Location Automata for Synchronised Shuffle Expressions

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Abstract

Several notions of synchronisation in concurrent systems can be modelled by regular shuffle operators. In this paper we consider regular expressions extended with three operators corresponding respectively to strong, arbitrary, and weak synchronisation. For these expressions, we define a location based position automaton. Furthermore, we show that the partial derivative automaton is still a quotient of the position automaton.

Keywords: Regular Expressions, Locations, Shuffle Operators, Position Automaton, Synchronisation, Partial Derivatives

1. Introduction

Several notions of synchronisation in concurrent systems can be modelled by regular shuffle operators. These operations range from the plain shuffle to intersection, which can be seen as two extreme cases, corresponding respectively to pure interleaving and strict synchronisation. If only a subset of letters is allowed to synchronise, several variants of synchronisation can be considered when shuffling two words [1]. In particular, one may require synchronising letters to always synchronise — strong synchronised shuffle; they may synchronise or not — arbitrary synchronised shuffle; or synchronising letters can optionally be interleaved from one word, but not from the other until the next synchronisation occurs — weak synchronised shuffle. Sulzmann and Thiemann [2, 3] introduced

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a general synchronised shuffling operator, that subsumes the above three operators, among others. For their general shuffling operator, the authors extended the notions of partial derivative and partial derivative automaton ($\mathcal{A}_{\mathrm{PD}}$) [4]. Another conversion from standard regular expressions to automata is the position automaton, of which the partial derivative automaton is a quotient. Positions correspond to the occurrences of letters in the expression, and there is a one-to-one correspondence between the set of states in the position automaton and the set of positions. That is no longer true if one considers regular expressions with shufflings. Broda et al. [5, 6] extended the notion of position to shuffle and to intersection by considering tree-like structures, called locations, that keep track of the set of positions, that correspond to the states of the automaton.

In this paper we show that locations can also be used to define a position automaton ($\mathcal{A}_{\mathrm{POS}}$) for the general shuffling operator, more precisely for each of the synchronised shuffle operators. This allows to extend the taxonomy of conversions from expressions to automata presented in [7] to include synchronised shuffle operators. Furthermore, we show that for expressions with synchronised shuffle operators, the partial derivative automaton is still a quotient of the position automaton.

The paper is organised as follows. Section 2 recalls standard notions regarding regular expressions and finite automata, as well as the different synchronised shuffle operators. In Section 3 we define locations for expressions with synchronised shuffle operators. Using locations we extend the construction of the position automaton to those expressions, and show its correctness. In Section 4 the partial derivative automaton, defined by Sulzmann and Thiemann [3], is shown to be a quotient of $\mathcal{A}_{\mathrm{POS}}$ introduced in Section 3. Section 5 concludes with some final remarks and points to future work. Due to their length, some of the proofs are omitted in the main text, and can be found in the appendix.

2. Preliminaries

In this section we review some basic definitions about regular expressions and finite automata and fix notation. More details can be found, e.g., in [8].

The set of standard regular expressions over an alphabet Σ is denoted by RE and contains $\emptyset$ plus all terms generated by the grammar

$$\alpha \quad\rightarrow\quad \varepsilon \mid \sigma \mid (\alpha+\alpha) \mid (\alpha\cdot\alpha) \mid (\alpha^{\star}) \quad (\sigma\in\Sigma).$$

The *language* associated with an expression $\alpha\in\mathrm{RE}$ is denoted by $\mathcal{L}(\alpha)$ and defined inductively as usual. The empty word is denoted by ε. We define $\varepsilon(\alpha)$ by $\varepsilon(\alpha)=\mathsf{true}$ if $\varepsilon\in\mathcal{L}(\alpha)$, and $\varepsilon(\alpha)=\mathsf{false}$ otherwise. Given a set of expressions S, the *language* associated with S is $\mathcal{L}(S)=\bigcup_{\alpha\in S}\mathcal{L}(\alpha)$. Moreover, we consider $\varepsilon S=S\varepsilon=S$ and $\emptyset S=S\emptyset=\emptyset$, for any set S of expressions. The *alphabetic size* of α, $|\alpha|_{\Sigma}$, is the number of occurrences of letters (alphabet symbols) in α. We denote the subset of Σ containing the symbols that occur in α by Σ_{α}.

A *nondeterministic finite automaton* (NFA) is a quintuple $A=\langle Q,\Sigma,\delta,I,F\rangle$ where Q is a finite set of states, Σ is a finite alphabet, $I\subseteq Q$ is the set of initial states, $F \subseteq Q$ is the set of final states, and $\delta : Q \times \Sigma \to 2^Q$ is the transition function. The *language* of A is denoted by $\mathcal{L}(A)$ and two automata are *equivalent* if they have the same language. Two automata $A_1 = \langle Q_1, \Sigma, \delta_1, I_1, F_1\rangle$ and $A_2 = \langle Q_2, \Sigma, \delta_2, I_2, F_2\rangle$ are *isomorphic*, $A_1 \simeq A_2$, if there is a bijection $\varphi : Q_1 \longrightarrow Q_2$ such that $\varphi(I_1) = I_2$, $\varphi(F_1) = F_2$, and $\varphi(\delta_1(q_1, \sigma)) = \delta_2(\varphi(q_1), \sigma)$, for all $q_1 \in Q_1$, $\sigma \in \Sigma$. Given an automaton A, one might be interested in obtaining an equivalent one with fewer states. This goal can be achieved by considering the quotient automaton of some right-invariant relation on the set of states of A. An equivalence relation $\equiv$ defined on the set of states Q is *right-invariant* w.r.t. A if and only if $\equiv \subseteq (Q - F)^2 \cup F^2$ and if $p \equiv q$, then $(\forall \sigma \in \Sigma)(\forall p' \in \delta(p, \sigma))(\exists q' \in \delta(q, \sigma))\,(p' \equiv q')$, for all $p, q \in Q$. If $\equiv$ is a right-invariant relation on Q, the *right-quotient automaton* $A/\equiv$ is given by $A/\equiv\, = \langle Q/\equiv, \Sigma, \delta/\equiv, I/\equiv, F/\equiv\rangle$, where $\delta/\equiv([p], \sigma) = \{\, [q] \mid q \in \delta(p, \sigma)\,\}$. Then, $\mathcal{L}\,(A/\equiv) = \mathcal{L}(A)$.

The Position Automaton. Given $\alpha \in$ RE, one can mark each occurrence of an alphabet symbol σ with its position in α, reading it from left to right. The resulting regular expression is a *marked* regular expression $\overline{\alpha}$ with all alphabet symbols occurring only once (linear) and belonging to $\Sigma_{\overline{\alpha}}$. The same notation is used for unmarking, i.e., $\overline{\overline{\alpha}} = \alpha$. In a marked expression $\overline{\alpha}$, a *position* $i \in [1, |\alpha|_\Sigma] = \{1, \dots, |\alpha|_\Sigma\}$ corresponds to a symbol σ_i, and thus to exactly one occurrence of σ in α. Given a position i, we denote by $\ell(i)$ the symbol $\sigma = \overline{\sigma_i}$. For instance, if $\alpha = a(bb + aba)^\star b$, then $\overline{\alpha} = a_1(b_2b_3 + a_4b_5a_6)^\star b_7$ and $\ell(4) = a$.

Let $\mathsf{Pos}(\alpha) = [1, |\alpha|_\Sigma]$, and $\mathsf{Pos}_0(\alpha) = \mathsf{Pos}(\alpha) \cup \{0\}$. Positions were used by Glushkov [9] to define an NFA equivalent to α, usually called the *position* or *Glushkov automaton*, $\mathcal{A}_{\mathrm{POS}}(\alpha)$. Each state of the automaton, except for the initial one, corresponds to a position, and there exists a transition from i to j by σ such that $\overline{\sigma_j} = \sigma$, if σ_i can be followed by σ_j in some word represented by $\overline{\alpha}$. The sets that are used to define the position automaton are $\mathsf{First}(\overline{\alpha}) = \{\, i \mid (\exists w \in \Sigma^\star_{\overline{\alpha}})\,(\sigma_i w \in \mathcal{L}(\overline{\alpha}))\,\}$, $\mathsf{Last}(\overline{\alpha}) = \{\, i \mid (\exists w \in \Sigma^\star_{\overline{\alpha}})\,(w\sigma_i \in \mathcal{L}(\overline{\alpha}))\,\}$ and, given $i \in \mathsf{Pos}(\alpha)$, $\mathsf{Follow}(\overline{\alpha}, i) = \{\, (\exists u, v \in \Sigma^\star_{\overline{\alpha}})\,(u\sigma_i\sigma_j v \in \mathcal{L}(\overline{\alpha}))\,\}$. For the sake of readability, whenever an expression α is not marked, we take $f(\alpha) = f(\overline{\alpha})$, for any function f that has marked expressions as arguments. We will not define the set Last explicitly, but just annotate each position i with a Boolean b, whose value is true iff $i \in \mathsf{Last}(\alpha)$. Those annotations can be done, when computing the set of positions of a given α as follows:

$$
\begin{aligned}
&\mathsf{Pos}(\varepsilon) = \emptyset, \quad \mathsf{Pos}(\sigma_i) = \{i : \mathsf{true}\}, \quad \mathsf{Pos}(\alpha^\star) = \mathsf{Pos}(\alpha),\\
&\mathsf{Pos}(\alpha_1 + \alpha_2) = \mathsf{Pos}(\alpha_1) \cup \mathsf{Pos}(\alpha_2),\\
&\mathsf{Pos}(\alpha_1\alpha_2) = \begin{cases} \mathsf{Pos}(\alpha_1) \cup \mathsf{Pos}(\alpha_2), & \text{if } \varepsilon(\alpha_2) = \mathsf{true},\\ \{\, i : \mathsf{false} \mid i : \mathsf{b} \in \mathsf{Pos}(\alpha_1)\,\} \cup \mathsf{Pos}(\alpha_2), & \text{otherwise.} \end{cases}
\end{aligned}
$$

In the remainder of the paper, we generally will omit the boolean in an annotated position $i : \mathsf{b}$, and we write $i \in \mathsf{Last}(\alpha)$ if and only if $\mathsf{b} = \mathsf{true}$, i.e., $i : \mathsf{true} \in \mathsf{Pos}(\alpha)$. The sets First and Follow are defined inductively as usual, but considering explicitly the letter associated with each position (this will be necessary when dealing with shuffle operators).

$$\begin{aligned}
\mathsf{First}(\varepsilon) = \emptyset, \quad \mathsf{First}(\sigma_i) &= \{(\overline{\sigma_i}, i)\}, \quad \mathsf{First}(\alpha^\star) = \mathsf{First}(\alpha),\\
\mathsf{First}(\alpha_1 + \alpha_2) &= \mathsf{First}(\alpha_1) \cup \mathsf{First}(\alpha_2),\\
\mathsf{First}(\alpha_1\alpha_2) &= \begin{cases} \mathsf{First}(\alpha_1) \cup \mathsf{First}(\alpha_2), & \text{if } \varepsilon(\alpha_1) = \mathsf{true},\\ \mathsf{First}(\alpha_1), & \text{otherwise.}\end{cases}\\
\mathsf{Follow}(\varepsilon, i) &= \mathsf{Follow}(\sigma_i, i) = \emptyset,\\
\mathsf{Follow}(\alpha_1 + \alpha_2, i) &= \begin{cases} \mathsf{Follow}(\alpha_1, i), & \text{if } i \in \mathsf{Pos}(\alpha_1),\\ \mathsf{Follow}(\alpha_2, i), & \text{if } i \in \mathsf{Pos}(\alpha_2),\end{cases}\\
\mathsf{Follow}(\alpha_1\alpha_2, i) &= \begin{cases} \mathsf{Follow}(\alpha_1, i), & \text{if } i \in \mathsf{Pos}(\alpha_1) \setminus \mathsf{Last}(\alpha_1),\\ \mathsf{Follow}(\alpha_1, i) \cup \mathsf{First}(\alpha_2), & \text{if } i \in \mathsf{Last}(\alpha_1),\\ \mathsf{Follow}(\alpha_2, i), & \text{if } i \in \mathsf{Pos}(\alpha_2),\end{cases}\\
\mathsf{Follow}(\alpha^\star, i) &= \begin{cases} \mathsf{Follow}(\alpha, i), & \text{if } i \notin \mathsf{Last}(\alpha),\\ \mathsf{Follow}(\alpha, i) \cup \mathsf{First}(\alpha), & \text{otherwise.}\end{cases}
\end{aligned}$$

We define the position automaton using the approach in Broda et al. [7], where the transition function is expressed as the composition of functions Select and Follow. Given a letter σ and a set of positions S, the function Select computes the subset of positions in S that correspond to letter σ. Formally, given $S \subseteq \mathsf{Pos}(\alpha)$ and $\sigma \in \Sigma$, let $\mathsf{Select}(S, \sigma) = \{\, i \mid i \in S \wedge \ell(i) = \sigma \,\}$. Then, the *position automaton* for α is

$$\mathcal{A}_{\mathrm{POS}}(\alpha) = \langle \mathsf{Pos}_0(\alpha), \Sigma, \delta_{\mathrm{POS}}, 0, \mathsf{Last}_0(\alpha)\rangle,$$

where $\delta_{\mathrm{POS}}(i, \sigma) = \mathsf{Select}(\mathsf{Follow}(\alpha, i), \sigma)$, $\mathsf{Follow}(\alpha, 0) = \mathsf{First}(\alpha)$, $\mathsf{Last}_0(\alpha) = \mathsf{Last}(\alpha) \cup \{0\}$ if $\varepsilon(\alpha) = \mathsf{true}$, and $\mathsf{Last}_0(\alpha) = \mathsf{Last}(\alpha)$, otherwise. Some examples can be found in [7].

2.1. Synchronised Shuffle Operators

In this section we review three synchronised shuffle operators, that were studied by Beek et al. [1] and by Sulzmann and Thiemann [3], presenting for each operator two equivalent definitions.

Strongly Synchronised Shuffle w.r.t. a set Γ*.* Given a set of alphabet symbols $\Gamma \subseteq \Sigma$, the strongly synchronised shuffle of two words w.r.t. Γ imposes synchronisation on all letters of Γ. The *strongly synchronised shuffle* of two words $u, v \in \Sigma^\star$ w.r.t. Γ, and denoted by $u \,{}^{\mathsf{s}}\|_\Gamma\, v$, is the finite set of words defined inductively as follows [3]:

$$\varepsilon \,{}^{\mathsf{s}}\|_{\Gamma}\, v \;=\; \begin{cases} \{v\}, & \text{if } \Sigma_v \cap \Gamma = \emptyset, \\ \emptyset, & \text{otherwise,} \end{cases} \qquad u \,{}^{\mathsf{s}}\|_{\Gamma}\, \varepsilon = \begin{cases} \{u\}, & \text{if } \Sigma_u \cap \Gamma = \emptyset, \\ \emptyset, & \text{otherwise,} \end{cases}$$

$$\sigma u \,{}^{\mathsf{s}}\|_{\Gamma}\, \tau v \;=\; \begin{cases} \{\, \sigma w \mid w \in u \,{}^{\mathsf{s}}\|_{\Gamma}\, v \,\}, & \text{if } \sigma = \tau \wedge \sigma \in \Gamma, \\ \emptyset, & \text{if } \sigma \neq \tau \wedge \sigma, \tau \in \Gamma, \\ \{\, \sigma w \mid w \in u \,{}^{\mathsf{s}}\|_{\Gamma}\, \tau v \,\}, & \text{if } \sigma \notin \Gamma \wedge \tau \in \Gamma, \\ \{\, \tau w \mid w \in \sigma u \,{}^{\mathsf{s}}\|_{\Gamma}\, v \,\}, & \text{if } \sigma \in \Gamma \wedge \tau \notin \Gamma, \\ \{\, \sigma w \mid w \in u \,{}^{\mathsf{s}}\|_{\Gamma}\, \tau v \,\} & \\ \quad \cup \{\, \tau w \mid w \in \sigma u \,{}^{\mathsf{s}}\|_{\Gamma}\, v \,\}, & \text{if } \sigma, \tau \notin \Gamma. \end{cases}$$

Note that for $\Gamma = \emptyset$ the operator ${}^{\mathsf{s}}\|_{\emptyset}$ coincides with the usual shuffle operator $⧢$, given by $u ⧢ \varepsilon = \varepsilon ⧢ u = \{u\}$ and $\sigma u ⧢ \tau v = \{\, \sigma w \mid w \in u ⧢ \tau v \,\} \cup \{\, \tau w \mid w \in \sigma u ⧢ v \,\}$.

Example 1. *In $abca \,{}^{\mathsf{s}}\|_{\{a\}}\, ada$ it is mandatory to synchronise the first occurrences of a in $abca$ and in ada, as well as the last occurrences. In between, there may be any word obtained by shuffling bc with d. Thus, $abca \,{}^{\mathsf{s}}\|_{\{a\}}\, ada = \{abcda, abdca, adbca\}$.*

The following is an equivalent definition of the strongly synchronised shuffle of two words [1],

$$\begin{aligned} u \,{}^{\mathsf{s}}\|_{\Gamma}\, v = \{\, x \mid & (\exists n \geq 1)(\forall i \in [1, n])\, (\sigma_i \in \Gamma \wedge u_i, v_i \in (\Sigma \setminus \Gamma)^{\star} \wedge \\ & u = u_1 \sigma_1 \cdots \sigma_{n-1} u_n \wedge v = v_1 \sigma_1 \cdots \sigma_{n-1} v_n \wedge \\ & x \in (u_1 ⧢ v_1) \sigma_1 \cdots \sigma_{n-1} (u_n ⧢ v_n)) \,\}. \end{aligned}$$

Note that ${}^{\mathsf{s}}\|_{\Gamma}$ is commutative and associative.

Arbitrary Synchronised Shuffle w.r.t. a set Γ. Given a set of alphabet symbols $\Gamma \subseteq \Sigma$, the arbitrary synchronised shuffle of two words w.r.t. Γ permits symbols in Γ to synchronise, but does not force their synchronisation. Formally, the *arbitrary synchronised shuffle* of words u and v, denoted by $u \,{}^{\mathsf{a}}\|_{\Gamma}\, v$, is defined as follows [3]:

$$\varepsilon \,{}^{\mathsf{a}}\|_{\Gamma}\, v \;=\; v \,{}^{\mathsf{a}}\|_{\Gamma}\, \varepsilon = \{v\},$$

$$\sigma u \,{}^{\mathsf{a}}\|_{\Gamma}\, \tau v \;=\; \begin{cases} \{\, \sigma w \mid w \in u \,{}^{\mathsf{a}}\|_{\Gamma}\, \tau v \,\} \cup \{\, \tau w \mid w \in \sigma u \,{}^{\mathsf{a}}\|_{\Gamma}\, v \,\}, & \text{if } \sigma \neq \tau \vee \sigma \notin \Gamma, \\ \{\, \sigma w \mid w \in u \,{}^{\mathsf{a}}\|_{\Gamma}\, \tau v \,\} \cup \{\, \tau w \mid w \in \sigma u \,{}^{\mathsf{a}}\|_{\Gamma}\, v \,\} & \\ \quad \cup \{\, \sigma w \mid w \in u \,{}^{\mathsf{a}}\|_{\Gamma}\, v \,\}, & \text{if } \sigma = \tau \wedge \sigma \in \Gamma. \end{cases}$$

Alternatively, we can define ${}^{\mathsf{a}}\|_{\Gamma}$ as follows [1]:

$$\begin{aligned} u \,{}^{\mathsf{a}}\|_{\Gamma}\, v = \{\, x \mid & (\exists n \geq 1)(\forall i \in [1, n])\, (\sigma_i \in \Gamma \wedge u_i, v_i \in \Sigma^{\star} \wedge \\ & u = u_1 \sigma_1 \cdots \sigma_{n-1} u_n \wedge v = v_1 \sigma_1 \cdots \sigma_{n-1} v_n \wedge \\ & x \in (u_1 ⧢ v_1) \sigma_1 \cdots \sigma_{n-1} (u_n ⧢ v_n)) \,\}. \end{aligned}$$

Note that ${}^{\mathsf{a}}\|_{\Gamma}$ is commutative and associative.

Example 2. *We have* $ab\,{}^{\mathsf{a}}\|_{\{a\}}\, da = \{abda, adba, adab, dab, daab, daba\}$.

Weak Synchronised Shuffle w.r.t. a set Γ. In the *weak synchronised shuffle* of two words u and v, letters in Γ can be synchronised or optionally be interleaved from one word, but not from the other, until the next synchronisation occurs. Given two words u and v, the definition of $u\,{}^{\mathsf{w}}\|_{\Gamma}\, v$ resorts to an auxiliary operator $\|_{(\Gamma,\Delta,\Lambda)}$. This operator memorises in Δ (resp. Λ) the elements in Γ, that have been interleaved from u (resp. v) since the last synchronisation has occured. Note that in every step during the computation of $u\,{}^{\mathsf{w}}\|_{\Gamma}\, v$ the operator $\|_{(\Gamma,\Delta,\Lambda)}$ has the invariant $\Delta\cap\Lambda=\emptyset \ \wedge\ \Delta\cup\Lambda\subseteq\Gamma$. Then,

$$
\begin{aligned}
u\,{}^{\mathsf{w}}\|_{\Gamma}\, v &= u\,\|_{(\Gamma,\emptyset,\emptyset)} v,\\
\varepsilon\,\|_{(\Gamma,\Delta,\Lambda)} v &= \begin{cases}\{v\}, & \text{if } \Sigma_v\cap\Delta=\emptyset,\\ \emptyset, & \text{otherwise.}\end{cases}\\
u\,\|_{(\Gamma,\Delta,\Lambda)} \varepsilon &= \begin{cases}\{u\}, & \text{if } \Sigma_u\cap\Lambda=\emptyset,\\ \emptyset, & \text{otherwise,}\end{cases}\\
\sigma u\,\|_{(\Gamma,\Delta,\Lambda)} \tau v &=
\end{aligned}
$$

$$
\begin{cases}
\{\,\sigma w \mid w\in u\,\|_{(\Gamma,\Delta,\Lambda)}\tau v\,\}\cup\{\,\tau w\mid w\in \sigma u\,\|_{(\Gamma,\Delta,\Lambda)} v\,\}, & \text{if } \sigma,\tau\notin\Gamma,\\
\{\,\sigma w\mid w\in u\,\|_{(\Gamma,\emptyset,\emptyset)} v\,\} & \\
\qquad\cup\ \{\,\sigma w\mid w\in u\,\|_{(\Gamma,\Delta\cup\{\sigma\},\Lambda)}\tau v\wedge \sigma\notin\Lambda\,\} & \\
\qquad\cup\ \{\,\tau w\mid w\in \sigma u\,\|_{(\Gamma,\Delta,\Lambda\cup\{\tau\})} v\wedge\tau\notin\Delta\,\}, & \text{if } \sigma=\tau\in\Gamma,\\
\{\,\sigma w\mid w\in u\,\|_{(\Gamma,\Delta\cup\{\sigma\},\Lambda)}\tau v\wedge \sigma\notin\Lambda\,\} & \\
\qquad\cup\ \{\,\tau w\mid w\in \sigma u\,\|_{(\Gamma,\Delta,\Lambda\cup\{\tau\})} v\wedge\tau\notin\Delta\,\}, & \text{if } \sigma\neq\tau, \sigma,\tau\in\Gamma,\\
\{\,\sigma w\mid w\in u\,\|_{(\Gamma,\Delta\cup\{\sigma\},\Lambda)}\tau v\wedge \sigma\notin\Lambda\,\} & \\
\qquad\cup\ \{\,\tau w\mid w\in \sigma u\,\|_{(\Gamma,\Delta,\Lambda)} v\,\}, & \text{if } \sigma\in\Gamma,\tau\notin\Gamma,\\
\{\,\sigma w\mid w\in u\,\|_{(\Gamma,\Delta,\Lambda)}\tau v\,\} & \\
\qquad\cup\ \{\,\tau w\mid w\in \sigma u\,\|_{(\Gamma,\Delta,\Lambda\cup\{\tau\})} v\wedge\tau\notin\Delta\,\}, & \text{if } \sigma\notin\Gamma,\tau\in\Gamma.
\end{cases}
$$

The operator $\|_{(\Gamma,\Delta,\Lambda)}$ as defined above coincides with the general synchronous shuffling [3] for the weak synchronised shuffle. However, in [3] the authors consider an extension of this operator, where Δ and Λ can be any subsets of Σ. For instance, for $\Delta=\Lambda=\Sigma$ the strong synchronised shuffle operator is obtained. In this paper, we choose to consider the three different forms of shuffling separately, as we feel that in this way functions for the strong and arbitrary synchronised shuffle are easier to understand and compute.

For $\Delta,\Lambda\subseteq\Gamma\subseteq\Sigma$, we have the following alternative definition of $u\,\|_{(\Gamma,\Delta,\Lambda)} v$, inspired by the one for $u\,{}^{\mathsf{w}}\|_{\Gamma}\, v$ in [1]:

$$
\begin{aligned}
u\,\|_{(\Gamma,\Delta,\Lambda)} v = \{\, x \mid {} & (\exists n\geq 1)(\forall i\in[1,n])\,(\sigma_i\in\Gamma\wedge u_i, v_i\in\Sigma^\star\wedge\\
& \Sigma_{u_i}\cap\Sigma_{v_i}\cap\Gamma=\Sigma_{u_1}\cap\Lambda=\Sigma_{v_1}\cap\Delta=\emptyset\ \wedge\\
& u=u_1\sigma_1\cdots\sigma_{n-1}u_n\ \wedge v=v_1\sigma_1\cdots\sigma_{n-1}v_n\ \wedge\\
& x\in(u_1\sqcup\!\sqcup\, v_1)\sigma_1\cdots\sigma_{n-1}(u_n\sqcup\!\sqcup\, v_n))\,\}.
\end{aligned}
$$

Note that ${}^{\mathsf{w}}\|_\Gamma$ is commutative but not associative.

Example 3. *We have* $abca\,{}^{\mathsf{w}}\|_{\{a,b\}}\, ada = \{abcda, abdca, adbca, abcada, adabca\}$.

Given two languages $L_1, L_2 \subseteq \Sigma^\star$ and $\circ \in \{\,{}^{\mathsf{s}}\|_\Gamma\,, {}^{\mathsf{a}}\|_\Gamma\,, {}^{\mathsf{w}}\|_\Gamma\,\}$ one has, as usual, $L_1 \circ L_2 = \bigcup_{u\in L_1, v\in L_2} u \circ v$. If L_1 and L_2 are regular, $L_1 \circ L_2$ is regular.[1] In the same way, we have for the auxiliary operator $\|_{(\Gamma,\Delta,\Lambda)}$ that $L_1 \|_{(\Gamma,\Delta,\Lambda)} L_2 = \bigcup_{u\in L_1, v\in L_2} u \|_{(\Gamma,\Delta,\Lambda)} v$.

2.2. Regular Expressions with Shuffle Operators

One can extend regular expressions to include the different shuffle operators defined above. The set of these extended regular expressions is denoted by RE($\|$), and contains $\emptyset$ plus all terms generated by the grammar

$$\alpha \quad\to\quad \varepsilon \mid \sigma \mid (\alpha+\alpha) \mid (\alpha\cdot\alpha) \mid (\alpha^\star) \mid (\alpha \sqcup\!\sqcup \alpha) \mid (\alpha\,{}^{\mathsf{s}}\|_\Gamma\,\alpha) \mid (\alpha\,{}^{\mathsf{a}}\|_\Gamma\,\alpha) \mid (\alpha\,{}^{\mathsf{w}}\|_\Gamma\,\alpha),$$

where $\sigma \in \Sigma$, $\Gamma \subseteq \Sigma$. The language of an expression $\alpha \in \mathrm{RE}(\|)$ is defined as usual, where for $\circ \in \{\sqcup\!\sqcup, {}^{\mathsf{s}}\|_\Gamma\,, {}^{\mathsf{a}}\|_\Gamma\,, {}^{\mathsf{w}}\|_\Gamma\,\}$ one has $\mathcal{L}(\alpha\circ\beta) = \mathcal{L}(\alpha)\circ\mathcal{L}(\beta)$.

3. A Location Based Position Automaton

In this section we define a new construction for a position automaton for extended regular expressions, which is based on the sets First, Last, and Follow. In order to define those sets for expressions containing the standard shuffle operator $\sqcup\!\sqcup$, Broda et al. [5, 6] considered more complex structures, called locations. Locations were defined in such a way that, given an expression with nested shuffles, it allows to specify how far a word has advanced in each of the components (shuffles) of this expression. In this paper, we define locations for expressions in RE($\|$). For operators ${}^{\mathsf{s}}\|_\Gamma$ and ${}^{\mathsf{a}}\|_\Gamma$ locations are defined as for $\sqcup\!\sqcup$. The definition of locations for expressions of the form $\alpha\,{}^{\mathsf{w}}\|_\Gamma\,\beta$ will have additional parameters corresponding to the sets Δ and Λ above. Additionally, we annotate each location p with a Boolean b, whose value is true iff the location belongs to Last. Given $\alpha \in \mathrm{RE}(\|)$, the set of *annotated locations* $\mathsf{Loc}(\alpha) = \mathsf{Loc}(\overline{\alpha})$ is

[1] Sulzmann and Thiermann [3] consider yet another concurrency operator, the synchronised composition of two languages L_1 and L_2, $L_1 \| L_2$. This operator can be defined using the general synchronous shuffling parameterised with $\Delta = \Lambda = \Sigma$ and $\Gamma = \Sigma_{L_1} \cap \Sigma_{L_2}$, where Σ_L is the set of alphabetic symbols that occur in L. However, as it is not straightforward to compute $\Sigma_{\mathcal{L}(\alpha)}$ for an expression α that contains synchronised shuffle operators, we do not consider the general synchronised shuffle operator in this paper.

defined inductively as follows:

$$
\begin{aligned}
\mathsf{Loc}(\varepsilon) &= \emptyset,\ \mathsf{Loc}(\sigma_i) = \{i : \mathsf{true}\},\ \mathsf{Loc}(\alpha^\star) = \mathsf{Loc}(\alpha),\\
\mathsf{Loc}(\alpha_1 + \alpha_2) &= \mathsf{Loc}(\alpha_1) \cup \mathsf{Loc}(\alpha_2),\\
\mathsf{Loc}(\alpha_1\alpha_2) &= \begin{cases} \mathsf{Loc}(\alpha_1) \cup \mathsf{Loc}(\alpha_2), & \text{if } \varepsilon(\alpha_2) = \mathsf{true},\\ \{\ p : \mathsf{false} \mid p : \mathsf{b} \in \mathsf{Loc}(\alpha_1)\ \} \cup \mathsf{Loc}(\alpha_2), & \text{otherwise,} \end{cases}\\
\mathsf{Loc}(\alpha_1 \sqcup\!\sqcup \alpha_2) &= \mathsf{Loc}(\alpha_1 \ {}^{\mathsf{s}}\|_\Gamma\ \alpha_2) = \mathsf{Loc}(\alpha_1 \ {}^{\mathsf{a}}\|_\Gamma\ \alpha_2)\\
&= \{\ (p,0) : \mathsf{b} \wedge \varepsilon(\alpha_2) \mid p : \mathsf{b} \in \mathsf{Loc}(\alpha_1)\ \}\\
&\cup\ \{\ (0,q) : \mathsf{b} \wedge \varepsilon(\alpha_1) \mid q : \mathsf{b} \in \mathsf{Loc}(\alpha_2)\ \}\\
&\cup\ \{\ (p,q) : \mathsf{b}_1 \wedge \mathsf{b}_2 \mid p : \mathsf{b}_1 \in \mathsf{Loc}(\alpha_1) \wedge q : \mathsf{b}_2 \in \mathsf{Loc}(\alpha_2)\ \},\\
\mathsf{Loc}(\alpha_1 \ {}^{\mathsf{w}}\|_\Gamma\ \alpha_2) &= \{\ (p^\Delta, 0^\emptyset) : \mathsf{b} \wedge \varepsilon(\alpha_2) \mid p : \mathsf{b} \in \mathsf{Loc}(\alpha_1) \wedge \Delta \subseteq \Gamma\ \}\\
&\cup\ \{\ (0^\emptyset, q^\Lambda) : \mathsf{b} \wedge \varepsilon(\alpha_1) \mid q : \mathsf{b} \in \mathsf{Loc}(\alpha_2) \wedge \Lambda \subseteq \Gamma\ \} \qquad (1)\\
&\cup\ \{\ (p^\Delta, q^\Lambda) : \mathsf{b}_1 \wedge \mathsf{b}_2 \mid p : \mathsf{b}_1 \in \mathsf{Loc}(\alpha_1) \wedge q : \mathsf{b}_2 \in \mathsf{Loc}(\alpha_2),\\
&\quad \wedge\ \Delta, \Lambda \subseteq \Gamma\ \wedge\ \Delta \cap \Lambda = \emptyset\ \}.
\end{aligned}
$$

It follows from this definition, that $p : \mathsf{b}, p : \mathsf{b}' \in \mathsf{Loc}(\alpha)$ implies $\mathsf{b} = \mathsf{b}'$. As before, we will omit the boolean of an annotated location whenever convenient. Note that each location p in α is either a position $i \in \mathsf{Pos}(\alpha)$, or of the form $(0,q)$, $(p,0)$, (p,q), $(0^\emptyset, q^\Lambda)$, $(p^\Delta, 0^\emptyset)$, or (p^Δ, q^Λ), where p, q are also locations in α and $\Delta, \Lambda \subseteq \Sigma$. In the remaining of the paper, we frequently use p^σ and $p^{\Delta,\sigma}$ as abbreviations for $p^{\{\sigma\}}$ and for $p^{\Delta\cup\{\sigma\}}$, respectively.

The set of positions of a location p, $\mathsf{lp}(p)$, is defined inductively by

$$
\begin{aligned}
\mathsf{lp}(i) &= \{i\},\\
\mathsf{lp}((0^\emptyset, p^\Lambda)) = \mathsf{lp}((0,p)) = \mathsf{lp}((p^\Delta, 0^\emptyset)) &= \mathsf{lp}((p,0)) = \mathsf{lp}(p),\\
\mathsf{lp}((p^\Delta, q^\Lambda)) = \mathsf{lp}((p,q)) &= \mathsf{lp}(p) \cup \mathsf{lp}(q).
\end{aligned}
$$

Example 4. *Consider* $\alpha = \alpha_1 \ {}^{\mathsf{w}}\|_{\{b\}}\ \alpha_2$*, where* $\alpha_1 = (ab)^\star \ {}^{\mathsf{s}}\|_{\{a\}}\ (ca)^\star$ *and* $\alpha_2 = (bc)^\star$*. The marked expression is* $\overline{\alpha} = ((a_1b_2)^\star \ {}^{\mathsf{s}}\|_{\{a\}}\ (c_3a_4)^\star) \ {}^{\mathsf{w}}\|_{\{b\}}\ (b_5c_6)^\star$*. The sets of (annotated) locations of* α_1*,* α_2*, and* α *are given below.*

$$
\begin{aligned}
\mathsf{Loc}(\alpha_1) &= \{\, p : \mathsf{false} \mid p \in \{(0,3),(1,0),(1,3),(2,3),(1,4)\}\,\}\\
&\quad \cup \{\, p : \mathsf{true} \mid p \in \{(0,4),(2,0),(2,4)\}\,\},\\
\mathsf{Loc}(\alpha_2) &= \{5 : \mathsf{false}, 6 : \mathsf{true}\},\\
\mathsf{Loc}(\alpha) &= \{\, (p^S, 0^\emptyset) : \mathsf{b} \mid p : \mathsf{b} \in \mathsf{Loc}(\alpha_1), S \subseteq \{b\}\,\}\\
&\quad \cup \{\, (0^\emptyset, p^S) : \mathsf{b} \mid p : \mathsf{b} \in \mathsf{Loc}(\alpha_2), S \subseteq \{b\}\,\}\\
&\quad \cup \{\, (p_1^{S_1}, p_2^{S_2}) : \mathsf{b}_1 \wedge \mathsf{b}_2 \mid p_i : \mathsf{b}_i \in \mathsf{Loc}(\alpha_i), S_i \subseteq \{b\},\\
&\qquad\qquad S_1 \cap S_2 = \emptyset, i = 1,2\,\}.
\end{aligned}
$$

We have $\mathsf{lp}(((2,3)^b, 0^\emptyset)) = \{2,3\}$*, and* $\mathsf{lp}(((2,3)^\emptyset, 5^b)) = \{2,3,5\}$*. For instance, the location* $((1,4)^\emptyset, 6^b)$ *corresponds to words for which the last letters read in the subexpressions* $(ab)^\star$*,* $(ca)^\star$ *and* $(bc)^\star$*, are respectively* a*,* a*, and* c*.*

As an example, consider $w = cabcbcbc \in cab\,{}^{\mathsf{w}}\|_{\{b\}}\, bcbcbc \subseteq \mathcal{L}(\alpha)$*. When processing* w *in a position automaton, the word can reach the final state* $((2,4)^{\emptyset}, 6^{b})$*, passing successively through states labelled respectively with locations* $0 \to^{c} ((0,3)^{\emptyset}, 0^{\emptyset}) \to^{a} ((1,4)^{\emptyset}, 0^{\emptyset}) \to^{b} ((1,4)^{\emptyset}, 5^{b}) \to^{c} ((1,4)^{\emptyset}, 6^{b}) \to^{b} ((2,4)^{\emptyset}, 5^{\emptyset}) \to^{c} ((2,4)^{\emptyset}, 6^{\emptyset}) \to^{b} ((2,4)^{\emptyset}, 5^{b}) \to^{c} ((2,4)^{\emptyset}, 6^{b})$*. This path corresponds to a synchronisation of the* b *in* cab *with the second* b *in* $bcbcbc$*. Synchronisations with either the first or third* b *in* $bcbcbc$ *lead to different paths.*

Given $\alpha \in \mathrm{RE}(\|)$, the states in the position automaton will be labelled by the elements in $\mathsf{Loc}(\alpha)$, except for the initial state labelled by 0. We now define the set $\mathsf{First}(\alpha) \subseteq \Sigma \times \mathsf{Loc}(\alpha)$ for the different shuffle operators as follows:

$$
\begin{aligned}
\mathsf{First}(\alpha_1\, {}^{\mathsf{s}}\|_{\Gamma}\, \alpha_2) &= \{\, (\sigma, (p, 0)) \mid \sigma \notin \Gamma \wedge (\sigma, p) \in \mathsf{First}(\alpha_1) \,\} \\
&\cup \{\, (\sigma, (0, q)) \mid \sigma \notin \Gamma \wedge (\sigma, q) \in \mathsf{First}(\alpha_2) \,\} \\
&\cup \{\, (\sigma, (p, q)) \mid \sigma \in \Gamma \wedge (\sigma, p) \in \mathsf{First}(\alpha_1) \wedge (\sigma, q) \in \mathsf{First}(\alpha_2) \,\}, \\
\mathsf{First}(\alpha_1\, {}^{\mathsf{a}}\|_{\Gamma}\, \alpha_2) &= \{\, (\sigma, (p, 0)) \mid (\sigma, p) \in \mathsf{First}(\alpha_1) \,\} \\
&\cup \{\, (\sigma, (0, q)) \mid (\sigma, q) \in \mathsf{First}(\alpha_2) \,\} \\
&\cup \{\, (\sigma, (p, q)) \mid \sigma \in \Gamma \wedge (\sigma, p) \in \mathsf{First}(\alpha_1), (\sigma, q) \in \mathsf{First}(\alpha_2) \,\}, \\
\mathsf{First}(\alpha_1\, {}^{\mathsf{w}}\|_{\Gamma}\, \alpha_2) &= \{\, (\sigma, (p^{\emptyset}, 0^{\emptyset})) \mid \sigma \notin \Gamma \wedge (\sigma, p) \in \mathsf{First}(\alpha_1) \,\} \\
&\cup \{\, (\sigma, (0^{\emptyset}, q^{\emptyset})) \mid \sigma \notin \Gamma \wedge (\sigma, q) \in \mathsf{First}(\alpha_2) \,\} \\
&\cup \{\, (\sigma, (p^{\emptyset}, q^{\emptyset})) \mid \sigma \in \Gamma \wedge (\sigma, p) \in \mathsf{First}(\alpha_1) \wedge (\sigma, q) \in \mathsf{First}(\alpha_2) \,\} \\
&\cup \{\, (\sigma, (p^{\{\sigma\}}, 0^{\emptyset})) \mid \sigma \in \Gamma \wedge (\sigma, p) \in \mathsf{First}(\alpha_1) \,\} \\
&\cup \{\, (\sigma, (0^{\emptyset}, q^{\{\sigma\}})) \mid \sigma \in \Gamma \wedge (\sigma, q) \in \mathsf{First}(\alpha_2) \,\}.
\end{aligned}
$$

Note that the definition of $\mathsf{First}(\alpha_1 \sqcup\!\sqcup\, \alpha_2)$ given in [5, 6] by

$$
\mathsf{First}(\alpha_1 \sqcup\!\sqcup\, \alpha_2) = \{\, (\sigma, (p, 0)) \mid (\sigma, p) \in \mathsf{First}(\alpha_1) \,\} \cup \{\, (\sigma, (0, q)) \mid (\sigma, q) \in \mathsf{First}(\alpha_2) \,\},
$$

coincides precisely with $\mathsf{First}(\alpha_1\, {}^{\mathsf{s}}\|_{\emptyset}\, \alpha_2)$.

Example 5. *For* $\alpha = ((ab)^{\star}\, {}^{\mathsf{s}}\|_{\{a\}}\, (ca)^{\star})\, {}^{\mathsf{w}}\|_{\{b\}}\, (bc)^{\star}$ *of Example 4, one has,*

$$
\begin{aligned}
&\mathsf{First}((ab)^{\star}) = \{(a, 1)\},\ \mathsf{First}((ca)^{\star}) = \{(c, 3)\},\ \mathsf{First}((bc)^{\star}) = \{(b, 5)\}, \\
&\mathsf{First}((ab)^{\star}\, {}^{\mathsf{s}}\|_{\{a\}}\, (ca)^{\star}) = \{(c, (0, 3))\}, \\
&\mathsf{First}(((ab)^{\star}\, {}^{\mathsf{s}}\|_{\{a\}}\, (ca)^{\star})\, {}^{\mathsf{w}}\|_{\{b\}}\, (bc)^{\star}) = \{(c, ((0, 3)^{\emptyset}, 0^{\emptyset})), (0^{\emptyset}, 5^{b})\}.
\end{aligned}
$$

Fact 1. *For every element* $(\sigma, p) \in \mathsf{First}(\alpha)$*,* $\ell(\mathsf{lp}(p)) = \{\sigma\}$*.*

Lemma 1. *Given* $\alpha \in \mathrm{RE}(\|)$*, if there is some* $\sigma w \in \mathcal{L}(\alpha)$*, then there is a location* p *such that* $(\sigma, p) \in \mathsf{First}(\alpha)$*.*

Proof. The proof is by induction on the structure of an expression α. For ε and alphabet symbols the result is obvious. For union, concatenation and Kleene star the proof is similar to the one for standard expressions.

Case $\alpha = \alpha_1 {}^{\mathsf{s}}\|_\Gamma\, \alpha_2$. Consider a word $\sigma w \in \mathcal{L}(\alpha_1 {}^{\mathsf{s}}\|_\Gamma\, \alpha_2)$. Then, there are words $u_1\sigma_1 \cdots \sigma_{n-1}u_n \in \mathcal{L}(\alpha_1)$ and $v_1\sigma_1 \cdots \sigma_{n-1}v_n \in \mathcal{L}(\alpha_2)$, where for $i \in [1, n]$ $\sigma_i \in \Gamma$ and $u_i, v_i \in (\Sigma \setminus \Gamma)^\star$, such that $\sigma w \in (u_1 \sqcup\!\sqcup v_1)\sigma_1 \cdots \sigma_{n-1}(u_n \sqcup\!\sqcup v_n)$. If $\sigma \in \Gamma$, then $u_1 = v_1 = \varepsilon$ and $\sigma = \sigma_1$. Thus, $\sigma u_2 \cdots \sigma_{n-1}u_n \in \mathcal{L}(\alpha_1)$ and $\sigma v_2 \cdots \sigma_{n-1}v_n \in \mathcal{L}(\alpha_2)$. If follows from the induction hypothesis that there are locations p_1 and p_2 such that $(\sigma, p_1) \in \mathsf{First}(\alpha_1)$ and $(\sigma, p_2) \in \mathsf{First}(\alpha_2)$. Consequently, $(\sigma, (p_1, p_2)) \in \mathsf{First}(\alpha_1 {}^{\mathsf{s}}\|_\Gamma\, \alpha_2)$. If $\sigma \notin \Gamma$ and $u_1 = \sigma u_1'$ (the case $v_1 = \sigma v_1'$ is analogous), then

$$\sigma u_1'\sigma_1 \cdots \sigma_{n-1}u_n \in \mathcal{L}(\alpha_1),$$

and by the induction hypothesis there is a location p such that $(\sigma, p) \in \mathsf{First}(\alpha_1)$. We conclude from the definition that $(\sigma, (p, 0)) \in \mathsf{First}(\alpha_1 {}^{\mathsf{s}}\|_\Gamma\, \alpha_2)$.

Case $\alpha = \alpha_1 {}^{\mathsf{a}}\|_\Gamma\, \alpha_2$. The proof is similar to the proof for ${}^{\mathsf{s}}\|_\Gamma$, dropping the assumption $\sigma \in \Gamma$.

Case $\alpha = \alpha_1 {}^{\mathsf{w}}\|_\Gamma\, \alpha_2$. Consider a word $\sigma w \in \mathcal{L}(\alpha_1 {}^{\mathsf{w}}\|_\Gamma\, \alpha_2)$. Then, there are words $u_1\sigma_1 \cdots \sigma_{n-1}u_n \in \mathcal{L}(\alpha_1)$ and $v_1\sigma_1 \cdots \sigma_{n-1}v_n \in \mathcal{L}(\alpha_2)$, where for $i \in [1, n]$ $\sigma_i \in \Gamma$ and $\Sigma_{u_i} \cap \Sigma_{v_i} \cap \Gamma = \emptyset$, such that $\sigma w \in (u_1 \sqcup\!\sqcup v_1)\sigma_1 \cdots \sigma_{n-1}(u_n \sqcup\!\sqcup v_n)$.

If $\sigma \notin \Gamma$ and $u_1 = \sigma u_1'$ (the case $v_1 = \sigma v_1'$ is analogous), then

$$\sigma u_1'\sigma_1 \cdots \sigma_{n-1}u_n \in \mathcal{L}(\alpha_1),$$

and by the induction hypothesis there is a location p such that $(\sigma, p) \in \mathsf{First}(\alpha_1)$. We conclude from the definition that $(\sigma, (p^\emptyset, 0^\emptyset)) \in \mathsf{First}(\alpha_1 {}^{\mathsf{w}}\|_\Gamma\, \alpha_2)$.

If $\sigma \in \Gamma$, then either $u_1 = v_1 = \varepsilon$ and $\sigma = \sigma_1$, or $u_1 = \sigma u_1'$ (the case $v_1 = \sigma v_1'$ is analogous) and $\sigma u_1'\sigma_1 \cdots \sigma_{n-1}u_n \in \mathcal{L}(\alpha_1)$. In latter case, we have by the induction hypothesis that there is a location p with $(\sigma, p) \in \mathsf{First}(\alpha_1)$. Thus, $(\sigma, (p^{\{\sigma\}}, 0^\emptyset)) \in \mathsf{First}(\alpha_1 {}^{\mathsf{s}}\|_\Gamma\, \alpha_2)$. In the former case, the induction hypothesis applies to $\sigma u_2 \cdots \sigma_{n-1}u_n \in \mathcal{L}(\alpha_1)$ and $\sigma v_2 \cdots \sigma_{n-1}v_n \in \mathcal{L}(\alpha_2)$. Thus, there are locations p_1 and p_2 such that $(\sigma, p_1) \in \mathsf{First}(\alpha_1)$ and $(\sigma, p_2) \in \mathsf{First}(\alpha_2)$. Consequently, $(\sigma, (p_1^\emptyset, p_2^\emptyset)) \in \mathsf{First}(\alpha_1 {}^{\mathsf{s}}\|_\Gamma\, \alpha_2)$. □

The set $\mathsf{Last}(\alpha) \subseteq \mathsf{Loc}(\alpha)$ is defined by $\mathsf{Last}(\alpha) = \{\, p \mid p : \mathsf{true} \in \mathsf{Loc}(\alpha) \,\}$. Furthermore, let $\mathsf{Last}_0(\alpha) = \mathsf{Last}(\alpha) \cup \{0\}$ if $\varepsilon(\alpha) = \mathsf{true}$, and $\mathsf{Last}_0(\alpha) = \mathsf{Last}(\alpha)$ otherwise.

Example 6. *For $\alpha = ((ab)^\star {}^{\mathsf{s}}\|_{\{a\}}\, (ca)^\star) {}^{\mathsf{w}}\|_{\{b\}}\, (bc)^\star$ of Example 4 we have,*

$$\begin{aligned}\mathsf{Last}(\alpha) \quad = \quad & \{\, (p^S, 0^\emptyset) \mid p \in L_2, S \subseteq \{b\} \,\} \cup \{\, (0^\emptyset, 6^S) \mid S \subseteq \{b\} \,\} \\ & \cup \{\, (p^{S_1}, 6^{S_2}) \mid p \in L_2, S_1, S_2 \subseteq \{b\}, S_1 \cap S_2 = \emptyset \,\},\end{aligned}$$

where $L_2 = \{(0, 4), (2, 0), (2, 4)\}$. Note that the only locations in $\mathsf{Last}(\alpha)$ reachable by words of $\mathcal{L}(\alpha)$ are $((2, 4)^b, 0^\emptyset)$ and $((2, 4)^\emptyset, 6^\emptyset)$. This happens because the only location reachable in $\mathsf{Last}((ab)^\star {}^{\mathsf{s}}\|_{\{a\}}\, (ca)^\star)$ is $(2, 4)$.

Finally, we define $\mathsf{Follow} : \mathrm{RE}(\|) \times \mathsf{Loc}_0(\alpha) \to 2^{\Sigma_\alpha \times \mathsf{Loc}(\alpha)}$, where $\mathsf{Loc}_0(\alpha) = \mathsf{Loc}(\alpha) \cup \{0\}$. Let $\mathsf{Follow}(\alpha, 0) = \mathsf{First}(\alpha)$, and for $p_1, q_1 \in \mathsf{Loc}_0(\alpha)$,

$$\begin{aligned}
\mathsf{Follow}(\alpha_1 {}^{\mathsf{s}}\|_\Gamma\, \alpha_2, (p_1, q_1)) = {} & \{ (\sigma, (p_2, q_1)) \mid (\sigma, p_2) \in \mathsf{Follow}(\alpha_1, p_1) \wedge \sigma \notin \Gamma \} \\
& \cup \{ (\sigma, (p_1, q_2)) \mid (\sigma, q_2) \in \mathsf{Follow}(\alpha_2, q_1) \wedge \sigma \notin \Gamma \} \\
& \cup \{ (\sigma, (p_2, q_2)) \mid (\sigma, p_2) \in \mathsf{Follow}(\alpha_1, p_1) \\
& \qquad \wedge (\sigma, q_2) \in \mathsf{Follow}(\alpha_2, q_1) \wedge \sigma \in \Gamma \}, \\
\mathsf{Follow}(\alpha_1 {}^{\mathsf{a}}\|_\Gamma\, \alpha_2, (p_1, q_1)) = {} & \{ (\sigma, (p_2, q_1)) \mid (\sigma, p_2) \in \mathsf{Follow}(\alpha_1, p_1) \} \\
& \cup \{ (\sigma, (p_1, q_2)) \mid (\sigma, q_2) \in \mathsf{Follow}(\alpha_2, q_1) \} \\
& \cup \{ (\sigma, (p_2, q_2)) \mid (\sigma, p_2) \in \mathsf{Follow}(\alpha_1, p_1) \\
& \qquad \wedge (\sigma, q_2) \in \mathsf{Follow}(\alpha_2, q_1) \wedge \sigma \in \Gamma \}, \\
\mathsf{Follow}(\alpha_1 {}^{\mathsf{w}}\|_\Gamma\, \alpha_2, (p_1^\Delta, q_1^\Lambda)) = {} & \{ (\sigma, (p_2^\Delta, q_1^\Lambda)) \mid (\sigma, p_2) \in \mathsf{Follow}(\alpha_1, p_1) \wedge \sigma \notin \Gamma \} \\
& \cup \{ (\sigma, (p_1^\Delta, q_2^\Lambda)) \mid (\sigma, q_2) \in \mathsf{Follow}(\alpha_2, q_1) \wedge \sigma \notin \Gamma \} \\
& \cup \{ (\sigma, (p_2^\emptyset, q_2^\emptyset)) \mid (\sigma, p_2) \in \mathsf{Follow}(\alpha_1, p_1) \\
& \qquad \wedge (\sigma, q_2) \in \mathsf{Follow}(\alpha_1, q_1) \wedge \sigma \in \Gamma \} \\
& \cup \{ (\sigma, (p_2^{\Delta,\sigma}, q_1^\Lambda)) \mid (\sigma, p_2) \in \mathsf{Follow}(\alpha_1, p_1) \wedge \sigma \in \Gamma \setminus \Lambda \} \\
& \cup \{ (\sigma, (p_1^\Delta, q_2^{\Lambda,\sigma})) \mid (\sigma, q_2) \in \mathsf{Follow}(\alpha_2, q_1) \wedge \sigma \in \Gamma \setminus \Delta \}.
\end{aligned}$$

Remark 1. *If $p, p' \in \mathsf{Loc}(\alpha)$ and $(\sigma, p) \in \mathsf{Follow}(\alpha, p')$, then $p \neq 0$.*

Furthermore, given $S \in 2^{\mathsf{Loc}_0(\alpha)}$ let $\mathsf{Follow}(\alpha, S) = \bigcup_{p \in S} \mathsf{Follow}(\alpha, p)$. Sometimes we use the abbreviation $\mathsf{Follow}(\alpha') = \{ (p, \sigma, q) \mid (\sigma, q) \in \mathsf{Follow}(\alpha', p) \}$.

Example 7. *For $\alpha = ((ab)^{\star} {}^{\mathsf{s}}\|_{\{a\}}\, (ca)^{\star}) {}^{\mathsf{w}}\|_{\{b\}}\, (bc)^{\star}$ of Example 4 and considering the set $\mathsf{First}(\alpha)$ given in Example 5, part of the Follow sets are given below.*

$$\begin{aligned}
\mathsf{Follow}((ab)^\star) &= \{(1, b, 2), (2, a, 1)\}, \\
\mathsf{Follow}((ca)^\star) &= \{(3, a, 4), (4, c, 3)\}, \\
\mathsf{Follow}((bc)^\star) &= \{(5, c, 6), (6, b, 5)\}, \\
\mathsf{Follow}((ab)^{\star} {}^{\mathsf{s}}\|_{\{a\}}\, (ca)^\star) &= \{((0,3), a, (1,4)), ((1,4), b, (2,4)), ((1,4), c, (1,3)), \\
&\quad ((1,3), b, (2,3)), ((2,3), a, (1,4)), ((2,4), c, (2,3))\}, \\
\mathsf{Follow}(\alpha, ((0,3)^\emptyset, 0^\emptyset)) &= \{(a, ((1,4)^\emptyset, 0^\emptyset)), (b, ((0,3)^\emptyset, 5^b))\}, \\
\mathsf{Follow}(\alpha, (0^\emptyset, 5^b)) &= \{(c, (0^\emptyset, 6^b)), (c, ((0,3)^\emptyset, 5^b))\}, \\
\mathsf{Follow}(\alpha, ((1,4)^\emptyset, 0^\emptyset)) &= \{(b, ((2,4)^b, 0^\emptyset)), (b, ((1,4)^\emptyset, 5^b)), (b, ((2,4)^\emptyset, 5^\emptyset)), \\
&\quad (c, ((1,3)^\emptyset, 0^\emptyset))\}, \\
\mathsf{Follow}(\alpha, ((0,3)^\emptyset, 5^b)) &= \{(a, ((1,4)^\emptyset, 5^b)), (c, ((0,3)^\emptyset, 6^b))\}, \\
\mathsf{Follow}(\alpha, (0^\emptyset, 6^b)) &= \{(b, (0^\emptyset, 5^b)), (c, ((0,3)^\emptyset, 6^b))\},
\end{aligned}$$

$$
\begin{aligned}
\mathsf{Follow}(\alpha, ((2,4)^{\emptyset}, 5^{\emptyset})) &= \{(c, ((2,3)^{\emptyset}, 5^{\emptyset})), (c, ((2,4)^{\emptyset}, 6^{\emptyset}))\},\\
\mathsf{Follow}(\alpha, ((1,3)^{\emptyset}, 0^{\emptyset})) &= \{(b, ((2,3)^{b}, 0^{\emptyset})), (b, ((1,3)^{\emptyset}, 5^{b})), (b, ((2,3)^{\emptyset}, 5^{\emptyset}))\},\\
\mathsf{Follow}(\alpha, ((2,4)^{b}, 0^{\emptyset})) &= \{(c, ((2,3)^{b}, 0^{\emptyset}))\},\\
\mathsf{Follow}(\alpha, ((1,4)^{\emptyset}, 5^{b})) &= \{(c, ((1,3)^{\emptyset}, 5^{b})), (c, ((1,4)^{\emptyset}, 6^{b}))\},\\
&\vdots
\end{aligned}
$$

For a set $S \subseteq \Sigma_\alpha \times \mathsf{Loc}(\alpha)$ and $\sigma \in \Sigma_\alpha$, let $\mathsf{Select}(S, \sigma) = \{\ p \mid (\sigma, p) \in S\ \}$. The *position automaton* for $\alpha \in \mathrm{RE}(\|)$ is

$$\mathcal{A}_{\mathrm{POS}}(\alpha) = \langle \mathsf{Loc}_0(\alpha), \Sigma, \delta_{\mathrm{POS}}, 0, \mathsf{Last}_0(\alpha)\rangle,$$

where $\delta_{\mathrm{POS}}(p, \sigma) = \mathsf{Select}(\mathsf{Follow}(\alpha, p), \sigma)$, for $p \in \mathsf{Loc}_0(\alpha), \sigma \in \Sigma$. The correctness of this construction follows from the following two lemmas, which are proved in the appendix.

Lemma 2. *Given $\alpha \in \mathrm{RE}(\|)$, if $x = \sigma_1 \cdots \sigma_n \in \mathcal{L}(\alpha)$ $(n \geq 0)$, then there is a sequence $p_0, p_1, \ldots, p_n \in \mathsf{Loc}_0(\alpha)$, such that $p_0 = 0$ and*

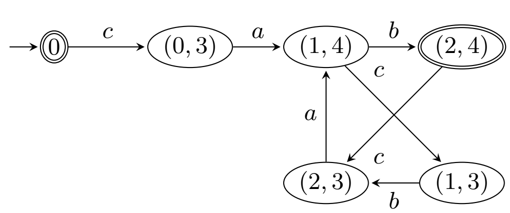
$$(\sigma_i, p_i) \in \mathsf{Follow}(\alpha, p_{i-1}) \quad (1 \leq i \leq n) \qquad \text{and} \qquad p_n \in \mathsf{Last}_0(\alpha).$$

Lemma 3. *Given $\alpha \in \mathrm{RE}(\|)$, if there are $r_0 = 0, r_1, \ldots, r_n \in \mathsf{Loc}_0(\alpha)$ and $\sigma_1, \ldots, \sigma_n \in \Sigma$ $(n \geq 0)$ such that (1) $(\sigma_i, r_i) \in \mathsf{Follow}(\alpha, r_{i-1})$, $1 \leq i \leq n$, and (2) $r_n \in \mathsf{Last}_0(\alpha)$, then $\sigma_1 \cdots \sigma_n \in \mathcal{L}(\alpha)$.*

As a consequence, we have the following proposition stating the correctness of $\mathcal{A}_{\mathrm{POS}}$.

Proposition 4. $\mathcal{L}(\mathcal{A}_{\mathrm{POS}}(\alpha)) = \mathcal{L}(\alpha)$.

Example 8. *Consider $\alpha = (ab)^{\star\,\mathsf{s}}\|_{\{a\}}\,(ca)^\star$ with $\overline{\alpha} = (a_1b_2)^{\star\,\mathsf{s}}\|_{\{a\}}\,(c_3a_4)^\star$. The sets $\mathsf{First}(\alpha)$ and $\mathsf{Follow}(\alpha)$ have been computed in Example 5 and Example 7, respectively. Note, that the only state attainable by a transition by letter a, is $(1,4)$ corresponding to the obligatory synchronisation of a_1 and a_4. The position automaton $\mathcal{A}_{\mathrm{POS}}(\alpha)$ is*



4. $\mathcal{A}_{\mathrm{PD}}$ as a quotient of $\mathcal{A}_{\mathrm{POS}}$

In this section, we relate the partial derivative automaton, defined by Sulzmann and Thiemann [3], and the position automaton in Section 3. For standard regular expressions extended with the shuffle operator $\sqcup\!\sqcup$, the former has been shown to be a quotient of the latter. Here, we extend this result to the synchronised shuffle operators. Sulzmann and Thiemann [3] defined the set of partial derivatives of expressions with the general shuffle operator w.r.t. an alphabet symbol $\sigma \in \Sigma$. However, since the parameters of the general operator change in each step of derivation, it is not straightforward to express the set of partial derivatives w.r.t. a word w in terms of the partial derivatives w.r.t. subwords of w. In Lemmas 5 and 7 we obtain explicit expressions for those sets for expressions containing synchronised shuffle operators. These are crucial to show that $\mathcal{A}_{\mathrm{PD}}(\alpha)$ is a quotient of $\mathcal{A}_{\mathrm{POS}}(\alpha)$, cf. Proposition 12. The proof of Proposition 12 follows the one in [5, 6]. To each location $p \in \mathsf{Loc}(\alpha)$ we associate a unique partial derivative of $\overline{\alpha}$. This expression is denoted by $\mathsf{c}(\overline{\alpha}, p)$ and called the c-*continuation of* p *in* α. Then, the partial derivative automaton $\mathcal{A}_{\mathrm{PD}}(\alpha)$ is obtained by merging in $\mathcal{A}_{\mathrm{POS}}(\alpha)$ states (locations) p and q, such that $\overline{\mathsf{c}(\overline{\alpha}, p)} = \overline{\mathsf{c}(\overline{\alpha}, q)}$. During the computation of the set of partial derivatives of an expression of the form $\alpha_1 \,{}^{\mathsf{w}}\!\|_\Gamma\, \alpha_2$ w.r.t. a word $w \in \Sigma^\star$, one has to consider all words $u_i \in \mathcal{L}(\alpha_i)$, $i = 1, 2$, such that $w \in u_1 \,{}^{\mathsf{w}}\!\|_\Gamma\, u_2$. At each step of the derivation, it is necessary to remember which symbols in Γ have been read solely from u_1 and also from u_2 since the last synchronisation happened. To this end, we consider expressions with the operator $\|_{(\Gamma,\Delta,\Lambda)}$, where this information is respectively stored in two additional parameters Δ and Λ, with $\Delta, \Lambda \subseteq \Gamma \subseteq \Sigma$. We denote the set of regular expressions with shuffle operators $\sqcup\!\sqcup$, ${}^{\mathsf{s}}\!\|_\Gamma$, ${}^{\mathsf{a}}\!\|_\Gamma$, and $\|_{(\Gamma,\Delta,\Lambda)}$ by $\mathrm{RE}_{\mathsf{gen}}(\|)$. In $\mathrm{RE}_{\mathsf{gen}}(\|)$ the natural counterpart of $\alpha_1 \,{}^{\mathsf{w}}\!\|_\Gamma\, \alpha_2$ is $\alpha_1 \|_{(\Gamma,\emptyset,\emptyset)} \alpha_2$. As such, we will from now on regard the set of expressions $\mathrm{RE}(\|)$ as a subset of $\mathrm{RE}_{\mathsf{gen}}(\|)$, and use ${}^{\mathsf{w}}\!\|_\Gamma$ and $\|_{(\Gamma,\emptyset,\emptyset)}$ interchangeably in $\mathrm{RE}_{\mathsf{gen}}(\|)$.

Since we don't consider the general shuffle operator [3], we define $\partial_\sigma(\alpha)$, for expressions containing the operators ${}^{\mathsf{s}}\!\|_\Gamma$, ${}^{\mathsf{a}}\!\|_\Gamma$, and $\|_{(\Gamma,\Delta,\Lambda)}$ separately as follows:

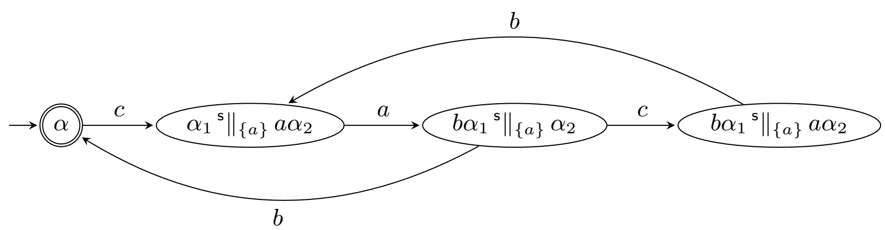
$$
\begin{aligned}
&\partial_\sigma(\emptyset) = \partial_\sigma(\varepsilon) = \emptyset,\ \partial_\sigma(\sigma') = \begin{cases} \{\varepsilon\} \text{ if } \sigma = \sigma', \\ \emptyset \text{ otherwise}, \end{cases} \quad \partial_\sigma(\alpha^\star) = \partial_\sigma(\alpha)\alpha^\star, \\
&\partial_\sigma(\alpha + \beta) = \partial_\sigma(\alpha) \cup \partial_\sigma(\beta), \quad \partial_\sigma(\alpha\beta) = \partial_\sigma(\alpha)\beta \cup \varepsilon(\alpha)\partial_\sigma(\beta), \\
&\partial_\sigma(\alpha \,{}^{\mathsf{s}}\!\|_\Gamma\, \beta) = \begin{cases} \partial_\sigma(\alpha) \,{}^{\mathsf{s}}\!\|_\Gamma\, \partial_\sigma(\beta), & \text{if } \sigma \in \Gamma, \\ \partial_\sigma(\alpha) \,{}^{\mathsf{s}}\!\|_\Gamma\, \{\beta\} \cup \{\alpha\} \,{}^{\mathsf{s}}\!\|_\Gamma\, \partial_\sigma(\beta), & \text{otherwise}, \end{cases} \\
&\partial_\sigma(\alpha \,{}^{\mathsf{a}}\!\|_\Gamma\, \beta) = (\sigma \in \Gamma)\partial_\sigma(\alpha) \,{}^{\mathsf{a}}\!\|_\Gamma\, \partial_\sigma(\beta) \cup \partial_\sigma(\alpha) \,{}^{\mathsf{a}}\!\|_\Gamma\, \{\beta\} \cup \{\alpha\} \,{}^{\mathsf{a}}\!\|_\Gamma\, \partial_\sigma(\beta), \\
&\partial_\sigma(\alpha \|_{(\Gamma,\Delta,\Lambda)} \beta) = \begin{cases} \partial_\sigma(\alpha) \|_{(\Gamma,\Delta,\Lambda)} \{\beta\} \cup \{\alpha\} \|_{(\Gamma,\Delta,\Lambda)} \partial_\sigma(\beta), & \text{if } \sigma \notin \Gamma, \\ \partial_\sigma(\alpha) \|_{(\Gamma,\emptyset,\emptyset)} \partial_\sigma(\beta) & \\ \qquad \cup (\sigma \notin \Lambda)\partial_\sigma(\alpha) \|_{(\Gamma,\Delta\cup\{\sigma\},\Lambda)} \{\beta\} & \\ \qquad \cup (\sigma \notin \Delta)\{\alpha\} \|_{(\Gamma,\Delta,\Lambda\cup\{\sigma\})} \partial_\sigma(\beta), & \text{otherwise}, \end{cases}
\end{aligned}
\tag{2}
$$

where, for any $S, T \subseteq \mathrm{RE}_{\mathsf{gen}}(\|) \setminus \{\emptyset\}$ and $\circ \in \{\, {}^{\mathsf{s}}\|_{\Gamma}\,, {}^{\mathsf{a}}\|_{\Gamma}\,, \|_{(\Gamma,\Delta,\Lambda)}\}$, we define $S \circ T = \{\, \alpha \circ \beta \mid \alpha \in S \wedge \beta \in T \,\}$, and for $\alpha' \neq \varepsilon$, $S\alpha' = \{\, \alpha\alpha' \mid \alpha \in S \wedge \alpha \neq \varepsilon \,\} \cup (\varepsilon \in S)\{\alpha'\}$. Moreover, $\mathsf{b}S = S$ if condition b is true, and $\mathsf{b}S = \emptyset$ otherwise. As usual, the set of partial derivatives of $\alpha \in \mathrm{RE}_{\mathsf{gen}}(\|)$ w.r.t. a word $w \in \Sigma^\star$ is inductively defined by $\partial_\varepsilon(\alpha) = \{\alpha\}$ and $\partial_{\sigma w}(\alpha) = \partial_w(\partial_\sigma(\alpha))$, where, given a set $S \subseteq \mathrm{RE}_{\mathsf{gen}}(\|)$, $\partial_\sigma(S) = \bigcup_{\alpha \in S} \partial_\sigma(\alpha)$. Moreover, $\mathcal{L}(\partial_w(\alpha)) = \{\, w_1 \mid ww_1 \in \mathcal{L}(\alpha) \,\}$. Let $\partial(\alpha) = \bigcup_{w \in \Sigma^\star} \partial_w(\alpha)$, and $\partial^+(\alpha) = \bigcup_{w \in \Sigma^+} \partial_w(\alpha)$. The partial derivative automaton of $\alpha \in \mathrm{RE}_{\mathsf{gen}}(\|)$ is

$$\mathcal{A}_{\mathrm{PD}}(\alpha) = \langle \partial(\alpha), \Sigma, \{\alpha\}, \delta_{\mathrm{PD}}, F_{\mathrm{PD}} \rangle,$$

with $F_{\mathrm{PD}} = \{\, \beta \in \partial(\alpha) \mid \varepsilon(\beta) = \varepsilon \,\}$ and $\delta_{\mathrm{PD}}(\beta, \sigma) = \partial_\sigma(\beta)$, for $\beta \in \partial(\alpha)$, $\sigma \in \Sigma$.

Example 9. *The partial derivative automaton for $\alpha = (ab)^\star \,{}^{\mathsf{s}}\|_{\{a\}}\, (ca)^\star$, where $\alpha_1 = (ab)^\star$ and $\alpha_2 = (ca)^\star$, from Example 8, is depicted below.*



Note that this automaton can be obtained from the position automaton $\mathcal{A}_{\mathrm{POS}}(\alpha)$, represented in Example 8, by merging the states (labelled with) 0 and $(2,4)$, as well as the states $(0,3)$ and $(2,3)$.

The following lemmas give explicit expressions for partial derivatives of $\alpha \in \mathrm{RE}_{\mathsf{gen}}(\|)$ w.r.t. a word. For the $\|_{(\Gamma,\Delta,\Lambda)}$ operator the situation differs slightly from the other two operators, so it is dealt with separately. Given $\Gamma \subseteq \Sigma$ and $w \in \Sigma^\star$, we compute the set of pairs of words u and v, such that $w \in u \circ v$, denoted by $\mathsf{p}(\circ, w)$, for $\circ \in \{\, {}^{\mathsf{s}}\|_{\Gamma}\,, {}^{\mathsf{a}}\|_{\Gamma}\,\}$. The definition of p for $\circ = {}^{\mathsf{s}}\|_{\Gamma}$ is as follows:

$$\begin{aligned}
\mathsf{p}({}^{\mathsf{s}}\|_{\Gamma}\,, \varepsilon) &= \{(\varepsilon, \varepsilon)\},\\
\mathsf{p}({}^{\mathsf{s}}\|_{\Gamma}\,, \sigma w) &= (\sigma \in \Gamma)\{\, (\sigma u, \sigma v) \mid (u, v) \in \mathsf{p}({}^{\mathsf{s}}\|_{\Gamma}\,, w) \,\}\\
&\quad \cup (\sigma \notin \Gamma)\{\, (\sigma u, v) \mid (u, v) \in \mathsf{p}({}^{\mathsf{s}}\|_{\Gamma}\,, w) \,\}\\
&\quad \cup (\sigma \notin \Gamma)\{\, (u, \sigma v) \mid (u, v) \in \mathsf{p}({}^{\mathsf{s}}\|_{\Gamma}\,, w) \,\}.
\end{aligned}$$

For $\circ = {}^{\mathsf{a}}\|_{\Gamma}$ we have the following:

$$\begin{aligned}
\mathsf{p}({}^{\mathsf{a}}\|_{\Gamma}\,, \varepsilon) &= \{(\varepsilon, \varepsilon)\},\\
\mathsf{p}({}^{\mathsf{a}}\|_{\Gamma}\,, \sigma w) &= (\sigma \in \Gamma)\{\, (\sigma u, \sigma v) \mid (u, v) \in \mathsf{p}({}^{\mathsf{a}}\|_{\Gamma}\,, w) \,\}\\
&\quad \cup \{\, (\sigma u, v) \mid (u, v) \in \mathsf{p}({}^{\mathsf{a}}\|_{\Gamma}\,, w) \,\}\\
&\quad \cup \{\, (u, \sigma v) \mid (u, v) \in \mathsf{p}({}^{\mathsf{a}}\|_{\Gamma}\,, w) \,\}.
\end{aligned}$$

Lemma 5. *For $\alpha, \beta \in \mathrm{RE}_{\mathsf{gen}}(\|)$, $w \in \Sigma^+$, and $\circ \in \{ {}^{\mathsf{s}}\|_\Gamma , {}^{\mathsf{a}}\|_\Gamma \}$, the following equality holds*

$$\partial_w(\alpha \circ \beta) = \bigcup_{(u,v) \in \mathsf{p}(\circ, w)} \partial_u(\alpha) \circ \partial_v(\beta).$$

Proof. By induction on the length of w. Let $w = \sigma \in \Gamma$. If $\circ$ is ${}^{\mathsf{s}}\|_\Gamma$, then $\mathsf{p}({}^{\mathsf{s}}\|_\Gamma, \sigma) = \{(\sigma, \sigma)\}$, and by definition, $\partial_\sigma(\alpha \,{}^{\mathsf{s}}\|_\Gamma\, \beta) = \partial_\sigma(\alpha)\,{}^{\mathsf{s}}\|_\Gamma\, \partial_\sigma(\beta)$. If $\circ$ is ${}^{\mathsf{a}}\|_\Gamma$, then $\mathsf{p}({}^{\mathsf{a}}\|_\Gamma, \sigma) = \{(\sigma, \sigma), (\sigma, \varepsilon), (\varepsilon, \sigma)\}$. By definition, $\partial_\sigma(\alpha \,{}^{\mathsf{a}}\|_\Gamma\, \beta) = \partial_\sigma(\alpha)\,{}^{\mathsf{a}}\|_\Gamma\, \partial_\sigma(\beta) \;\cup\; \partial_\sigma(\alpha)\,{}^{\mathsf{a}}\|_\Gamma\, \{\beta\} \;\cup\; \{\alpha\}\,{}^{\mathsf{a}}\|_\Gamma\, \partial_\sigma(\beta)$. Finally, if $w = \sigma \notin \Gamma$, then $\mathsf{p}(\circ, \sigma) = \{(\sigma, \varepsilon), (\varepsilon, \sigma)\}$, and by definition, $\partial_\sigma(\alpha \circ \beta) = \partial_\sigma(\alpha) \circ \{\beta\} \cup \{\alpha\} \circ \partial_\sigma(\beta) = \partial_\sigma(\alpha) \circ \partial_\varepsilon(\beta) \cup \partial_\varepsilon(\alpha) \circ \partial_\sigma(\beta)$.

Now, consider $w = \sigma w'$. We have

$$\begin{aligned}
\partial_{\sigma w'}(\alpha \circ \beta) = \partial_{w'}(\partial_\sigma(\alpha \circ \beta)) &= \bigcup_{\substack{(u',v') \in \mathsf{p}(\circ, w') \\ \alpha' \circ \beta' \in \partial_\sigma(\alpha \circ \beta)}} \partial_{u'}(\alpha') \circ \partial_{v'}(\beta') \\
&= \bigcup_{\substack{(u',v') \in \mathsf{p}(\circ, w') \\ (\alpha' \circ \beta') \in X}} \partial_{u'}(\alpha') \circ \partial_{v'}(\beta') \\
&= \bigcup_{(u',v') \in \mathsf{p}(\circ, w')} Y = \bigcup_{(u,v) \in \mathsf{p}(\circ, \sigma w')} \partial_u(\alpha) \circ \partial_v(\beta),
\end{aligned}$$

where

- for $\sigma \in \Gamma$ and $\circ = {}^{\mathsf{s}}\|_\Gamma$, $X = \partial_\sigma(\alpha)\,{}^{\mathsf{s}}\|_\Gamma\, \partial_\sigma(\beta)$ and $Y = \partial_{\sigma u'}(\alpha)\,{}^{\mathsf{s}}\|_\Gamma\, \partial_{\sigma v'}(\beta)$;
- for $\sigma \in \Gamma$ and $\circ = {}^{\mathsf{a}}\|_\Gamma$,

  $$\begin{aligned}
  X &= \partial_\sigma(\alpha)\,{}^{\mathsf{a}}\|_\Gamma\, \partial_\sigma(\beta) \cup \partial_\sigma(\alpha)\,{}^{\mathsf{a}}\|_\Gamma\, \{\beta\} \cup \{\alpha\}\,{}^{\mathsf{a}}\|_\Gamma\, \partial_\sigma(\beta), \text{ and} \\
  Y &= \partial_{\sigma u'}(\alpha)\,{}^{\mathsf{a}}\|_\Gamma\, \partial_{\sigma v'}(\beta) \cup \partial_{\sigma u'}(\alpha)\,{}^{\mathsf{a}}\|_\Gamma\, \partial_{v'}(\beta) \cup \partial_{u'}(\alpha)\,{}^{\mathsf{a}}\|_\Gamma\, \partial_{\sigma v'}(\beta);
  \end{aligned}$$

- and for $\sigma \notin \Gamma$,

  $$X = \partial_\sigma(\alpha) \circ \{\beta\} \cup \{\alpha\} \circ \partial_\sigma(\beta) \text{ and } Y = \partial_{\sigma u'}(\alpha) \circ \partial_{v'}(\beta) \cup \partial_{u'}(\alpha) \circ \partial_{\sigma v'}(\beta).$$

$\square$

For ${}^{\mathsf{w}}\|_\Gamma$ we also need to consider the generalised operator $\|_{(\Gamma,\Delta,\Lambda)}$, where $\Delta, \Lambda \subseteq \Gamma \subseteq \Sigma$. Given $w \in \Sigma^\star$, we want to compute the set of words u and v, such that $w \in u \,\|_{(\Gamma,\Delta,\Lambda)} v$. To this end we define the function p as follows:

$$\begin{aligned}
\mathsf{p}(\,\|_{(\Gamma,\Delta,\Lambda)}, \varepsilon) &= \{(\Delta, \Lambda, \varepsilon, \varepsilon)\} \\
\mathsf{p}(\,\|_{(\Gamma,\Delta,\Lambda)}, \sigma w) &= (\sigma \in \Gamma)\{\, (\Delta_u, \Lambda_v, \sigma u, \sigma v) \mid (\Delta_u, \Lambda_v, u, v) \in \mathsf{p}(\,\|_{(\Gamma,\emptyset,\emptyset)}, w) \,\} \\
&\cup (\sigma \in \Gamma \setminus \Lambda)\{\, (\Delta_u, \Lambda_v, \sigma u, v) \mid (\Delta_u, \Lambda_v, u, v) \in \mathsf{p}(\,\|_{(\Gamma,\Delta\cup\sigma,\Lambda)}, w) \,\} \\
&\cup (\sigma \in \Gamma \setminus \Delta)\{\, (\Delta_u, \Lambda_v, u, \sigma v) \mid (\Delta_u, \Lambda_v, u, v) \in \mathsf{p}(\,\|_{(\Gamma,\Delta,\Lambda\cup\sigma)}, w) \,\} \\
&\cup (\sigma \notin \Gamma)\{\, (\Delta_u, \Lambda_v, \sigma u, v) \mid (\Delta_u, \Lambda_v, u, v) \in \mathsf{p}(\,\|_{(\Gamma,\Delta,\Lambda)}, w) \,\} \\
&\cup (\sigma \notin \Gamma)\{\, (\Delta_u, \Lambda_v, u, \sigma v) \mid (\Delta_u, \Lambda_v, u, v) \in \mathsf{p}(\,\|_{(\Gamma,\Delta,\Lambda)}, w) \,\}.
\end{aligned}$$

Lemma 6. *For $w \in \Sigma^\star$, $\Delta, \Lambda \subseteq \Gamma$ such that $\Delta \cap \Lambda = \emptyset$, we have*

$$(\Delta_u, \Lambda_v, u, v) \in \mathsf{p}(\,\|_{(\Gamma,\Delta,\Lambda)}, w) \quad \textit{iff} \quad w \in u \,\|_{(\Gamma,\Delta,\Lambda)} v.$$

Example 10. *For $\Gamma = \Sigma = \{a\}$ and $w = aaa$, we have*

$$\begin{aligned}\mathsf{p}(\,\|_{(\Gamma,\emptyset,\emptyset)}, w) = \{&(\emptyset, \emptyset, aaa, aaa), (\emptyset, \emptyset, aaa, aa), (\emptyset, \emptyset, aa, aaa),\\ &(\emptyset, \emptyset, aaa, a), (\emptyset, \emptyset, a, aaa), (\{a\}, \emptyset, aaa, aa),\\ &(\{a\}, \emptyset, aaa, a), (\{a\}, \emptyset, aaa, \varepsilon), (\{a\}, \emptyset, aa, aa),\\ &(\emptyset, \{a\}, aa, aaa), (\emptyset, \{a\}, a, aaa), (\emptyset, \{a\}, \varepsilon, aaa), (\emptyset, \{a\}, aa, aa)\}.\end{aligned}$$

Lemma 7. *For $\alpha, \beta \in \mathrm{RE}_{\mathsf{gen}}(\|)$ and $w \in \Sigma^+$, the following equality holds*

$$\partial_w(\alpha \,\|_{(\Gamma,\Delta,\Lambda)} \beta) = \bigcup_{(\Delta_u,\Lambda_v,u,v) \in \mathsf{p}(\,\|_{(\Gamma,\Delta,\Lambda)}, w)} \partial_u(\alpha) \,\|_{(\Gamma,\Delta_u,\Lambda_v)} \partial_v(\beta).$$

Proof. By induction on the length of w. For $w = \sigma \notin \Gamma$, by definition,

$$\partial_\sigma(\alpha \,\|_{(\Gamma,\Delta,\Lambda)} \beta) = \partial_\sigma(\alpha) \,\|_{(\Gamma,\Delta,\Lambda)} \{\beta\} \cup \{\alpha\} \,\|_{(\Gamma,\Delta,\Lambda)} \partial_\sigma(\beta).$$

Furthermore,

$$\mathsf{p}(\,\|_{(\Gamma,\Delta,\Lambda)}, \sigma) = \{(\Delta, \Lambda, \sigma, \varepsilon), (\Delta, \Lambda, \varepsilon, \sigma)\}.$$

Thus, the result holds. For $w = \sigma \in \Gamma$, we only consider the case $\sigma \in \Delta$ and $\sigma \notin \Lambda$. The remaining cases are analogous. In this case, $\mathsf{p}(\,\|_{(\Gamma,\Delta,\Lambda)}, \sigma) = \{(\Delta, \Lambda, \sigma, \sigma), (\Delta \cup \{\sigma\}, \Lambda, \sigma, \varepsilon)\}$. Also, by definition we have $\partial_\sigma(\alpha \,\|_{(\Gamma,\Delta,\Lambda)} \beta) = \partial_\sigma(\alpha) \,\|_{(\Gamma,\emptyset,\emptyset)} \partial_\sigma(\beta) \cup \partial_\sigma(\alpha) \,\|_{(\Gamma,\Delta\cup\{\sigma\},\Lambda)} \{\beta\}$.

Now, consider the word σw, with $|w| \geq 1$ and $\sigma \notin \Gamma$. One has

$$\begin{aligned}
\partial_{\sigma w}(\alpha \,\|_{(\Gamma,\Delta,\Lambda)} \beta) &= \partial_w(\partial_\sigma(\alpha \,\|_{(\Gamma,\Delta,\Lambda)} \beta))\\
&= \partial_w\left(\partial_\sigma(\alpha) \,\|_{(\Gamma,\Delta,\Lambda)} \{\beta\} \cup \{\alpha\} \,\|_{(\Gamma,\Delta,\Lambda)} \partial_\sigma(\beta)\right)\\
&= \bigcup_{(\Delta_u,\Lambda_v,u,v) \in \mathsf{p}(\,\|_{(\Gamma,\Delta,\Lambda)}, w)} \partial_u(\partial_\sigma(\alpha)) \,\|_{(\Gamma,\Delta_u,\Lambda_v)} \partial_v(\beta)\\
&\cup \bigcup_{(\Delta_u,\Lambda_v,u,v) \in \mathsf{p}(\,\|_{(\Gamma,\Delta,\Lambda)}, w)} \partial_u(\alpha) \,\|_{(\Gamma,\Delta_u,\Lambda_v)} \partial_v(\partial_\sigma(\beta))\\
&= \bigcup_{(\Delta_u,\Lambda_v,\sigma u,v) \in \mathsf{p}(\,\|_{(\Gamma,\Delta,\Lambda)}, \sigma w)} \partial_{\sigma u}(\alpha) \,\|_{(\Gamma,\Delta_u,\Lambda_v)} \partial_v(\beta)\\
&\cup \bigcup_{(\Delta_u,\Lambda_v,u,\sigma v) \in \mathsf{p}(\,\|_{(\Gamma,\Delta,\Lambda)}, \sigma w)} \partial_u(\alpha) \,\|_{(\Gamma,\Delta_u,\Lambda_v)} \partial_{\sigma v}(\beta)\\
&= \bigcup_{(\Delta_{u'},\Lambda_{v'},u',v') \in \mathsf{p}(\,\|_{(\Gamma,\Delta,\Lambda)}, \sigma w)} \partial_{u'}(\alpha) \,\|_{(\Gamma,\Delta_{u'},\Lambda_{v'})} \partial_{v'}(\beta).
\end{aligned}$$

Finally, consider σw, with $|w| \geq 1$ and $\sigma \in \Gamma$. Again, we show the result for the case $\sigma \in \Delta$ and $\sigma \notin \Lambda$, since the remaining cases are shown analogously. Then,

$$\begin{aligned}
\partial_{\sigma w}(\alpha \,\|_{(\Gamma,\Delta,\Lambda)} \beta) &= \partial_w(\partial_\sigma(\alpha \,\|_{(\Gamma,\Delta,\Lambda)} \beta)) \\
&= \partial_w\left(\partial_\sigma(\alpha) \,\|_{(\Gamma,\emptyset,\emptyset)} \partial_\sigma(\beta) \cup \partial_\sigma(\alpha) \,\|_{(\Gamma,\Delta\cup\{\sigma\},\Lambda)} \{\beta\}\right) \\
&= \bigcup_{(\Delta_u,\Lambda_v,u,v)\in \mathsf{p}(\,\|_{(\Gamma,\emptyset,\emptyset)},w)} \partial_u(\partial_\sigma(\alpha)) \,\|_{(\Gamma,\Delta_u,\Lambda_v)} \partial_v(\partial_\sigma(\beta)) \\
&\cup \bigcup_{(\Delta_u,\Lambda_v,u,v)\in \mathsf{p}(\,\|_{(\Gamma,\Delta\cup\{\sigma\},\Lambda)},w)} \partial_u(\partial_\sigma(\alpha)) \,\|_{(\Gamma,\Delta_u,\Lambda_v)} \partial_v(\beta) \\
&= \bigcup_{(\Delta_u,\Lambda_v,\sigma u,\sigma v)\in \mathsf{p}(\,\|_{(\Gamma,\emptyset,\emptyset)},\sigma w)} \partial_{\sigma u}(\alpha) \,\|_{(\Gamma,\Delta_u,\Lambda_v)} \partial_{\sigma v}(\beta) \\
&\cup \bigcup_{(\Delta_u,\Lambda_v,\sigma u,v)\in \mathsf{p}(\,\|_{(\Gamma,\Delta\cup\{\sigma\},\Lambda)},\sigma w)} \partial_{\sigma u}(\alpha) \,\|_{(\Gamma,\Delta_u,\Lambda_v)} \partial_v(\beta) \\
&= \bigcup_{(\Delta_{u'},\Lambda_{v'},u',v')\in \mathsf{p}(\,\|_{(\Gamma,\Delta,\Lambda)},\sigma w)} \partial_{u'}(\alpha) \,\|_{(\Gamma,\Delta_{u'},\Lambda_{v'})} \partial_{v'}(\beta).
\end{aligned}$$

$\square$

Following [10], Broda et al. [5, 6] showed for regular expressions with the shuffle operator $\sqcup\!\sqcup$, given a location $p \in \mathsf{Loc}(\alpha)$, that there exists a unique expression $\mathsf{c}(\overline{\alpha}, p)$, called the c-continuation of p in $\overline{\alpha}$, such that for all $w\sigma_i \in \Sigma^+_{\overline{\alpha}}$ with $i \in \mathsf{lp}(p)$, either $\partial_{w\sigma_i}(\overline{\alpha}) = \emptyset$, or $\partial_{w\sigma_i}(\overline{\alpha}) = \{\mathsf{c}(\overline{\alpha}, p)\}$. Whenever synchronisation is present, this result doesn't hold anymore, since differently marked symbols σ_i and σ_j, with $\overline{\sigma_i} = \overline{\sigma_j}$, are synchronised. However, one can still extend the notion of c-continuation to expressions with synchronised shuffle operators, such that the set of c-continuations relates to the set $\partial^+(\alpha)$. Let $\mathsf{c}(\overline{\alpha}, 0) = \overline{\alpha}$. For $\alpha \in \mathrm{RE}(\|)$, the c-*continuation*$\mathsf{c}(\overline{\alpha}, p)$ of a location $p \in \mathsf{Loc}(\alpha)$ in $\overline{\alpha}$ is defined as follows:

$$\begin{aligned}
\mathsf{c}(\sigma_i, i) &= \varepsilon, \quad \mathsf{c}(\alpha^\star, p) = \mathsf{c}(\alpha, p)\alpha^\star, \\
\mathsf{c}(\alpha_1 + \alpha_2, p) &= \begin{cases} \mathsf{c}(\alpha_1, p), & \text{if } p \in \mathsf{Loc}(\alpha_1), \\ \mathsf{c}(\alpha_2, p), & \text{if } p \in \mathsf{Loc}(\alpha_2), \end{cases} \\
\mathsf{c}(\alpha_1\alpha_2, p) &= \begin{cases} \mathsf{c}(\alpha_1, p)\alpha_2, & \text{if } p \in \mathsf{Loc}(\alpha_1), \\ \mathsf{c}(\alpha_2, p), & \text{if } p \in \mathsf{Loc}(\alpha_2), \end{cases} \\
\mathsf{c}(\alpha_1 \,{}^{\mathsf{s}}\|_\Gamma\, \alpha_2, (p_1, p_2)) &= \mathsf{c}(\alpha_1, p_1) \,{}^{\mathsf{s}}\|_\Gamma\, \mathsf{c}(\alpha_2, p_2), \\
\mathsf{c}(\alpha_1 \,{}^{\mathsf{a}}\|_\Gamma\, \alpha_2, (p_1, p_2)) &= \mathsf{c}(\alpha_1, p_1) \,{}^{\mathsf{a}}\|_\Gamma\, \mathsf{c}(\alpha_2, p_2), \\
\mathsf{c}(\alpha_1 \,{}^{\mathsf{w}}\|_\Gamma\, \alpha_2, (p_1^\Delta, p_2^\Lambda)) &= \mathsf{c}(\alpha_1, p_1) \,\|_{(\Gamma,\Delta,\Lambda)} \mathsf{c}(\alpha_2, p_2).
\end{aligned}$$

Example 11. *Consider* $\alpha = (ab)^\star \,{}^{\mathsf{s}}\|_{\{a\}}\, (ca)^\star$ *with* $\overline{\alpha} = (a_1b_2)^\star \,{}^{\mathsf{s}}\|_{\{a\}}\, (c_3a_4)^\star$ *from Example 8. For* $p \in \mathsf{Loc}_0(\alpha)$*, we have* $\mathsf{c}(\overline{\alpha}, 0) = \mathsf{c}(\overline{\alpha}, (2, 4)) = \overline{\alpha}$*,* $\mathsf{c}(\overline{\alpha}, (0, 3)) = \mathsf{c}(\overline{\alpha}, (2, 3)) = (a_1b_2)^\star \,{}^{\mathsf{s}}\|_{\{a\}}\, a_4(c_3a_4)^\star$*,* $\mathsf{c}(\overline{\alpha}, (1, 3)) = b_2(a_1b_2)^\star \,{}^{\mathsf{s}}\|_{\{a\}}\, a_4(c_3a_4)^\star$*, and* $\mathsf{c}(\overline{\alpha}, (1, 4)) = b_2(a_1b_2)^\star \,{}^{\mathsf{s}}\|_{\{a\}}\, (c_3a_4)^\star$*.*

We now relate the set of c-continuations with $\partial^+(\alpha)$ and, furthermore, characterise the ones that correspond to final states. For readability, we denote $\overline{\mathsf{c}(\overline{\alpha}, p)}$ by $\mathsf{d}(\alpha, p)$, for any location $p \in \mathsf{Loc}_0(\alpha)$.

Lemma 8. *For $\alpha \in \mathrm{RE}(\|)$, $\partial^+(\alpha) \subseteq \{\, \mathsf{d}(\alpha, p) \mid p \in \mathsf{Loc}(\alpha)\,\}$.*

Proof. We show by structural induction on α that for every $w \in \Sigma^+$, if $\beta \in \partial_w(\alpha)$, then there is some $p \in \mathsf{Loc}(\alpha)$, such that $\beta = \mathsf{d}(\alpha, p)$. We only consider the cases ${}^{\mathsf{s}}\|_\Gamma$, ${}^{\mathsf{a}}\|_\Gamma$, and ${}^{\mathsf{w}}\|_\Gamma$ since for the remaining operators the result follows from the one for marked expressions in [5, 6, Lemma 8].

Let $\alpha = \alpha_1 \,{}^{\mathsf{s}}\|_\Gamma\, \alpha_2$, $w \in \Sigma^+$, and $\beta \in \partial_w(\alpha)$. By Lemma 5, there are words u, v, such that $w \in u \,{}^{\mathsf{s}}\|_\Gamma\, v$ and $\beta = \beta_1 \,{}^{\mathsf{s}}\|_\Gamma\, \beta_2 \in \partial_u(\alpha_1) \,{}^{\mathsf{s}}\|_\Gamma\, \partial_v(\alpha_2)$, i.e., $\beta_1 \in \partial_u(\alpha_1)$ and $\beta_2 \in \partial_v(\alpha_2)$. By induction, there exist $p_i \in \mathsf{Loc}(\alpha_i)$, for $i = 1, 2$, such that $\beta_i = \mathsf{d}(\alpha_i, p_i)$. Furthermore, $(p_1, p_2) \in \mathsf{Loc}(\alpha)$ and $\mathsf{d}(\alpha_1 \,{}^{\mathsf{s}}\|_\Gamma\, \alpha_2, (p_1, p_2)) = \mathsf{d}(\alpha_1, p_1) \,{}^{\mathsf{s}}\|_\Gamma\, \mathsf{d}(\alpha_2, p_2) = \beta_1 \,{}^{\mathsf{s}}\|_\Gamma\, \beta_2 = \beta$. The proof for the operator ${}^{\mathsf{a}}\|_\Gamma$ is identical, due to Lemma 5 and because continuations for operators ${}^{\mathsf{s}}\|_\Gamma$ and ${}^{\mathsf{a}}\|_\Gamma$ are defined in the same way.

Now, consider $\alpha = \alpha_1 \,{}^{\mathsf{w}}\|_\Gamma\, \alpha_2 = \alpha_1 \,\|_{(\Gamma,\emptyset,\emptyset)} \alpha_2$. Let $w \in \Sigma^+$ and $\beta \in \partial_w(\alpha)$. By Lemma 7, there is a tuple $(\Delta_u, \Lambda_v, u, v) \in \mathsf{p}(\,\|_{(\Gamma,\emptyset,\emptyset)}, w)$ such that $\beta = \beta_1 \,\|_{(\Gamma,\Delta_u,\Lambda_v)} \beta_2$, where $\beta_1 \in \partial_u(\alpha_1)$ and $\beta_2 \in \partial_v(\alpha_2)$. By induction, there exist $p_i \in \mathsf{Loc}(\alpha_i)$, for $i = 1, 2$, such that $\beta_i = \mathsf{d}(\alpha_i, p_i)$. Furthermore, $(p_1^{\Delta_u}, p_2^{\Lambda_v}) \in \mathsf{Loc}(\alpha)$ and

$$\mathsf{d}(\alpha, (p_1^{\Delta_u}, p_2^{\Delta_v})) = \mathsf{d}(\alpha_1, p_1) \,\|_{(\Gamma,\Delta_u,\Lambda_v)} \mathsf{d}(\alpha_2, p_2) = \beta.$$

$\square$

Lemma 9. *For $\alpha \in \mathrm{RE}(\|)$ and $p \in \mathsf{Loc}(\alpha)$, one has*

$$\varepsilon(\mathsf{d}(\alpha, p)) = \varepsilon \iff p \in \mathsf{Last}(\alpha).$$

Proof. By structural induction on α. $\square$

The next proposition relates derivatives of $\mathsf{d}(\alpha, p)$ with $\mathsf{Follow}(\alpha, p)$. The proof is in the appendix.

Proposition 10. *For $\alpha \in \mathrm{RE}(\|)$, $p \in \mathsf{Loc}_0(\alpha)$, and $\sigma \in \Sigma_\alpha$, one has*

$$\partial_\sigma(\mathsf{d}(\alpha, p)) = \{\, \mathsf{d}(\alpha, q) \mid (\sigma, q) \in \mathsf{Follow}(\alpha, p)\,\}.$$

Consider the equivalence relation $\equiv_c \subseteq \mathsf{Loc}_0(\alpha) \times \mathsf{Loc}_0(\alpha)$ defined by $p \equiv_c q$ iff $\mathsf{d}(\alpha, p) = \mathsf{d}(\alpha, q)$.

Lemma 11. *The relation $\equiv_c$ is right-invariant w.r.t. $\mathcal{A}_{\mathrm{POS}}(\alpha)$.*

Proof. Consider $p, q \in \mathsf{Loc}_0(\alpha)$ such that $p \equiv_c q$, i.e., $\mathsf{d}(\alpha, p) = \mathsf{d}(\alpha, q)$. By Lemma 9, we have $p \in \mathsf{Last}(\alpha)$ if and only if $q \in \mathsf{Last}(\alpha)$. Let $(\sigma, p') \in \mathsf{Follow}(\alpha, p)$ and consider $\beta = \mathsf{d}(\alpha, p')$. By Proposition 10 and by $p \equiv_c q$, we have $\beta \in \partial_\sigma(\mathsf{d}(\alpha, p)) = \partial_\sigma(\mathsf{d}(\alpha, q))$. Again by Proposition 10, there exists $q' \in \mathsf{Loc}_0(\alpha)$ such that $\beta = \mathsf{d}(\alpha, q')$, i.e. $p' \equiv_c q'$, and $(\sigma, q') \in \mathsf{Follow}(\alpha, q)$. $\square$

From the above, we have the following result for $\alpha \in \mathrm{RE}(\|)$.

Proposition 12. *$\mathcal{A}_{\mathrm{POS}}(\alpha)/{\equiv_c} \simeq \mathcal{A}_{\mathrm{PD}}(\alpha)$.*

Proof. We show that the function $\varphi_{\mathrm{c}} : \mathsf{Loc}_0(\alpha)/{\equiv_c} \longrightarrow \partial^+(\alpha)$, defined by $\varphi_{\mathrm{c}}([p]) = \mathsf{d}(\alpha, p)$, is an isomorphism. Injectivity follows from Lemma 11 and surjectivity from Lemma 8. For the initial state we have $\varphi_{\mathrm{c}}([0]) = \mathsf{d}(\alpha, 0) = \alpha$. Furthermore, by Lemma 9, $[p]$ is a final state in $\mathcal{A}_{\mathrm{POS}}(\alpha)/{\equiv_c}$ if and only if $\varphi_{\mathrm{c}}([p])$ is a final state in $\mathcal{A}_{\mathrm{PD}}(\alpha)$. Finally,

$$
\begin{aligned}
\varphi_{\mathrm{c}}({}^{\delta_{\mathrm{POS}}}\!/_{\equiv_c}([p], \sigma)) &= \varphi_{\mathrm{c}}(\{\, [q] \mid (\sigma, q) \in \mathsf{Follow}(\alpha, p) \,\}) \\
&= \{\, \mathsf{d}(\alpha, q) \mid (\sigma, q) \in \mathsf{Follow}(\alpha, p) \,\} \\
&= \partial_\sigma(\mathsf{d}(\alpha, p)) = \delta_{\mathrm{PD}}(\varphi_{\mathrm{c}}([p]), \sigma).
\end{aligned}
$$

□

Example 12. *It follows from the* c*-continuations computed for expression $\alpha = (ab)^{\star\,\mathsf{s}}\|_\Gamma\,(bc)^\star$ in Example 11 that $0 \equiv_c (2, 4)$ and $(0, 3) \equiv_c (2, 3)$. Thus, the partial derivative automaton in Example 9 can be obtained by merging those states of $\mathcal{A}_{\mathrm{POS}}(\alpha)$.*

5. Conclusions

The notion of location introduced in [5, 6] provides a suitable framework for the definition of a position automaton for several synchronised shuffle operators. For future work, we will study the average behaviour of synchronised shuffle expressions, and compare the results with those for regular expressions with (standard) shuffle and intersection [11, 12].

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Some Proofs Omitted in the Main Text

For the proof of Lemma 2 we need the following notion.

Coloured words. Given $\sigma \in \Sigma$ one can consider a *coloured letter* σ^c, where $c \in \{0,1,2\}$. Coloured words are denoted by $\widehat{x}$. We define a version of the shuffle operator $\widehat{\sqcup\!\sqcup}$ as follows:

$$\begin{aligned}
&\sigma_1 \cdots \sigma_n \mathbin{\widehat{\sqcup\!\sqcup}} \varepsilon = \{\, \sigma_1^1 \cdots \sigma_n^1 \,\},\\
&\varepsilon \mathbin{\widehat{\sqcup\!\sqcup}} \sigma_1 \cdots \sigma_n = \{\, \sigma_1^2 \cdots \sigma_n^2 \,\},\\
&\qquad \sigma u \mathbin{\widehat{\sqcup\!\sqcup}} \tau v = \{\, \sigma^1 \widehat{w} \mid \widehat{w} \in u \mathbin{\widehat{\sqcup\!\sqcup}} \tau v \,\} \cup \{\, \tau^2 \widehat{w} \mid \widehat{w} \in \sigma u \mathbin{\widehat{\sqcup\!\sqcup}} v \,\}.
\end{aligned}$$

Now, consider a word $x \in (u_1 \sqcup\!\sqcup v_1)\tau_1 \cdots \tau_{n-1}(u_n \sqcup\!\sqcup v_n)$ for words $u = u_1\tau_1 \cdots \tau_{n-1}u_n$ and $v = v_1\tau_1 \cdots \tau_{n-1}v_n$, and $\tau_1, \ldots, \tau_{n-1} \in \Sigma$. Then, there is a coloured word $\widehat{x} = \widehat{w_1}\tau_1^0 \cdots \tau_{n-1}^0 \widehat{w_n}$, with $\widehat{w_i} \in u_i \widehat{\sqcup} v_i$ and such that the word obtained by erasing all colours in $\widehat{x}$ is x. Furthermore, the number of letters in $\widehat{x}$, coloured with either 1 or 0 is precisely $|u|$, while the number of letters in $\widehat{x}$ coloured with 0 or 2 is $|v|$.

Lemma 2. *Given $\alpha \in \mathrm{RE}(\|)$, if $x = \sigma_1 \cdots \sigma_n \in \mathcal{L}(\alpha)$ $(n \geq 0)$, then there is a sequence $p_0, p_1, \ldots, p_n \in \mathsf{Loc}_0(\alpha)$, such that $p_0 = 0$ and*

$$(\sigma_i, p_i) \in \mathsf{Follow}(\alpha, p_{i-1}) \quad (1 \leq i \leq n) \qquad \textit{and} \qquad p_n \in \mathsf{Last}_0(\alpha).$$

Proof. We proceed by structural induction on $\alpha \in \mathrm{RE}(\|)$. As in Lemma 1, we only consider the case $\alpha = \alpha_1 \circ \alpha_2$, with $\circ \in \{ {}^{\mathsf{s}}\|_\Gamma, {}^{\mathsf{a}}\|_\Gamma, {}^{\mathsf{w}}\|_\Gamma \}$. Let $x \in u \circ v$ with $u = \mu_1 \cdots \mu_{|u|} \in \mathcal{L}(\alpha_1)$ and $v = \nu_1 \cdots \nu_{|v|} \in \mathcal{L}(\alpha_2)$, such that there is a sequence $p_0 = 0, \ldots, p_{|u|} \in \mathsf{Loc}_0(\alpha_1)$, with $(\mu_i, p_i) \in \mathsf{Follow}(\alpha_1, p_{i-1})$ $(1 \leq i \leq |u|)$ and $p_{|u|} \in \mathsf{Last}_0(\alpha_1)$, as well as a sequence $q_0 = 0, \ldots, q_{|v|} \in \mathsf{Loc}_0(\alpha_2)$, with $(\nu_i, q_i) \in \mathsf{Follow}(\alpha_2, q_{i-1})$ $(1 \leq i \leq |v|)$ and $q_{|v|} \in \mathsf{Last}_0(\alpha_2)$.

If $|v| = 0$, i.e., $v = \varepsilon$, then $|u| > 0$ and $x = u$. For $\circ \in \{ {}^{\mathsf{s}}\|_\Gamma, {}^{\mathsf{a}}\|_\Gamma \}$ the sequence $0, (p_1, 0), \ldots, (p_{|u|}, 0)$ satisfies the result. For $\circ = {}^{\mathsf{w}}\|_\Gamma$ the sequence is $0, (p_1^{\Delta_1}, 0^{\emptyset}), \ldots, (p_{|u|}^{\Delta_{|u|}}, 0^{\emptyset})$, where $\Delta_i = \{ \mu_j \mid \mu_j \in \Gamma \wedge j \leq i \}$, for $1 \leq i \leq |u|$. The case $|u| = 0$ is analogous.

Now consider $|u|, |v| > 0$. Then, there is a coloured word $\widehat{x} = \sigma_1^{c_1} \cdots \sigma_n^{c_n}$, such that the word obtained by erasing all colours in $\widehat{x}$ is x. In the following, we define a sequence of locations $r_0, r_1, \ldots, r_n \in \mathsf{Loc}_0(\alpha)$, such that $r_0 = 0$, $(\sigma_i, r_i) \in \mathsf{Follow}(\alpha, r_{i-1})$ $(1 \leq i \leq n)$, and $r_n \in \mathsf{Last}_0(\alpha)$.

We consider the cases $\circ \in \{ {}^{\mathsf{s}}\|_\Gamma, {}^{\mathsf{a}}\|_\Gamma \}$ and $\circ = {}^{\mathsf{w}}\|_\Gamma$ separately. For the former let

$$r_1 \quad = \quad \begin{cases} (p_1, 0), & \text{if } c_1 = 1; \\ (p_1, q_1), & \text{if } c_1 = 0; \\ (0, q_1), & \text{if } c_1 = 2, \end{cases}$$

and for $i \in \{2, \ldots, n\}$ and $r_{i-1} = (r_{1,i-1}, r_{2,i-1})$,

$$r_i \quad = \quad \begin{cases} (p_{j+1}, r_{2,i-1}), & \text{if } c_i = 1 \text{ and } r_{1,i-1} = p_j; \\ (p_{j+1}, q_{k+1}), & \text{if } c_i = 0, r_{1,i-1} = p_j \text{ and } r_{2,i-1} = q_k; \\ (r_{1,i-1}, q_{k+1}), & \text{if } c_i = 2 \text{ and } r_{2,i-1} = q_k. \end{cases}$$

First, we show that $r_n \in \mathsf{Last}_0(\alpha)$. Since the number of c_i's equal to 1 or 0 equals $|u|$, and the number of c_i's equal to 0 or 2 equals $|v|$, it follows from the definition of the sequence $r_0, r_1, \ldots, r_n$ that $r_n = (p_{|u|}, q_{|v|})$. Since $p_{|u|} \in \mathsf{Last}_0(\alpha_1)$ and $q_{|v|} \in \mathsf{Last}_0(\alpha_2)$, we have that $r_n = (p_{|u|}, q_{|v|}) \in \mathsf{Last}_0(\alpha)$.

It remains to show that $(\sigma_i, r_i) \in \mathsf{Follow}(\alpha, r_{i-1})$ $(1 \leq i \leq n)$. Here, we consider the two shuffle operators separately.

In the case $\alpha = \alpha_1 \, {}^{\mathsf{s}}\|_\Gamma \, \alpha_2$, the definition of ${}^{\mathsf{s}}\|_\Gamma$ guarantees that the colours in $\widehat{x}$ are such that $c_i = 1$ iff $\sigma_i = \mu_j \notin \Gamma$ for some $j \in [1, |u|]$, $c_i = 2$ iff $\sigma_i = \nu_k \notin \Gamma$ for some $k \in [1, |v|]$, and finally $c_i = 0$ iff $\sigma_i = \mu_j = \nu_k \in \Gamma$ for some $j \in [1, |u|]$

and $k \in [1, |v|]$. We show that for $i \in [1, n]$ one has $(\sigma_i, r_i) \in \mathsf{Follow}(\alpha, r_{i-1})$ for $r_{i-1} = (r_{1,i-1}, r_{2,i-1})$, considering the cases $c_i = 1$ and $c_i = 0$. The remaining case, $c_i = 2$, is analogous to the first one. If $c_i = 1$ and $r_{1,i-1} = p_j$, then $\sigma_i = \mu_{j+1} \notin \Gamma$ and $(\sigma_i, p_{j+1}) \in \mathsf{Follow}(\alpha_1, p_j)$. Thus, $(\sigma_i, (p_{j+1}, r_{2,i-1})) \in \mathsf{Follow}(\alpha, (r_{1,i-1}, r_{2,i-1}))$. If $c_i = 0$ and $(r_{1,i-1}, r_{2,i-1}) = (p_j, q_k)$, then $\sigma_i = \mu_{j+1} = \nu_{k+1} \in \Gamma$, $(\sigma_i, p_{j+1}) \in \mathsf{Follow}(\alpha_1, p_j)$ and $(\sigma_i, q_{k+1}) \in \mathsf{Follow}(\alpha_2, q_k)$. Thus, $(\sigma_i, (p_{j+1}, q_{k+1})) \in \mathsf{Follow}(\alpha, (r_{1,i-1}, r_{2,i-1}))$.
The proof for ${}^{\mathsf{a}}\|_\Gamma$ is analogous, skipping the conditions $\mu_{j+1} \notin \Gamma$ and $\nu_{k+1} \notin \Gamma$.

Now, consider $\alpha = \alpha_1 \, {}^{\mathsf{w}}\|_\Gamma \, \alpha_2$. Let,

$$r_1 = \begin{cases} (p_1^{\emptyset}, 0^{\emptyset}), & \text{if } c_1 = 1 \text{ and } \mu_1 \notin \Gamma, \\ (p_1^{\{\mu_1\}}, 0^{\emptyset}), & \text{if } c_1 = 1 \text{ and } \mu_1 \in \Gamma, \\ (p_1^{\emptyset}, q_1^{\emptyset}), & \text{if } c_1 = 0, \\ (0^{\emptyset}, q_1^{\emptyset}), & \text{if } c_1 = 2 \text{ and } \nu_1 \notin \Gamma, \\ (0^{\emptyset}, q_1^{\{\nu_1\}}), & \text{if } c_1 = 2 \text{ and } \nu_1 \in \Gamma, \end{cases}$$

and for $i \in \{2, \ldots, n\}$ and $r_{i-1} = (p_j^{\Delta}, q_k^{\Lambda})$,

$$r_i = \begin{cases} (p_{j+1}^{\Delta}, q_k^{\Lambda}), & \text{if } c_i = 1 \text{ and } \mu_{j+1} \notin \Gamma, \\ (p_{j+1}^{\Delta,\mu_{j+1}}, q_k^{\Lambda}), & \text{if } c_i = 1 \text{ and } \mu_{j+1} \in \Gamma, \\ (p_{j+1}^{\emptyset}, q_{k+1}^{\emptyset}), & \text{if } c_i = 0, \\ (p_j^{\Delta}, q_{k+1}^{\Lambda}), & \text{if } c_i = 2 \text{ and } \nu_{k+1} \notin \Gamma, \\ (p_j^{\Delta}, q_{k+1}^{\Lambda,\nu_{k+1}}), & \text{if } c_i = 2 \text{ and } \nu_{k+1} \in \Gamma. \end{cases}$$

First, we show that $r_n \in \mathsf{Last}_0(\alpha)$. Again, since the number of c_i's equal to 1 or 0 equals $|u|$, and the number of c_i's equal to 0 or 2 equals $|v|$, it follows from the definition of the sequence $r_0, r_1, \ldots, r_n$ that $r_n = (p_{|u|}^{\Delta}, q_{|v|}^{\Lambda})$, for some $\Delta, \Lambda \subseteq \Gamma$ such that $\Delta \cap \Lambda = \emptyset$. Since $p_{|u|} \in \mathsf{Last}_0(\alpha_1)$ and $q_{|v|} \in \mathsf{Last}_0(\alpha_2)$, we have by (1) that $r_n = (p_{|u|}^{\Delta}, q_{|v|}^{\Lambda}) \in \mathsf{Last}_0(\alpha)$.

It remains to show that $(\sigma_i, r_i) \in \mathsf{Follow}(\alpha, r_{i-1})$ $(1 \le i \le n)$. We show that for $i \in [1, n]$ one has $(\sigma_i, r_i) \in \mathsf{Follow}(\alpha, r_{i-1})$ for $r_{i-1} = (p_j^{\Delta}, q_k^{\Lambda})$, considering the cases $c_i = 1$ and $c_i = 0$. The remaining case, $c_i = 2$, is analogous to the first one.

If $c_i = 1$ and $\sigma_i = \mu_{j+1} \notin \Gamma$, then by the definition of Follow we have $(\sigma_i, p_{j+1}) \in \mathsf{Follow}(\alpha_1, p_j)$. Thus, $(\sigma_i, (p_{j+1}^{\Delta}, q_k^{\Lambda})) \in \mathsf{Follow}(\alpha, r_{i-1})$.

If $c_i = 1$ and $\sigma_i = \mu_{j+1} \in \Gamma$, then by the definition of Follow we have $(\sigma_i, p_{j+1}) \in \mathsf{Follow}(\alpha_1, p_j)$ and $\sigma_i \notin \Lambda$. Thus, $(\sigma_i, (p_{j+1}^{\Delta,\sigma_i}, q_k^{\Lambda})) \in \mathsf{Follow}(\alpha, r_{i-1})$.

If $c_i = 0$, then $\sigma_i = \mu_{j+1} = \nu_{k+1} \in \Gamma$, $(\sigma_i, p_{j+1}) \in \mathsf{Follow}(\alpha_1, p_j)$ and $(\sigma_i, q_{k+1}) \in \mathsf{Follow}(\alpha_2, q_k)$. Thus, $(\sigma_i, (p_{j+1}^{\emptyset}, q_{k+1}^{\emptyset})) \in \mathsf{Follow}(\alpha, r_{i-1})$. $\square$

Lemma 3. *Given $\alpha \in \mathrm{RE}(\|)$, if there are $r_0 = 0, r_1, \ldots, r_n \in \mathsf{Loc}_0(\alpha)$ and $\sigma_1, \ldots, \sigma_n \in \Sigma$ $(n \ge 0)$ such that (1) $(\sigma_i, r_i) \in \mathsf{Follow}(\alpha, r_{i-1})$, $1 \le i \le n$, and (2) $r_n \in \mathsf{Last}_0(\alpha)$, then $\sigma_1 \cdots \sigma_n \in \mathcal{L}(\alpha)$.*

Proof. The proof is by structural induction on α. As before, only the shuffle operators are dealt with. Let $\alpha = \alpha_1 \circ \alpha_2$, with $\circ \in \{{}^{\mathsf{s}}\|_\Gamma, {}^{\mathsf{a}}\|_\Gamma\}$. Then, each location r_i is of the form (p_i, q_i), for $1 \leq i \leq n$. If $r_n = (p_n, 0)$ then $\varepsilon \in \mathcal{L}(\alpha_2)$ (analogously $r_n = (0, q_n)$ implies that $\varepsilon \in \mathcal{L}(\alpha_1)$, and the proof is similar). Moreover, if the operator $\circ$ is ${}^{\mathsf{s}}\|_\Gamma$, then it follows, by the definition of Follow, that $\sigma_1, \ldots, \sigma_n \notin \Gamma$. For both operators, we have that the sequences of locations $p_1, \ldots, p_n$ and of letters $\sigma_1, \ldots, \sigma_n$ satisfy the conditions in this lemma for α_1. Thus, by induction, $\sigma_1 \cdots \sigma_n \in \mathcal{L}(\alpha_1)$. On the other hand, $\sigma_1 \cdots \sigma_n \in \sigma_1 \cdots \sigma_n \circ \varepsilon \subseteq \mathcal{L}(\alpha_1 \circ \alpha_2)$.

For the remaining of the proof we suppose that $p_n, q_n \neq 0$ in $r_n = (p_n, q_n)$. We will use sequences of the form $L = (\sigma_1, r_1) \cdot (\sigma_2, r_2) \cdots (\sigma_n, r_n)$ and define some functions on $(\Sigma \times \mathsf{Loc}(\alpha))^\star$. The function $\mathsf{tr} : (\Sigma \times \mathsf{Loc}(\alpha))^\star \longrightarrow \Sigma^\star$ is given by $\mathsf{tr}((\sigma_1, r_1) \cdots (\sigma_n, r_n)) = \sigma_1 \cdots \sigma_n$. The function $\mathsf{seq} : (\Sigma \times \mathsf{Loc}(\alpha_1 \circ \alpha_2))^\star \longrightarrow (\Sigma \times \mathsf{Loc}(\alpha_1))^\star \times (\Sigma \times \mathsf{Loc}(\alpha_2))^\star$ which definition follows. If $L = \varepsilon$, then $\mathsf{seq}(L) = (\varepsilon, \varepsilon)$. If $L = (\sigma, r) \cdot L'$, then $\mathsf{seq}(L) = (L_1, L_2)$ is obtained from L'_1 and L'_2, where $\mathsf{seq}(L') = (L'_1, L'_2)$, as explained below.

In each step i of the computation it is ensured that for $u' = \mathsf{tr}(L'_1)$, $v' = \mathsf{tr}(L'_2)$, and $w = \mathsf{tr}(L')$, if $w \in u' \circ v'$, then $\sigma_i \cdot w \in u \circ v$, where $u = \mathsf{tr}(L_1)$ and $v = \mathsf{tr}(L_2)$. For $L \neq \varepsilon$ we define $\mathsf{seq}(L)$ inductively as follows:

1. If $L = (\sigma_i, (p_i, q_i)) \cdot L'$, $(\sigma_i, p_i) \in \mathsf{Follow}(\alpha_1, p_{i-1})$, $(\sigma_i, q_i) \in \mathsf{Follow}(\alpha_2, q_{i-1})$, and $\sigma_i \in \Gamma$, then
$$\mathsf{seq}(L) = ((\sigma_i, p_i) \cdot L'_1, (\sigma_i, q_i) \cdot L'_2).$$
We have that $u = \sigma_i u'$, $v = \sigma_i v'$, and $\sigma_i w \in u \circ v$.

2. If $L = (\sigma_i, (p_i, q_i)) \cdot L'$, $(\sigma_i, p_i) \in \mathsf{Follow}(\alpha_1, p_{i-1})$ and $q_i = q_{i-1}$, then
$$\mathsf{seq}(L) = ((\sigma_i, p_i) \cdot L'_1, L'_2).$$
If $\circ = {}^{\mathsf{s}}\|_\Gamma$ then $\sigma_i \notin \Gamma$. For both operators, we have that $u = \sigma_i u'$ and $v = v'$, and $\sigma_i w \in u \circ v$.

3. Finally, if $L = (\sigma_i, (p_i, q_i)) \cdot L'$, $(\sigma_i, q_i) \in \mathsf{Follow}(\alpha_2, q_{i-1})$ and $p_i = p_{i-1}$, then
$$\mathsf{seq}(L) = (L'_1, (\sigma_i, q_i) \cdot L'_2).$$
Again, if $\circ = {}^{\mathsf{s}}\|_\Gamma$ then $\sigma_i \notin \Gamma$. For both operators, we have that $u = u'$ and $v = \sigma_i v'$, and $\sigma_i w \in u \circ v$.

Now, consider
$$\mathsf{seq}(L) = (L_1, L_2) = ((\mu_1, p'_1) \cdots (\mu_{k_1}, p'_{k_1}), (\nu_1, q'_1) \cdots (\nu_{k_2}, q'_{k_2})).$$

It follows from $p_n, q_n \neq 0$ that $k_1, k_2 \geq 1$. In particular, we have $p'_{k_1}, q'_{k_2} \neq 0$, since (μ_{k_1}, p'_{k_1}) and (ν_{k_2}, q'_{k_2}) are members of Follow sets. In order to apply the induction hypothesis to the sequences of locations $p'_1, \ldots, p'_{k_1}$ and letters

$\mu_1, \ldots, \mu_{k_1}$ (analogously for $q'_1, \ldots, q'_{k_2}$ and letters $\nu_1, \ldots, \nu_{k_2}$), we have to show that conditions (1) and (2) hold.

Condition (1) is true, since by construction $(\mu_l, p_l) \in \mathsf{Follow}(\alpha_1, p_{l-1})$ for $2 \leq l \leq k_1$. Condition (2) follows directly from the definition of $\mathsf{Last}_0(\alpha_1 \circ \alpha_2)$ for $\circ \in \{ {}^{\mathsf{s}}\|_\Gamma \,, {}^{\mathsf{a}}\|_\Gamma \}$, i.e., definition of Loc, and the fact that $p'_{k_1}, q'_{k_2} \neq 0$.

Now, we can apply the induction hypothesis and conclude that $\mathsf{tr}(L_1) = \mu_1 \cdots \mu_{k_1} \in \mathcal{L}(\alpha_1)$ and $\mathsf{tr}(L_2) = \nu_1 \cdots \nu_{k_2} \in \mathcal{L}(\alpha_2)$, and consequently $\mu_1 \cdots \mu_{k_1} \circ \nu_1 \cdots \nu_{k_2} \subseteq \mathcal{L}(\alpha_1 \circ \alpha_2)$. It follows from the definition of seq that $\sigma_1 \cdots \sigma_n = \mathsf{tr}(L) \in \mathsf{tr}(L_1) \circ \mathsf{tr}(L_2)$. Thus, $\sigma_1 \cdots \sigma_n \in \mathcal{L}(\alpha_1 \circ \alpha_2)$.

Next, we consider the case $\alpha = \alpha_1 \, {}^{\mathsf{w}}\|_\Gamma \, \alpha_2$. The proof is similar to the proof for the two other operators, but superscripts in pairs of locations have to be taken into account. In fact, each location r_i is now of the form $(p_i^{\Delta_i}, q_i^{\Lambda_i})$ with $\Delta_i, \Lambda_i \subseteq \Gamma$, for $1 \leq i \leq n$. Note that $r_n = (p_n^{\Delta_n}, 0^\emptyset)$ implies that $\varepsilon \in \mathcal{L}(\alpha_2)$ (analogously $r_n = (0^\emptyset, q_n^{\Lambda_n})$ implies that $\varepsilon \in \mathcal{L}(\alpha_1)$). Then, the sequences of locations $p_1, \ldots, p_n$ and of letters $\sigma_1, \ldots, \sigma_n$ satisfy the conditions in this lemma for α_1. Thus, by induction $\sigma_1 \cdots \sigma_n \in \mathcal{L}(\alpha_1)$. On the other hand, $\sigma_1 \cdots \sigma_n \in \sigma_1 \cdots \sigma_n \, {}^{\mathsf{w}}\|_\Gamma \, \varepsilon \subseteq \mathcal{L}(\alpha_1 \, {}^{\mathsf{w}}\|_\Gamma \, \alpha_2)$.

Considering the sequence $L = (\sigma_1, r_1) \cdot (\sigma_2, r_2) \cdots (\sigma_n, r_n)$ with $p_n^{\Delta_n}, q_n^{\Lambda_n} \neq 0^\emptyset$ in $r_n = (p_n^{\Delta_n}, q_n^{\Lambda_n})$ we define $\mathsf{seq}(L)$. For $L = \varepsilon$ let $\mathsf{seq}(L) = (\varepsilon, \varepsilon)$. For $L \neq \varepsilon$ the pair $\mathsf{seq}(L) = (L_1, L_2)$ is defined below. The function tr is defined as above. Again, in each step of the computation it is ensured that for $u' = \mathsf{tr}(L'_1)$, $v' = \mathsf{tr}(L'_2)$, $u = \mathsf{tr}(L_1)$, $v = \mathsf{tr}(L_2)$, and $w = \mathsf{tr}(L')$, if $w \in u' \,\|_{(\Gamma, \Delta_i, \Lambda_i)} v'$ then the following holds. If $i \geq 2$, then $\sigma_i \cdot w \in u \,\|_{(\Gamma, \Delta_{i-1}, \Lambda_{i-1})} v$, and if $i = 1$, then $\sigma_1 \cdot w \in u \,\|_{(\Gamma, \emptyset, \emptyset)} v = u \, {}^{\mathsf{w}}\|_\Gamma \, v$. Since $\varepsilon \in \varepsilon \,\|_{(\Gamma, \Delta, \Lambda)} \varepsilon$, for any $\Delta, \Lambda \subseteq \Gamma$, it follows that for the initial sequence L one has $\sigma_1 \cdots \sigma_n \in u \,\|_{(\Gamma, \emptyset, \emptyset)} v = u \, {}^{\mathsf{w}}\|_\Gamma \, v$. The pair of sequences $\mathsf{seq}(L) = (L_1, L_2)$, for $L \neq \varepsilon$, is now defined as follows:

1. If $L = (\sigma_i, (p_i^\emptyset, q_i^\emptyset)) \cdot L'$, $(\sigma_i, p_i) \in \mathsf{Follow}(\alpha_1, p_{i-1})$, $(\sigma_i, q_i) \in \mathsf{Follow}(\alpha_2, q_{i-1})$, and $\sigma_i \in \Gamma$, then

$$\mathsf{seq}(L) = ((\sigma_i, p_i) \cdot L'_1, (\sigma_i, q_i) \cdot L'_2).$$

   We have that $u = \sigma_i u'$ and $v = \sigma_i v'$. Let $w \in u' \,\|_{(\Gamma, \emptyset, \emptyset)} v'$. If $i \geq 2$, then $\sigma_i \cdot w \in u \,\|_{(\Gamma, \Delta_{i-1}, \Lambda_{i-1})} v$. If $i = 1$, then $w \in u' \,\|_{(\Gamma, \emptyset, \emptyset)} v'$ also implies that $\sigma_1 \cdot w \in u \,\|_{(\Gamma, \emptyset, \emptyset)} v = u \, {}^{\mathsf{w}}\|_\Gamma \, v$.

2. If $L = (\sigma_i, (p_i^{\Delta_i}, q_i^{\Lambda_i})) \cdot L'$, $(\sigma_i, p_i) \in \mathsf{Follow}(\alpha_1, p_{i-1})$, $\sigma_i \in \Delta_i$, and $\sigma_i \notin \Lambda_i$, then

$$\mathsf{seq}(L) = ((\sigma_i, p_i) \cdot L'_1, L'_2).$$

   We have that $u = \sigma_i u'$, $v = v'$, $\Delta_i = \Delta_{i-1} \cup \{\sigma_i\}$, and $\Lambda_i = \Lambda_{i-1}$. Let $w \in u' \,\|_{(\Gamma, \Delta_{i-1} \cup \{\sigma_i\}, \Lambda_{i-1})} v'$. If $i \geq 2$, then $\sigma_i \cdot w \in u \,\|_{(\Gamma, \Delta_{i-1}, \Lambda_{i-1})} v$. If $i = 1$, then $\Delta_1 = \{\sigma_1\}$ and $\Lambda_1 = \emptyset$. Furthermore, $w \in u' \,\|_{(\Gamma, \{\sigma_1\}, \emptyset)} v'$ implies that $\sigma_1 \cdot w \in u \,\|_{(\Gamma, \emptyset, \emptyset)} v = u \, {}^{\mathsf{w}}\|_\Gamma \, v$.

   The case $(\sigma_i, q_i) \in \mathsf{Follow}(\alpha_2, q_{i-1})$, $\sigma_i \in \Lambda_i$, and $\sigma_i \notin \Delta_i$ is analogous.

3. If $L = (\sigma_i, (p_i^{\Delta_i}, q_i^{\Lambda_i})) \cdot L'$, $(\sigma_i, p_i) \in \mathsf{Follow}(\alpha_1, p_{i-1})$, $\sigma_i \notin \Gamma$, $\Delta_i = \Delta_{i-1}$ and $q_i^{\Lambda_i} = q_{i-1}^{\Lambda_{i-1}}$ (i.e., $\Lambda_i = \Lambda_{i-1}$) then

$$\mathsf{seq}(L) = ((\sigma_i, p_i) \cdot L'_1, L'_2).$$

We have that $u = \sigma_i u'$ and $v = v'$. Let $w \in u' \|_{(\Gamma,\Delta_i,\Lambda_i)} v'$. If $i \geq 2$, then $\sigma_i \cdot w \in u \|_{(\Gamma,\Delta_i,\Lambda_i)} v$. If $i = 1$, then $\Delta_1 = \Lambda_1 = \emptyset$. Furthermore, $w \in u' \|_{(\Gamma,\emptyset,\emptyset)} v'$ implies that $\sigma_1 \cdot w \in u \|_{(\Gamma,\emptyset,\emptyset)} v = u \,{}^{\mathsf{w}}\|_\Gamma\, v$. Again, the case $(\sigma_i, q_i) \in \mathsf{Follow}(\alpha_2, q_{i-1})$, $\sigma_i \notin \Gamma$, $\Lambda_i = \Lambda_{i-1}$ and $p_i^{\Delta_i} = p_{i-1}^{\Delta_{i-1}}$ is analogous.

The rest of the proof for this case is identical to the one for $\circ \in \{ {}^{\mathsf{s}}\|_\Gamma, {}^{\mathsf{a}}\|_\Gamma \}$. $\square$

Proposition 10. *For $\alpha \in \mathrm{RE}(\|)$, $p \in \mathsf{Loc}_0(\alpha)$, and $\sigma \in \Sigma_\alpha$, one has*

$$\partial_\sigma(\mathsf{d}(\alpha, p)) = \{ \mathsf{d}(\alpha, q) \mid (\sigma, q) \in \mathsf{Follow}(\alpha, p) \}.$$

Proof. It is sufficient to show the result for expressions $\alpha = \alpha_1 \circ \alpha_2$ with $\circ \in \{ {}^{\mathsf{s}}\|_\Gamma, {}^{\mathsf{a}}\|_\Gamma, {}^{\mathsf{w}}\|_\Gamma \}$. In these cases the proof follows the structure of the one for $\sqcup\!\sqcup$ in [5]. ($\subseteq$) First, suppose that $p = 0$. Let $\sigma \notin \Gamma$ and $\beta \in \partial_\sigma(\mathsf{d}(\alpha, 0)) = \partial_\sigma(\alpha) = \partial_\sigma(\alpha_1) \circ \{\alpha_2\} \cup \{\alpha_1\} \circ \partial_\sigma(\alpha_2)$. If $\beta = \beta_1 \circ \alpha_2$ with $\beta_1 \in \partial_\sigma(\alpha_1) = \partial_\sigma(\mathsf{d}(\alpha_1, 0))$, then by induction there exists $q \in \mathsf{Loc}(\alpha_1)$ such that $\beta_1 = \mathsf{d}(\alpha_1, q)$ and $(\sigma, q) \in \mathsf{Follow}(\alpha_1, 0) = \mathsf{First}(\alpha_1)$. If $\circ \in \{ {}^{\mathsf{s}}\|_\Gamma, {}^{\mathsf{a}}\|_\Gamma \}$ then $(\sigma, (q, 0)) \in \mathsf{First}(\alpha_1 \circ \alpha_2) = \mathsf{Follow}(\alpha, 0)$. Furthermore, $\beta = \beta_1 \circ \alpha_2 = \mathsf{d}(\alpha_1, q) \circ \alpha_2 = \mathsf{d}(\alpha_1, q) \circ \mathsf{d}(\alpha_2, 0) = \mathsf{d}(\alpha, (q, 0))$. If $\circ = {}^{\mathsf{w}}\|_\Gamma$, then $(\sigma, (q^\emptyset, 0^\emptyset)) \in \mathsf{First}(\alpha_1 \,{}^{\mathsf{w}}\|_\Gamma\, \alpha_2) = \mathsf{Follow}(\alpha, 0)$. Furthermore, $\beta = \beta_1 \,{}^{\mathsf{w}}\|_\Gamma\, \alpha_2 = \mathsf{d}(\alpha_1, q) \,{}^{\mathsf{w}}\|_\Gamma\, \alpha_2 = \mathsf{d}(\alpha_1, q) \|_{(\Gamma,\emptyset,\emptyset)} \mathsf{d}(\alpha_2, 0) = \mathsf{d}(\alpha, (q^\emptyset, 0^\emptyset))$. The case $\beta = \alpha_1 \circ \beta_2$ with $\beta_2 \in \partial_\sigma(\alpha_2)$ is identical.

Now, let $\sigma \in \Gamma$ and $\beta \in \partial_\sigma(\mathsf{d}(\alpha, 0)) = \partial_\sigma(\alpha)$. Consider $\circ \in \{ {}^{\mathsf{s}}\|_\Gamma, {}^{\mathsf{a}}\|_\Gamma \}$. Thus $\beta = \beta_1 \circ \beta_2 \in \partial_\sigma(\alpha_1) \circ \partial_\sigma(\alpha_2)$ with $\beta_i \in \partial_\sigma(\alpha_i) = \partial_\sigma(\mathsf{d}(\alpha_i, 0))$, for $i = 1, 2$. Then, by induction there exists $q_i \in \mathsf{Loc}(\alpha_i)$ such that $\beta_i = \mathsf{d}(\alpha_i, q_i)$ and $(\sigma, q_i) \in \mathsf{Follow}(\alpha_i, 0) = \mathsf{First}(\alpha_i)$. Thus, $(\sigma, (q_1, q_2)) \in \mathsf{First}(\alpha_1 \circ \alpha_2) = \mathsf{Follow}(\alpha, 0)$. Furthermore, $\beta = \beta_1 \circ \beta_2 = \mathsf{d}(\alpha_1, q_1) \circ \mathsf{d}(\alpha_2, q_2) = \mathsf{d}(\alpha, (q_1, q_2))$. When $\circ$ is ${}^{\mathsf{a}}\|_\Gamma$, one has to consider the additional case corresponding to the situation above, i.e., $\beta = \beta_1 \,{}^{\mathsf{a}}\|_\Gamma\, \alpha_2$, etc. If $\circ = {}^{\mathsf{w}}\|_\Gamma$ then one has,

$$\partial_\sigma(\alpha) = \partial_\sigma(\alpha_1) \,{}^{\mathsf{w}}\|_\Gamma\, \partial_\sigma(\alpha_2) \cup \partial_\sigma(\alpha_1) \|_{(\Gamma,\{\sigma\},\emptyset)} \{\alpha_2\} \cup \{\alpha_1\} \|_{(\Gamma,\emptyset,\{\sigma\})} \partial_\sigma(\alpha_2).$$

If $\beta = \beta_1 \,{}^{\mathsf{w}}\|_\Gamma\, \beta_2$ with $\beta_i \in \partial_\sigma(\alpha_i)$ for $i = 1, 2$, then it follows by induction that there are positions $q_i \in \mathsf{Loc}(\alpha_i)$ such that $\beta_i = \mathsf{d}(\alpha_i, q_i)$ and $(\sigma, q_i) \in \mathsf{Follow}(\alpha_i, 0) = \mathsf{First}(\alpha_i)$. Hence, $(\sigma, (q_1^\emptyset, q_2^\emptyset)) \in \mathsf{First}(\alpha_1 \,{}^{\mathsf{w}}\|_\Gamma\, \alpha_2) = \mathsf{Follow}(\alpha, 0)$. If $\beta = \beta_1 \|_{(\Gamma,\{\sigma\},\emptyset)} \alpha_2$ with $\beta_1 \in \partial_\sigma(\alpha_1)$, then it follows by induction that there is a positions $q_1 \in \mathsf{Loc}(\alpha_1)$ such that $\beta_1 = \mathsf{d}(\alpha_1, q_1)$ and $(\sigma, q_1) \in \mathsf{Follow}(\alpha_1, 0) = \mathsf{First}(\alpha_1)$. Hence, $(\sigma, (q_1^{\{\sigma\}}, 0^\emptyset)) \in \mathsf{First}(\alpha_1 \,{}^{\mathsf{w}}\|_\Gamma\, \alpha_2) = \mathsf{Follow}(\alpha, 0)$. The remaining case is analogous.

Let $p = (p_1, p_2)$, $\circ \in \{ {}^{\mathsf{s}}\|_\Gamma, {}^{\mathsf{a}}\|_\Gamma \}$, and

$$\beta \in \partial_\sigma(\mathsf{d}(\alpha_1 \circ \alpha_2, (p_1, p_2))) = \partial_\sigma(\mathsf{d}(\alpha_1, p_1) \circ \mathsf{d}(\alpha_2, p_2)).$$

If $\sigma \notin \Gamma$, then

$$\partial_\sigma(\mathsf{d}(\alpha_1,p_1)\circ\mathsf{d}(\alpha_2,p_2)) = \partial_\sigma(\mathsf{d}(\alpha_1,p_1))\circ\{\mathsf{d}(\alpha_2,p_2)\}\cup\{\mathsf{d}(\alpha_1,p_1)\}\circ\partial_\sigma(\mathsf{d}(\alpha_2,p_2)).$$

If $\beta \in \partial_\sigma(\mathsf{d}(\alpha_1,p_1))\circ\{\mathsf{d}(\alpha_2,p_2)\}$, then $\beta = \beta_1\circ\mathsf{d}(\alpha_2,p_2)$ with $\beta_1 \in \partial_\sigma(\mathsf{d}(\alpha_1,p_1))$. By induction, there exists $q_1 \in \mathsf{Loc}(\alpha_1)$ such that $\beta_1 = \mathsf{d}(\alpha_1,q_1)$ and $(\sigma,q_1) \in \mathsf{Follow}(\alpha_1,p_1)$. We have $(q_1,p_2) \in \mathsf{Loc}(\alpha_1\circ\alpha_2)$,

$$\beta = \mathsf{d}(\alpha_1,q_1)\circ\mathsf{d}(\alpha_2,p_2) = \mathsf{d}(\alpha,(q_1,p_2)),$$

and $(\sigma,(q_1,p_2)) \in \mathsf{Follow}(\alpha,(p_1,p_2))$. The remaining case is analogous.

If $\sigma \in \Gamma$ and $\beta \in \partial_\sigma(\mathsf{d}(\alpha_1,p_1))\circ\partial_\sigma(\mathsf{d}(\alpha_2,p_2))$ then $\beta = \beta_1\circ\beta_2$, with $\beta_i \in \partial_\sigma(\mathsf{d}(\alpha_i,p_i))$ for $i = 1,2$. By induction, there exist $q_i \in \mathsf{Loc}(\alpha_i)$ such that $\beta_i = \mathsf{d}(\alpha_i,q_i)$, and $(\sigma,q_i) \in \mathsf{Follow}(\alpha_i,p_i)$. We have that $(q_1,q_2) \in \mathsf{Loc}(\alpha_1\circ\alpha_2)$,

$$\beta = \mathsf{d}(\alpha_1,q_1)\circ\mathsf{d}(\alpha_2,q_2) = \mathsf{d}(\alpha,(q_1,q_2)),$$

and $(\sigma,(q_1,q_2)) \in \mathsf{Follow}(\alpha,(p_1,p_2))$.

For $\circ = {}^{\mathsf{a}}\|_\Gamma$ we have the additional case

$$\beta \in \partial_\sigma(\mathsf{d}(\alpha_1,p_1))\,{}^{\mathsf{a}}\|_\Gamma\,\{\mathsf{d}(\alpha_2,p_2)\}\cup\{\mathsf{d}(\alpha_1,p_1)\}\,{}^{\mathsf{a}}\|_\Gamma\,\partial_\sigma(\mathsf{d}(\alpha_2,p_2)),$$

which is analogous to the case where $\sigma \notin \Gamma$.

Now, let $p = (p_1^\Delta,p_2^\Lambda)$, $\circ = {}^{\mathsf{w}}\|_\Gamma$, and

$$\beta \in \partial_\sigma(\mathsf{d}(\alpha_1\,{}^{\mathsf{w}}\|_\Gamma\,\alpha_2,(p_1^\Delta,p_2^\Lambda))) = \partial_\sigma(\mathsf{d}(\alpha_1,p_1)\,\|_{(\Gamma,\Delta,\Lambda)}\mathsf{d}(\alpha_2,p_2)).$$

If $\sigma \notin \Gamma$, then

$$\begin{aligned}\partial_\sigma(\mathsf{d}(\alpha_1,p_1)\,\|_{(\Gamma,\Delta,\Lambda)}\mathsf{d}(\alpha_2,p_2)) &= \partial_\sigma(\mathsf{d}(\alpha_1,p_1))\,\|_{(\Gamma,\Delta,\Lambda)}\{\mathsf{d}(\alpha_2,p_2)\}\\ &\cup\ \{\mathsf{d}(\alpha_1,p_1)\}\,\|_{(\Gamma,\Delta,\Lambda)}\partial_\sigma(\mathsf{d}(\alpha_2,p_2)).\end{aligned}$$

Consider $\beta = \beta_1\,\|_{(\Gamma,\Delta,\Lambda)}\mathsf{d}(\alpha_2,p_2)$ with $\beta_1 \in \partial_\sigma(\mathsf{d}(\alpha_1,p_1))$. By induction, there exists $q_1 \in \mathsf{Loc}(\alpha_1)$ such that $\beta_1 = \mathsf{d}(\alpha_1,q_1)$ and $(\sigma,q_1) \in \mathsf{Follow}(\alpha_1,p_1)$. We have $(q_1^\Delta,p_2^\Lambda) \in \mathsf{Loc}(\alpha_1\,{}^{\mathsf{w}}\|_\Gamma\,\alpha_2)$,

$$\beta = \mathsf{d}(\alpha_1,q_1)\,\|_{(\Gamma,\Delta,\Lambda)}\mathsf{d}(\alpha_2,p_2) = \mathsf{d}(\alpha_1\,{}^{\mathsf{w}}\|_\Gamma\,\alpha_2,(q_1^\Delta,p_2^\Lambda)),$$

and $(\sigma,(q_1^\Delta,p_2^\Lambda)) \in \mathsf{Follow}(\alpha,(p_1^\Delta,p_2^\Lambda))$. The remaining case is analogous.

If $\sigma \in \Gamma$ and $\beta \in \partial_\sigma(\mathsf{d}(\alpha_1,p_1))\,\|_{(\Gamma,\emptyset,\emptyset)}\partial_\sigma(\mathsf{d}(\alpha_2,p_2))$, then $\beta = \beta_1\,\|_{(\Gamma,\emptyset,\emptyset)}\beta_2$ with $\beta_i \in \partial_\sigma(\mathsf{d}(\alpha_i,p_i))$, for $i = 1,2$. By induction, there exist $q_i \in \mathsf{Loc}(\alpha_i)$, such that $\beta_i = \mathsf{d}(\alpha_i,q_i)$ and $(\sigma,q_i) \in \mathsf{Follow}(\alpha_i,p_i)$. We have that $(q_1^\emptyset,q_2^\emptyset) \in \mathsf{Loc}(\alpha_1\,{}^{\mathsf{w}}\|_\Gamma\,\alpha_2)$,

$$\beta = \mathsf{d}(\alpha_1,q_1)\,\|_{(\Gamma,\emptyset,\emptyset)}\mathsf{d}(\alpha_2,q_2) = \mathsf{d}(\alpha_1\,{}^{\mathsf{w}}\|_\Gamma\,\alpha_2,(q_1^\emptyset,q_2^\emptyset)),$$

and $(\sigma,(q_1^\emptyset,q_2^\emptyset)) \in \mathsf{Follow}(\alpha,(p_1^\Delta,p_2^\Lambda))$.

If $\sigma \in \Gamma$, $\sigma \notin \Lambda$, and $\beta \in \partial_\sigma(\mathsf{d}(\alpha_1,p_1))\,\|_{(\Gamma,\Delta\cup\{\sigma\},\Lambda)}\mathsf{d}(\alpha_2,p_2)$, then $\beta = \beta_1\,\|_{(\Gamma,\Delta\cup\{\sigma\},\Lambda)}\mathsf{d}(\alpha_2,p_2)$ with $\beta_1 \in \partial_\sigma(\mathsf{d}(\alpha_1,p_1))$. By induction, there exists

$q_1 \in \mathsf{Loc}(\alpha_1)$, such that $\beta_1 = \mathsf{d}(\alpha_1, q_1)$, and $(\sigma, q_1) \in \mathsf{Follow}(\alpha_1, p_1)$. We have that $(q_1^{\Delta\cup\{\sigma\}}, p_2^{\Lambda}) \in \mathsf{Loc}(\alpha_1 \,{}^{\mathsf{w}}\|_\Gamma\, \alpha_2)$, and $(\sigma, (q_1^{\Delta\cup\{\sigma\}}, p_2^{\Lambda})) \in \mathsf{Follow}(\alpha, (p_1^{\Delta}, p_2^{\Lambda}))$. The remaining case, $\sigma \in \Gamma$, $\sigma \notin \Delta$, and $\beta \in \mathsf{d}(\alpha_1, p_1) \,\|_{(\Gamma,\Delta,\Lambda\cup\{\sigma\})} \partial_\sigma(\mathsf{d}(\alpha_2, p_2))$, is analogous.

($\supseteq$) Let $\circ \in \{\,{}^{\mathsf{s}}\|_\Gamma\,, {}^{\mathsf{a}}\|_\Gamma\,\}$. Consider an expression of the form $\alpha = \alpha_1 \circ \alpha_2$, and suppose that there exists $q = (q_1, q_2) \in \mathsf{Loc}(\alpha_1 \circ \alpha_2)$ such that $(\sigma, q) \in \mathsf{Follow}(\alpha_1 \circ \alpha_2, p)$. Let

$$\beta = \mathsf{d}(\alpha_1 \circ \alpha_2, (q_1, q_2)) = \mathsf{d}(\alpha_1, q_1) \circ \mathsf{d}(\alpha_2, q_2) = \beta_1 \circ \beta_2.$$

If $p = 0$, then $\mathsf{Follow}(\alpha, 0) = \mathsf{First}(\alpha)$. Thus, either both $(\sigma, q_i) \in \mathsf{First}(\alpha_i)$ for $i = 1, 2$ and $\sigma \in \Gamma$, or one of q_1 or q_2 is 0. We just consider the former case. It follows by induction that $\mathsf{d}(\alpha_i, q_i) \in \partial_\sigma(\mathsf{d}(\alpha_i, 0)) = \partial_\sigma(\alpha_i)$. Thus, $\beta \in \partial_\sigma(\alpha_1) \circ \partial_\sigma(\alpha_2) = \partial_\sigma(\alpha)$.

Now, suppose that $p = (p_1, p_2) \in \mathsf{Loc}(\alpha_1 \circ \alpha_2)$, i.e., $p_i \in \mathsf{Loc}_0(\alpha_i)$ for $i = 1, 2$. First, let $\sigma \notin \Gamma$ and $q_1 = p_1$, $(\sigma, q_2) \in \mathsf{Follow}(\alpha_2, p_2)$ (the case $q_2 = p_2$ is identical). By induction, $\beta_2 \in \partial_\sigma(\mathsf{d}(\alpha_2, p_2))$. Thus,

$$\begin{aligned}\beta &= \mathsf{d}(\alpha_1, p_1) \circ \beta_2 \in \{\mathsf{d}(\alpha_1, p_1)\} \circ \partial_\sigma(\mathsf{d}(\alpha_2, p_2))\\ &\subseteq \partial_\sigma(\mathsf{d}(\alpha_1, p_1) \circ \mathsf{d}(\alpha_2, p_2)) = \partial_\sigma(\mathsf{d}(\alpha_1 \circ \alpha_2, p)).\end{aligned}$$

Now, let $\sigma \in \Gamma$ and $(\sigma, q_i) \in \mathsf{Follow}(\alpha_i, p_i)$, for $i = 1, 2$. By induction, $\beta_i \in \partial_\sigma(\mathsf{d}(\alpha_i, p_i))$ and thus,

$$\begin{aligned}\beta &= \beta_1 \circ \beta_2 \in \partial_\sigma(\mathsf{d}(\alpha_1, p_1)) \circ \partial_\sigma(\mathsf{d}(\alpha_2, p_2))\\ &\subseteq \partial_\sigma(\mathsf{d}(\alpha_1, p_1) \circ \mathsf{d}(\alpha_2, p_2)) = \partial_\sigma(\mathsf{d}(\alpha_1 \circ \alpha_2, p)).\end{aligned}$$

For the operator $\circ = {}^{\mathsf{a}}\|_\Gamma$ we also need to consider the case $q = (p_1, q_2)$ and $(\sigma, q_2) \in \mathsf{Follow}(\alpha_2, p_2)$ (as well as the case $q = (q_1, p_2)$). In these cases the proof is identical to the one above for $\circ$ and $\sigma \notin \Gamma$.

Finally, let $\alpha = \alpha_1 \,{}^{\mathsf{w}}\|_\Gamma\, \alpha_2$. First, we consider $p = 0$ and $(\sigma, q) \in \mathsf{Follow}(\alpha, 0) = \mathsf{First}(\alpha)$. If $\sigma \notin \Gamma$ and $q = (q_1^{\emptyset}, 0^{\emptyset})$, with $(\sigma, q_1) \in \mathsf{First}(\alpha_1) = \mathsf{Follow}(\alpha_1, 0)$, then it follows by induction that $\mathsf{d}(\alpha_1, q_1) \in \partial_\sigma(\mathsf{d}(\alpha_1, 0)) = \partial_\sigma(\alpha_1)$. Thus,

$$\begin{aligned}\mathsf{d}(\alpha_1 \,{}^{\mathsf{w}}\|_\Gamma\, \alpha_2, (q_1^{\emptyset}, 0^{\emptyset})) &= \mathsf{d}(\alpha_1, q_1) \,\|_{(\Gamma,\emptyset,\emptyset)} \mathsf{d}(\alpha_2, 0) = \mathsf{d}(\alpha_1, q_1) \,\|_{(\Gamma,\emptyset,\emptyset)} \alpha_2\\ &\in \partial_\sigma(\alpha_1) \,\|_{(\Gamma,\emptyset,\emptyset)} \{\alpha_2\} \subseteq \partial_\sigma(\alpha).\end{aligned}$$

The remaining case is analogous. If $\sigma \in \Gamma$ and $q = (q_1^{\emptyset}, q_2^{\emptyset})$, with $(\sigma, q_i) \in \mathsf{First}(\alpha_i) = \mathsf{Follow}(\alpha_i, 0)$ for $i = 1, 2$, then by induction $\mathsf{d}(\alpha_i, q_i) \in \partial_\sigma(\mathsf{d}(\alpha_i, 0)) = \partial_\sigma(\alpha_i)$. Thus,

$$\begin{aligned}\mathsf{d}(\alpha_1 \,{}^{\mathsf{w}}\|_\Gamma\, \alpha_2, (q_1^{\emptyset}, q_2^{\emptyset})) &= \mathsf{d}(\alpha_1, q_1) \,\|_{(\Gamma,\emptyset,\emptyset)} \mathsf{d}(\alpha_2, q_2)\\ &\in \partial_\sigma(\alpha_1) \,\|_{(\Gamma,\emptyset,\emptyset)} \partial_\sigma(\alpha_2) \subseteq \partial_\sigma(\alpha) = \partial_\sigma(\mathsf{d}(\alpha, 0)).\end{aligned}$$

If $\sigma \in \Gamma$ and $q = (q_1^{\{\sigma\}}, 0^{\emptyset})$, with $(\sigma, q_1) \in \mathsf{First}(\alpha_1) = \mathsf{Follow}(\alpha_1, 0)$, then by induction $\mathsf{d}(\alpha_1, q_1) \in \partial_\sigma(\mathsf{d}(\alpha_1, 0)) = \partial_\sigma(\alpha_1)$. Thus,

$$\begin{aligned}\mathsf{d}(\alpha_1 \,{}^{\mathsf{w}}\|_\Gamma\, \alpha_2, (q_1^{\{\sigma\}}, 0^{\emptyset})) &= \mathsf{d}(\alpha_1, q_1) \,\|_{(\Gamma,\{\sigma\},\emptyset)} \mathsf{d}(\alpha_2, 0)\\ &\in \partial_\sigma(\alpha_1) \,\|_{(\Gamma,\{\sigma\},\emptyset)} \{\alpha_2\} \subseteq \partial_\sigma(\alpha) = \partial_\sigma(\mathsf{d}(\alpha, 0)).\end{aligned}$$

The remaining case is analogous.

Now, consider $p = (p_1^\Delta, p_2^\Lambda)$ and $(\sigma, q) \in \mathsf{Follow}(\alpha, p)$. If $\sigma \notin \Gamma$, $q = (q_1^\Delta, p_2^\Lambda)$ and $(\sigma, q_1) \in \mathsf{Follow}(\alpha_1, p_1)$, then by induction $\mathsf{d}(\alpha_1, q_1) \in \partial_\sigma(\mathsf{d}(\alpha_1, p_1))$. Thus,

$$\begin{aligned}\mathsf{d}(\alpha_1 {}^{\mathsf{w}}\|_\Gamma\, \alpha_2, (q_1^\Delta, p_2^\Lambda)) &= \mathsf{d}(\alpha_1, q_1) \,\|_{(\Gamma,\Delta,\Lambda)} \mathsf{d}(\alpha_2, p_2) \\ &\in \partial_\sigma(\mathsf{d}(\alpha_1, p_1)) \,\|_{(\Gamma,\Delta,\Lambda)} \mathsf{d}(\alpha_2, p_2) \subseteq \\ &\partial_\sigma(\mathsf{d}(\alpha_1, p_1) \,\|_{(\Gamma,\Delta,\Lambda)} \mathsf{d}(\alpha_2, p_2)) = \partial_\sigma(\mathsf{d}(\alpha, (p_1^\Delta, p_2^\Lambda)).\end{aligned}$$

The remaining case is analogous. If $\sigma \in \Gamma$ and $q = (q_1^\emptyset, q_2^\emptyset)$, with $(\sigma, q_i) \in \mathsf{Follow}(\alpha_i, p_i)$ then by induction $\mathsf{d}(\alpha_i, q_i) \in \partial_\sigma(\mathsf{d}(\alpha_i, p_i))$, for $i = 1, 2$. Thus,

$$\begin{aligned}\mathsf{d}(\alpha_1 {}^{\mathsf{w}}\|_\Gamma\, \alpha_2, (q_1^\emptyset, q_2^\emptyset)) &= \mathsf{d}(\alpha_1, q_1) \,\|_{(\Gamma,\emptyset,\emptyset)} \mathsf{d}(\alpha_2, q_2) \\ &\in \partial_\sigma(\mathsf{d}(\alpha_1, p_1)) \,\|_{(\Gamma,\emptyset,\emptyset)} \partial_\sigma(\mathsf{d}(\alpha_2, p_2)) \subseteq \partial_\sigma(\mathsf{d}(\alpha, (p_1^\Delta, p_2^\Lambda))).\end{aligned}$$

If $\sigma \in \Gamma$ and $q = (q_1^{\Delta\cup\{\sigma\}}, p_2^\Lambda)$, with $(\sigma, q_1) \in \mathsf{Follow}(\alpha_1, p_1)$ and $\sigma \notin \Lambda$, then by induction $\mathsf{d}(\alpha_1, q_1) \in \partial_\sigma(\mathsf{d}(\alpha_1, p_1))$. Thus,

$$\begin{aligned}\mathsf{d}(\alpha_1 {}^{\mathsf{w}}\|_\Gamma\, \alpha_2, (q_1^{\Delta\cup\{\sigma\}}, p_2^\Lambda)) &= \mathsf{d}(\alpha_1, q_1) \,\|_{(\Gamma,\Delta\cup\{\sigma\},\Lambda)} \mathsf{d}(\alpha_2, p_2) \\ &\in \partial_\sigma(\mathsf{d}(\alpha_1, p_1)) \,\|_{(\Gamma,\Delta\cup\{\sigma\},\Lambda)} \{\alpha_2\} \subseteq \partial_\sigma(\mathsf{d}(\alpha, (p_1^\Delta, p_2^\Lambda))).\end{aligned}$$

The remaining case is analogous.

$\square$