-formal specification of theories of interest
 - A theory defines a vocabulary of sorts and functions. The meaning of the theory symbols are specified in the theory declaration.
- Catalog of logic declarations - semi-formal specification of fragments of (combinations of) theories
 - A logic consists of one or more theories, together with some restrictions on the kinds of expressions that may be used within that logic.
- Library of benchmarks
- Utility tools (parsers, converters, ...)
- Useful links (documentation, solvers, ...)
- SMT-LIB language expresses logical statements in a many-sorted first-order logic.

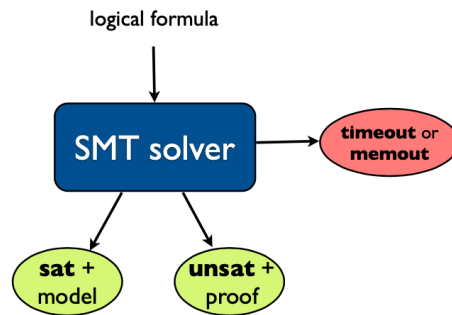
pySMT

- The pySMT library allows a Python program to communicate with several SMT solvers based on a common language.
- This makes it possible to code a problem independently of the SMT solver, and run the same problem with several SMT solvers.



Theorem Provers

$$\varphi \text{ valid} \iff \neg\varphi \text{ unsatisfiable}$$



If a solver cannot find a solution perhaps other can.

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