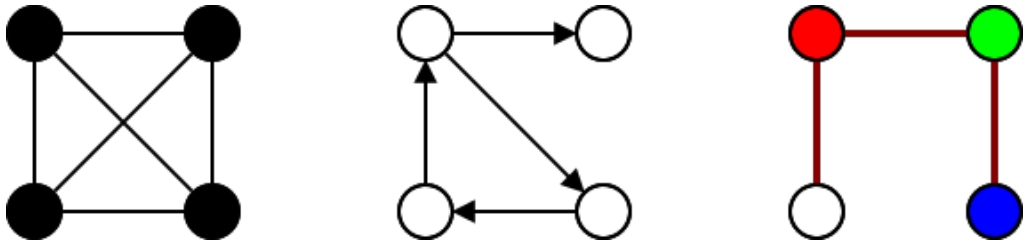
Graphs: Introduction

L.EIC

Algorithms and Data Structures

2024/2025



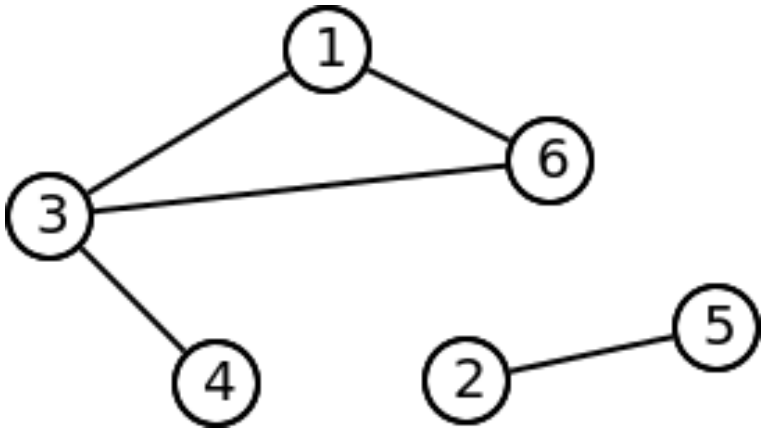
P Ribeiro, AP Tomas

Concept

Graph Definition

Formally, a **graph** is:

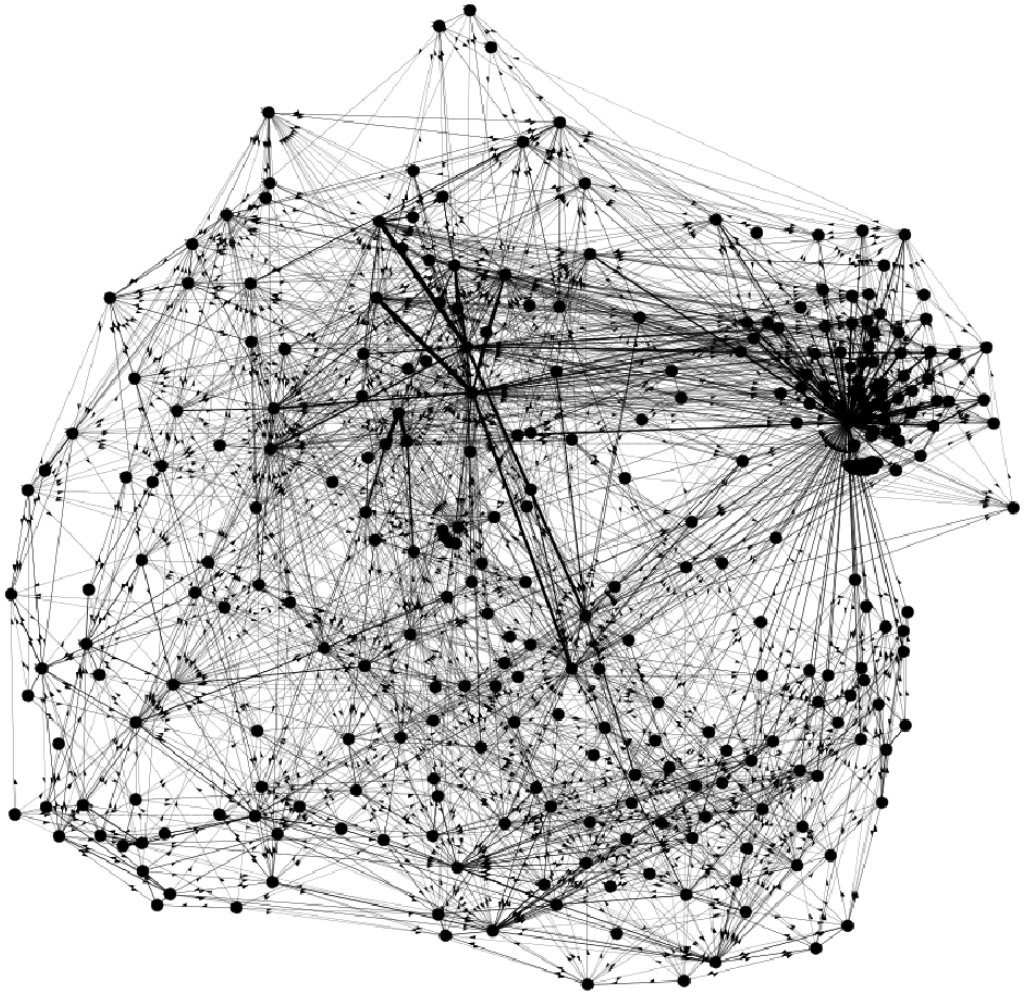
- A set of **nodes**/**vertices** (**V**).
- A set of **links**/**edges**/**connection** (**E**), that connect pairs of vertices



- $V = \{1, 2, 3, 4, 5, 6\}$
- $E = \{(1, 6), (1, 3), (3, 6), (3, 4), (2, 5)\}$

What are graphs for?

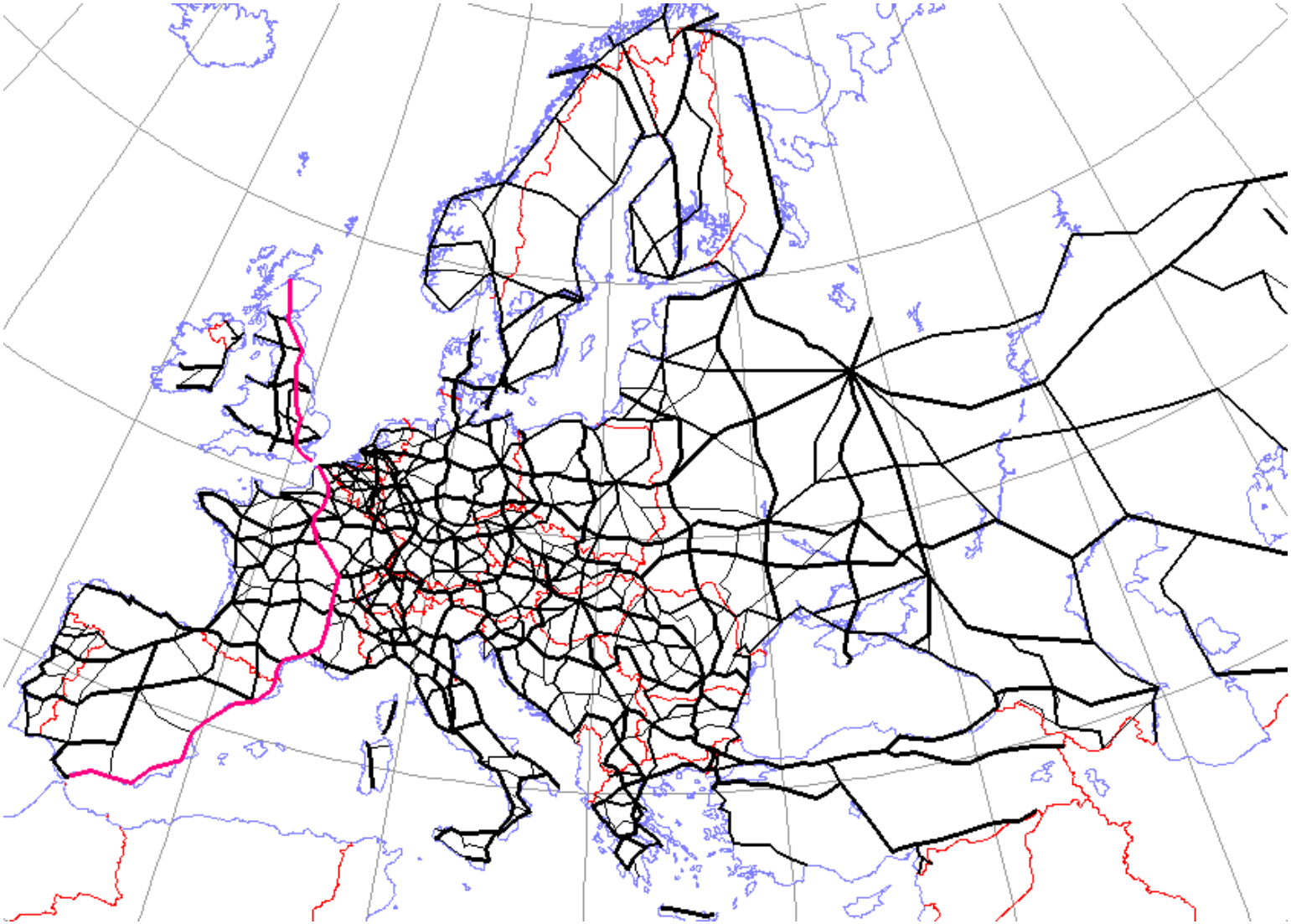
- Graphs are **ubiquitous** in Computer Science and they are present, implicitly or explicitly in many algorithms.
- They can be used in a multitude of applications.



Graph Examples

Networks that exist in the real "physical" world

- Road Network



Graph Examples

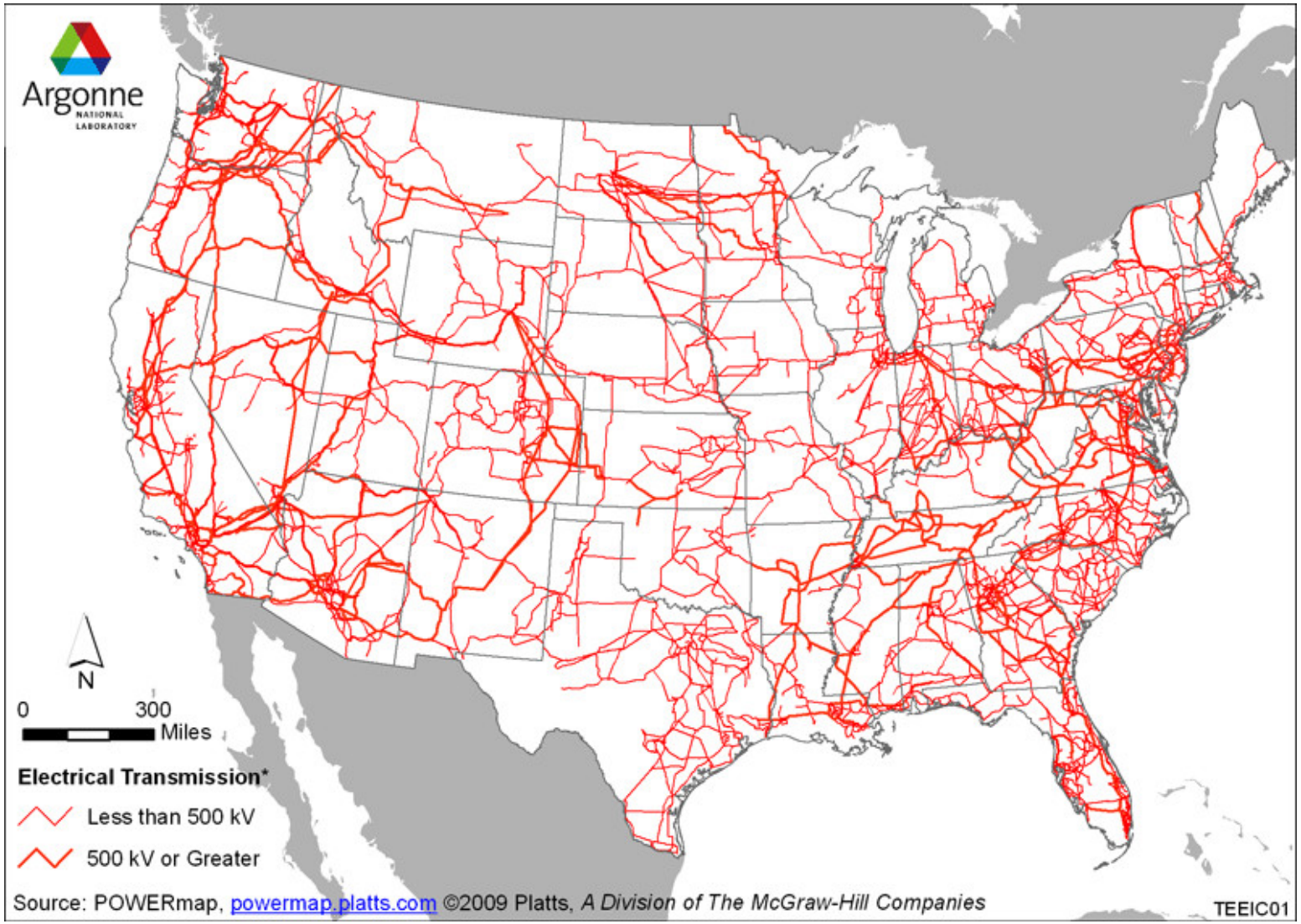
Networks that exist in the real "physical" world

- Public Transportation (ex: subway, train)

Graph Examples

Networks that exist in the real "physical" world

- Power Grid



Graph Examples

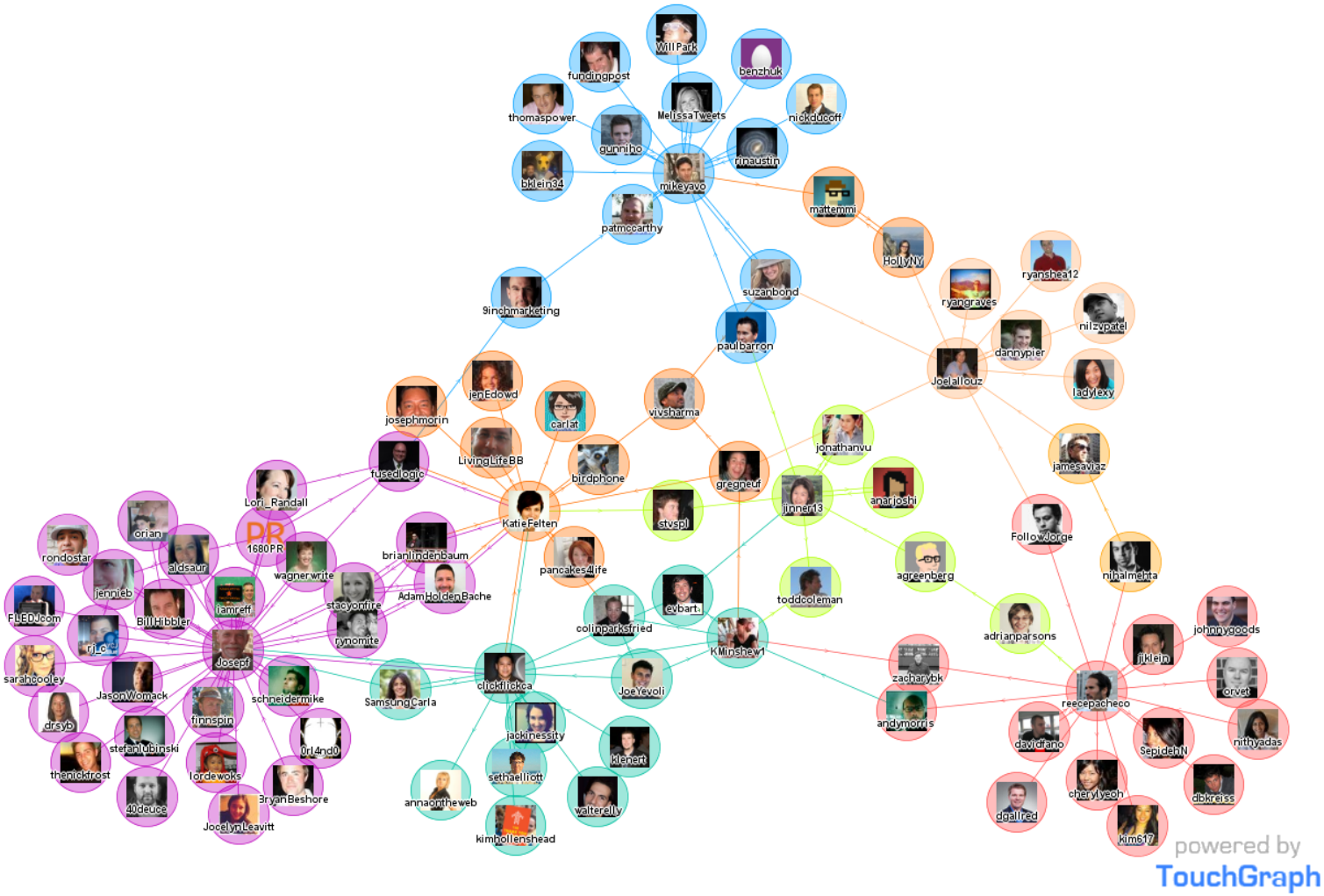
Networks that exist in the real "physical" world

- Computer Network

Graph Examples

Social Network

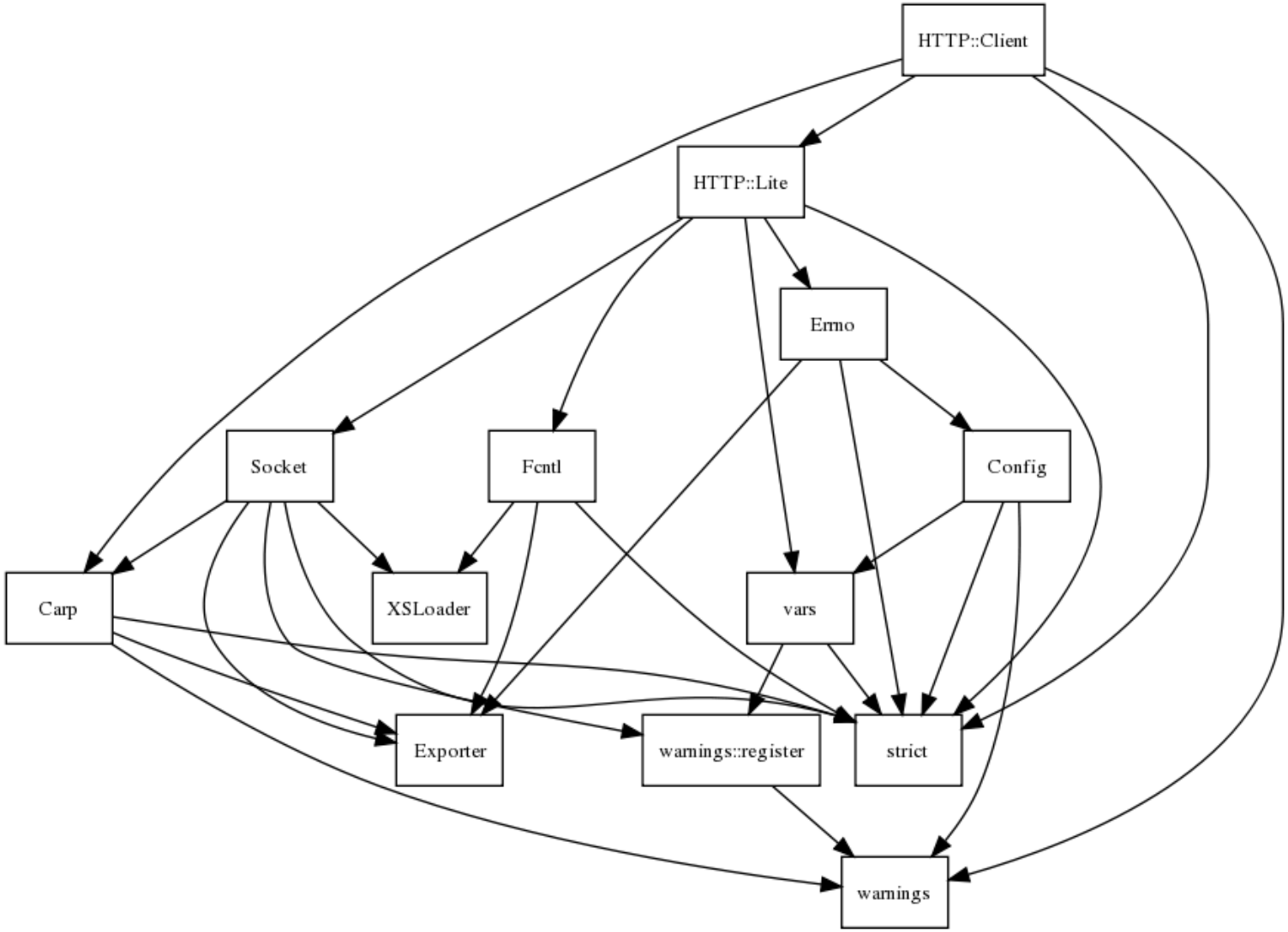
- Facebook (others: Twitter, emails, co-authorship of articles, ...)



Graph Examples

Software Networks

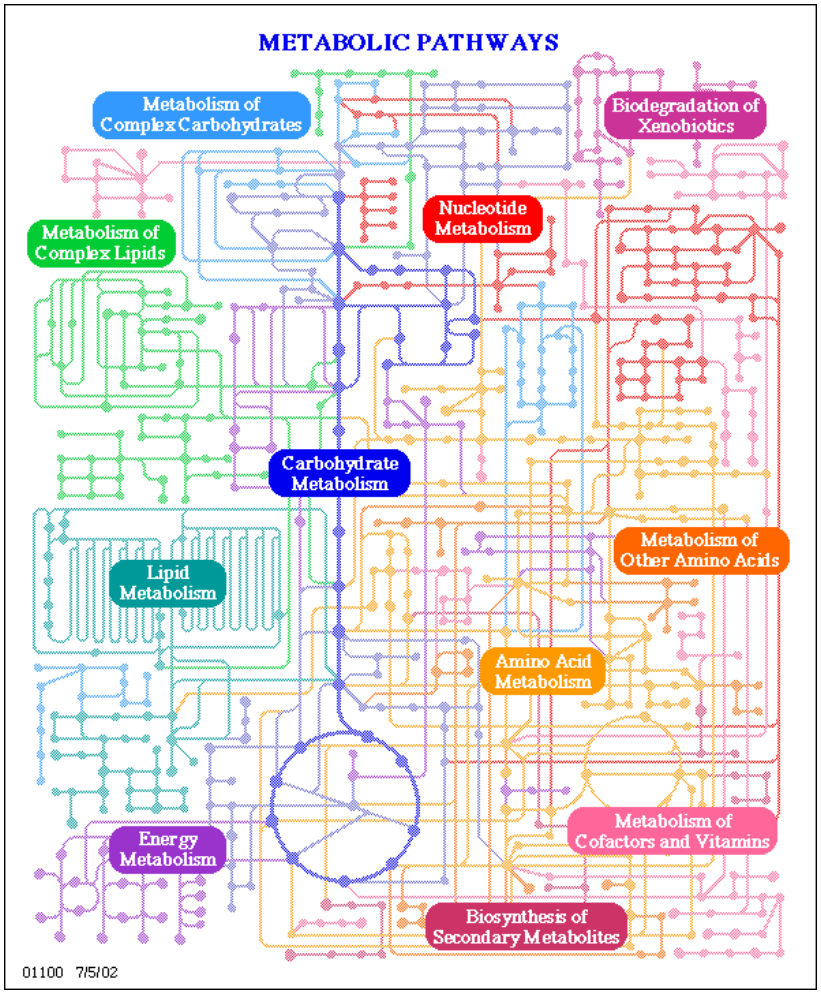
- Module Dependencies (other examples: state, information flow, ...)



Graph Examples

Biological Networks

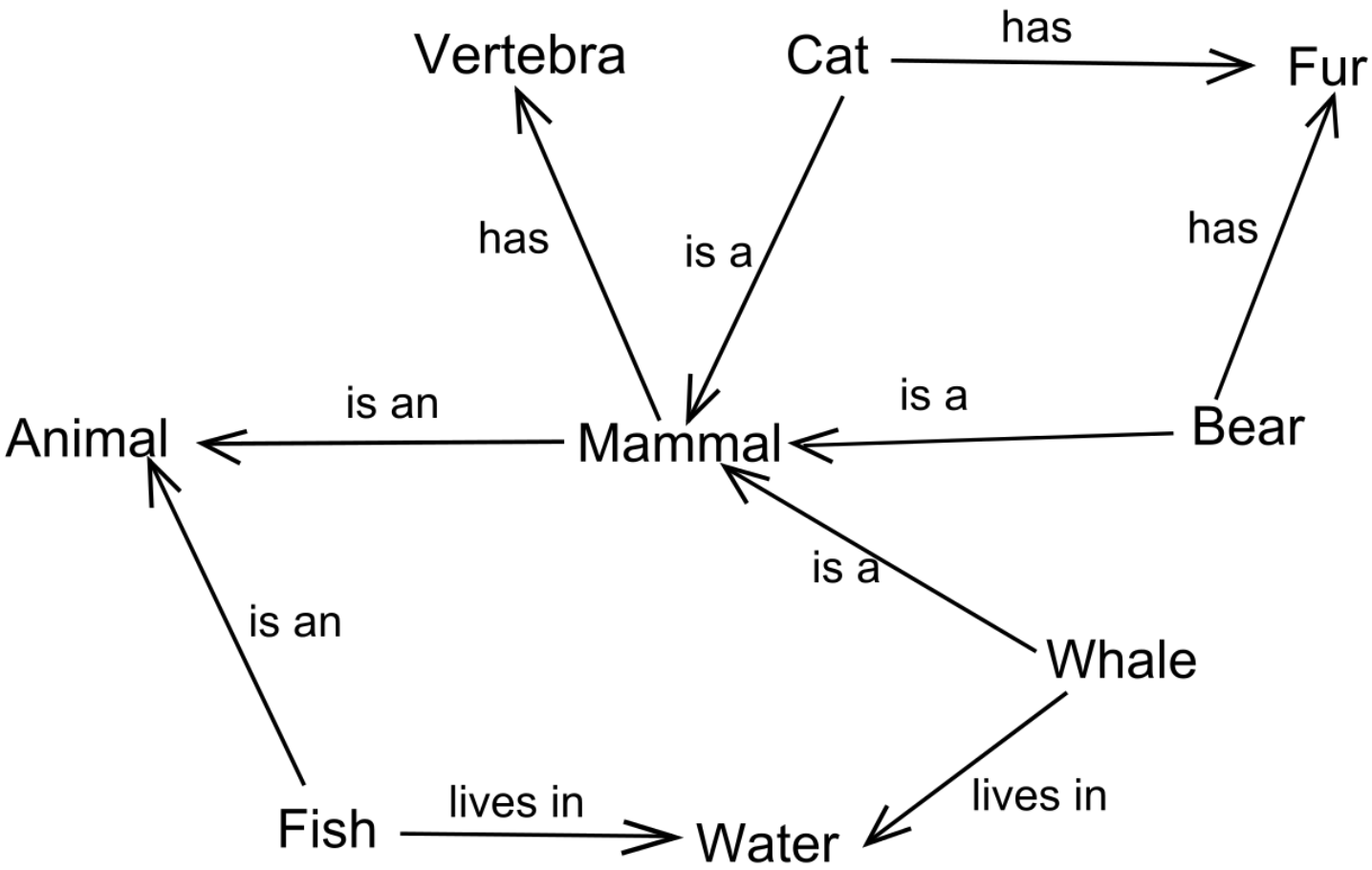
- Metabolic Networks (other examples: protein interaction, brain networks, food webs, phylogenetic trees, ...)



Graph Examples

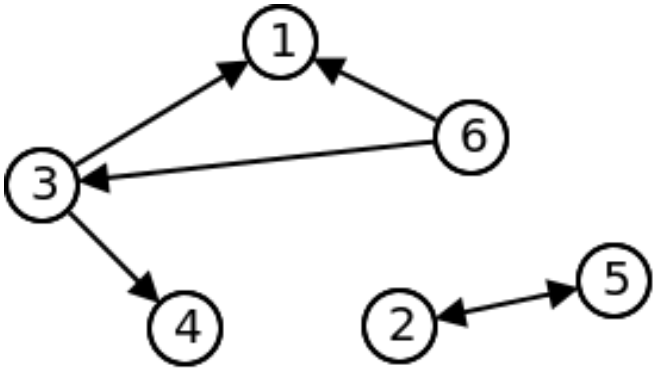
Other Graphs

- Semantic Networks (other examples: world wide web, ...)

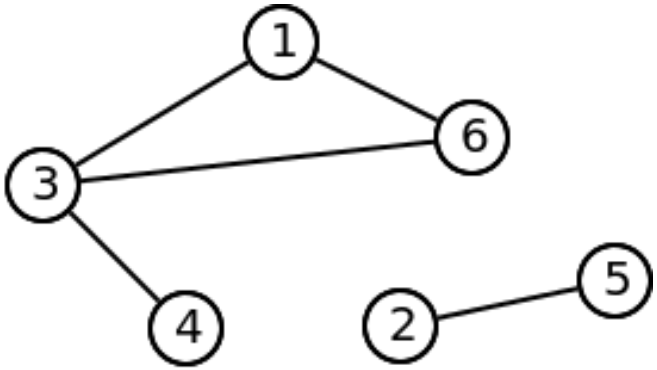


Terminology

- **Directed** graph - each link has a starting node (**origin**) and an **end** node (order matters!). Usually we use arrows to indicate the direction.
- **Undirected** graphs - There is no origin or end, but just a connection



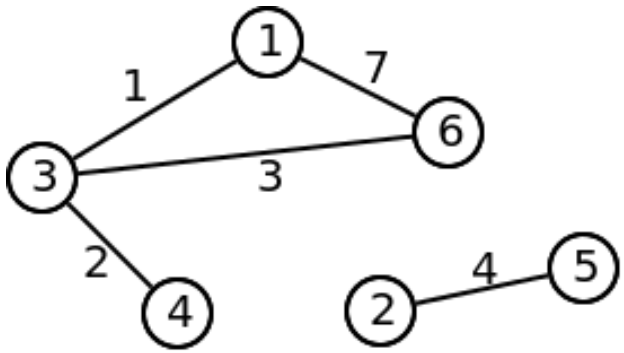
Directed Graph



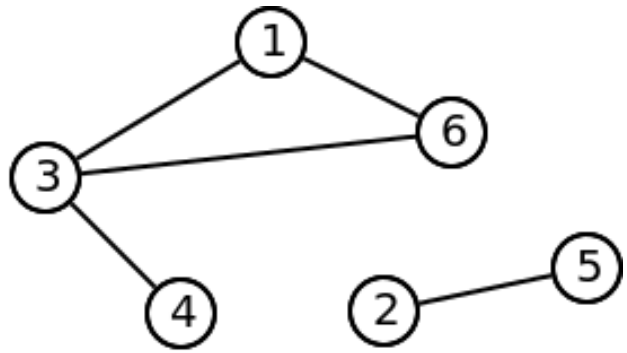
Undirected Graph

Terminology

- **Weighted** graph - there is a value associated with each link (it could be distance, cost, ...)
- **Unweighted** - there are no weights associated with a link



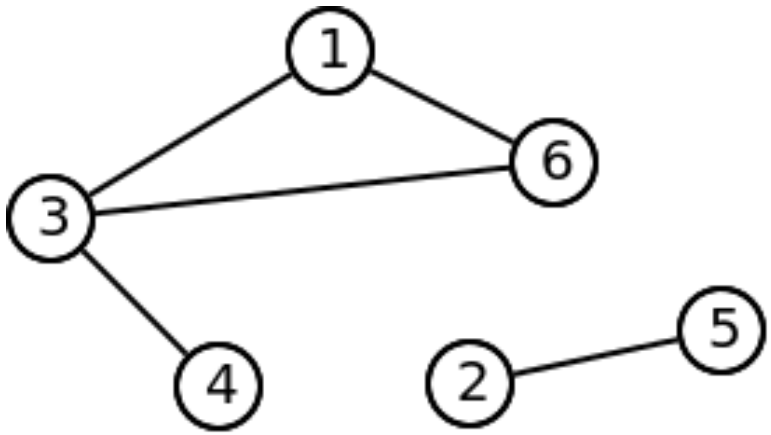
Weighted Graph



Unweighted Graph

Terminology

- **Degree** - number of connections of a node
- In directed graphs we can distinguish between **indegree** and **outdegree**



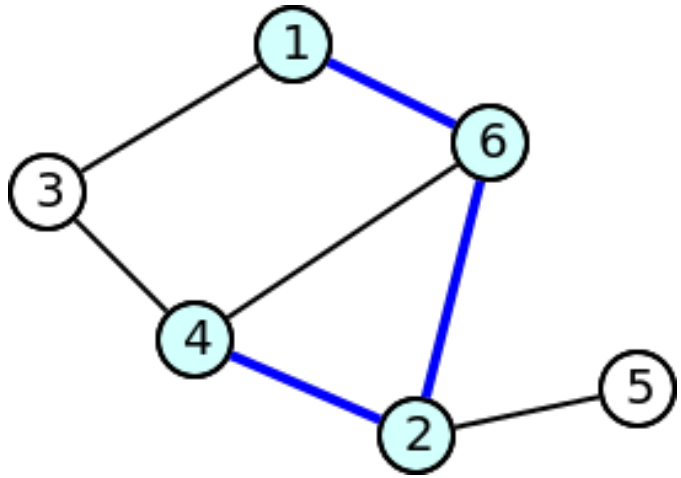
1 has degree 2
2 has degree 1
3 has degree 3
4 has degree 1
5 has degree 1
6 has degree 2

Terminology

- **Adjacent/neighbor** node: two nodes are neighbors if they are linked
- **Trivial graph**: graph with no edges and a single node
- **Self-loop**: link from a node to itself
- **Multigraph:** graph with multiple links between the same node pair
- **Simple graph:** graph without self-loops and without repeated links (we are mostly going to work with simple graphs)
- **Dense graph:** with many links when compared with the maximum possible - $|E|$ of the order of $\mathcal{O}(|V|^2)$
- **Sparse graph:** with few links when compared with the maximum possible - $|E|$ with lower order than $\mathcal{O}(|V|^2)$

Terminology

- **Path**: sequence of alternating nodes and edges, such that two consecutive nodes are linked (and respect the direction, in case of a directed graph).
- In simple graphs we typically describe a path using just the nodes.

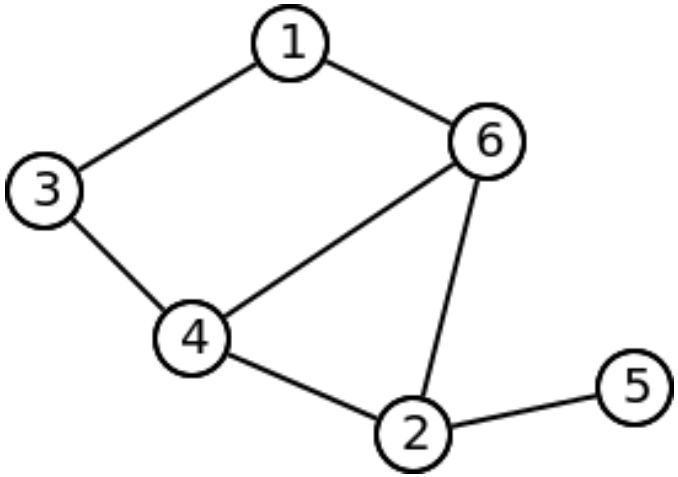


$1 \rightarrow 6 \rightarrow 2 \rightarrow 4$

- **Cycle**: path that starts and ends on the same node (ex: for the above graph, $1 \rightarrow 6 \rightarrow 4 \rightarrow 3 \rightarrow 1$ is a cycle)
- **Acyclic graph**: graph without cycles
- **DAG**: directed acyclic graph

Terminology

- **Size** of a path: number of edges in the path
- **Cost** of a path: if the graph is weighted, we can talk about the cost, which is the sum of the edge weights
- **Distance**: size/cost of the minimum path between two nodes
- **Diameter** of a graph: max distance between two nodes of a graph



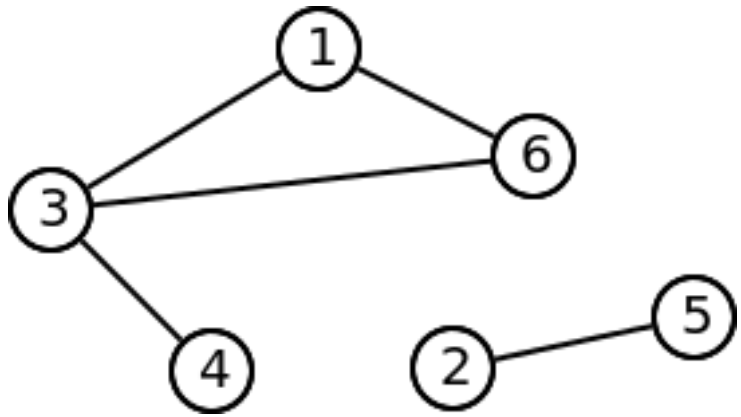
Diameter = 3

	1	**2**	**3**	**4**	**5**	**6**
1	0	2	1	2	3	1
2	2	0	2	1	1	1
3	1	2	0	1	3	2
4	2	1	1	0	2	1
5	3	1	3	2	0	2
6	1	1	2	1	2	0

Distances between nodes

Terminology

- **Connected Component**: Subset of nodes where there is at least one path between each of them
- **Connected Graph**: Graph with just one connected component (there is a path between all pairs of nodes)

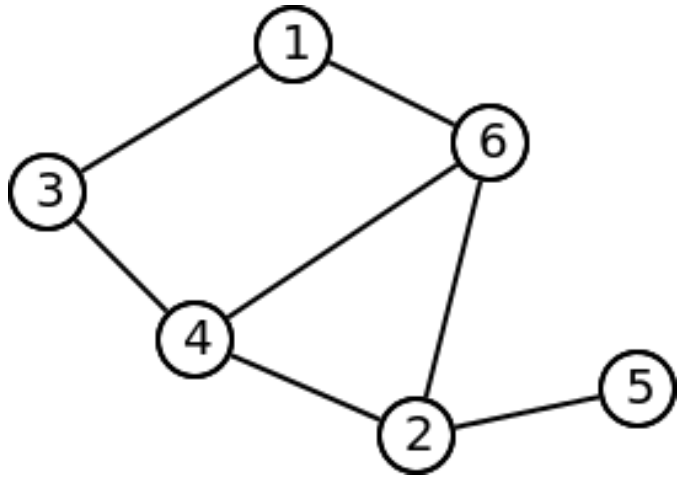


Graph with two connected components: $\{1, 3, 4, 6\}$ e $\{2, 5\}$

(this drawing is from an undirected graph; for directed graphs we can talk about **strongly connected components** - we will talk about them later)

Terminology

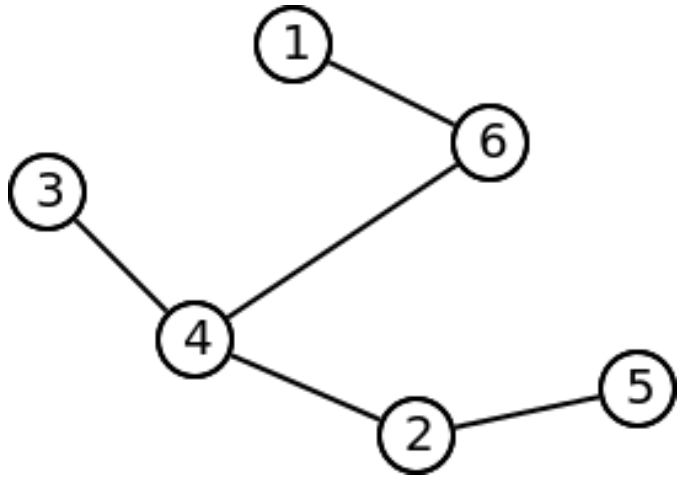
- **Subgraph**: subset of nodes and the edges between them
- **Complete** graph: with links between all pairs of nodes
- **Clique**: a complete subgraph
- **Triangle**: a clique with 3 nodes



Subgraph examples: $\{1, 3\}$, $\{1, 6, 2\}$, $\{2, 4, 5, 6\}$, etc
Example clique: $\{2, 4, 6\}$ (a triangle)

Terminology

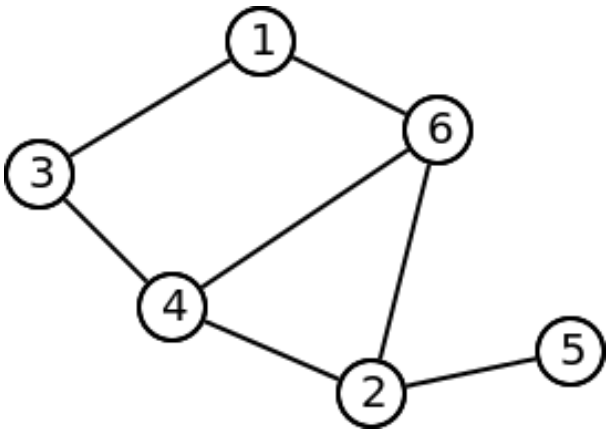
- **Tree**: simple, connected acyclic graph
  (if it has n nodes, then it will have $n - 1$ edges)
- **Forest**: set of multiple disconnected trees



Graph Representation

How to represent a graph?

- **Adjacency Matrix**: $|V| \times |V|$ matrix where the (i, j) cell indicates if there is a link between nodes i and j *(an undirected graph has a symmetric matrix)*; if the graph is weighted we can store the weight
- **Adjacency list**: each node stores a list of its neighbors; if the graph is weighted we have to store pairs (destination,weight)



	1	2	3	4	5	6
1			X			X
2				X	X	X
3	X			X		
4		X	X			X
5		X				
6	X	X		X		

Adjacency Matrix

(e.g. bool[][])

1: 3 → 6
2: 4 → 5 → 6
3: 1 → 4
4: 2 → 3 → 6
5: 2
6: 1 → 2 → 4

Adjacency List

(e.g. vector<list>)

Graph Representation

Some pros and cons:

- **Adjacency Matrix**:
  - Very simple to implement
  - Quick to check if there is a connection between two nodes - $\mathcal{O}(1)$
  - Slow to traverse the neighbors - $\mathcal{O}(|V|)$
  - Lots of memory wasted (in sparse graphs) - $\mathcal{O}(|V|^2)$
  - Weighted graph implies simply to store the weight in the matrix
  - Adding/Removing edges is simply changing a cell - $\mathcal{O}(1)$

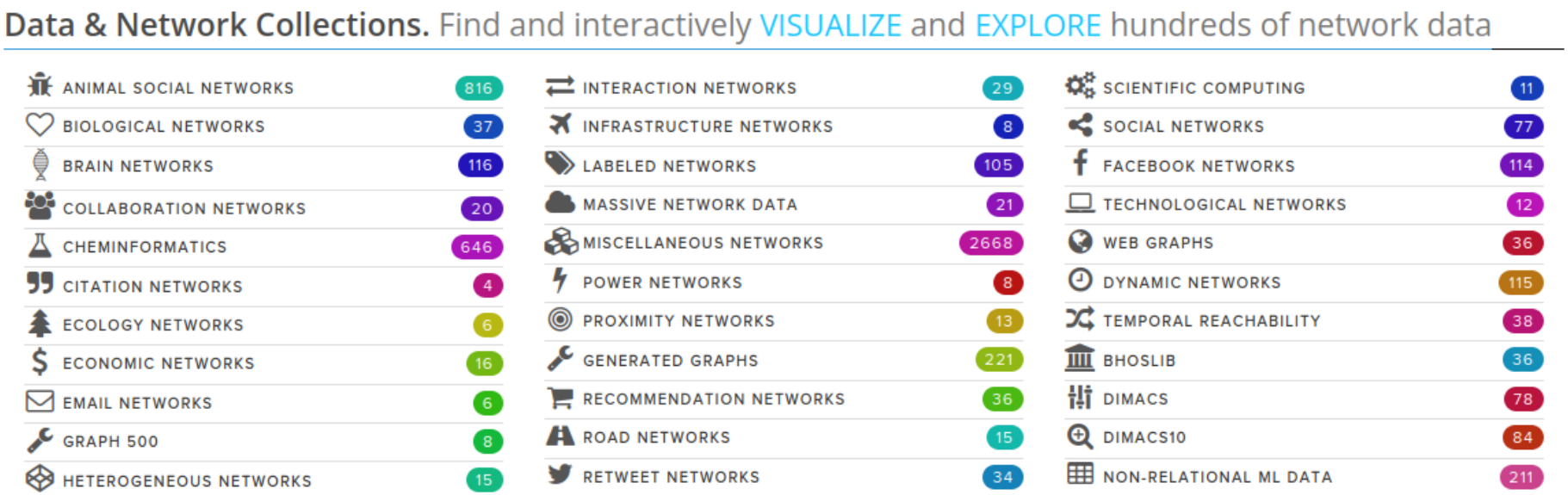
- **Adjacency List**:
  - Slow to see if there is a link between u and v - $\mathcal{O}(\text{degree(u)})$
  - Quick to traverse the neighbors - $\mathcal{O}(\text{degree(u)})$
  - Efficient usage of memory - $\mathcal{O}(|V| + |E|)$
  - Weighted graph implies adding an attribute to the list
  - Removing edge (u, v) implies traversing the list - $\mathcal{O}(\text{degree(u)})$

    Note: we can use for instante BSTs (set/map) to improve the efficiency of searching and removing to $\mathcal{O}(\log \mathit{degree}(u))$

Graph datasets

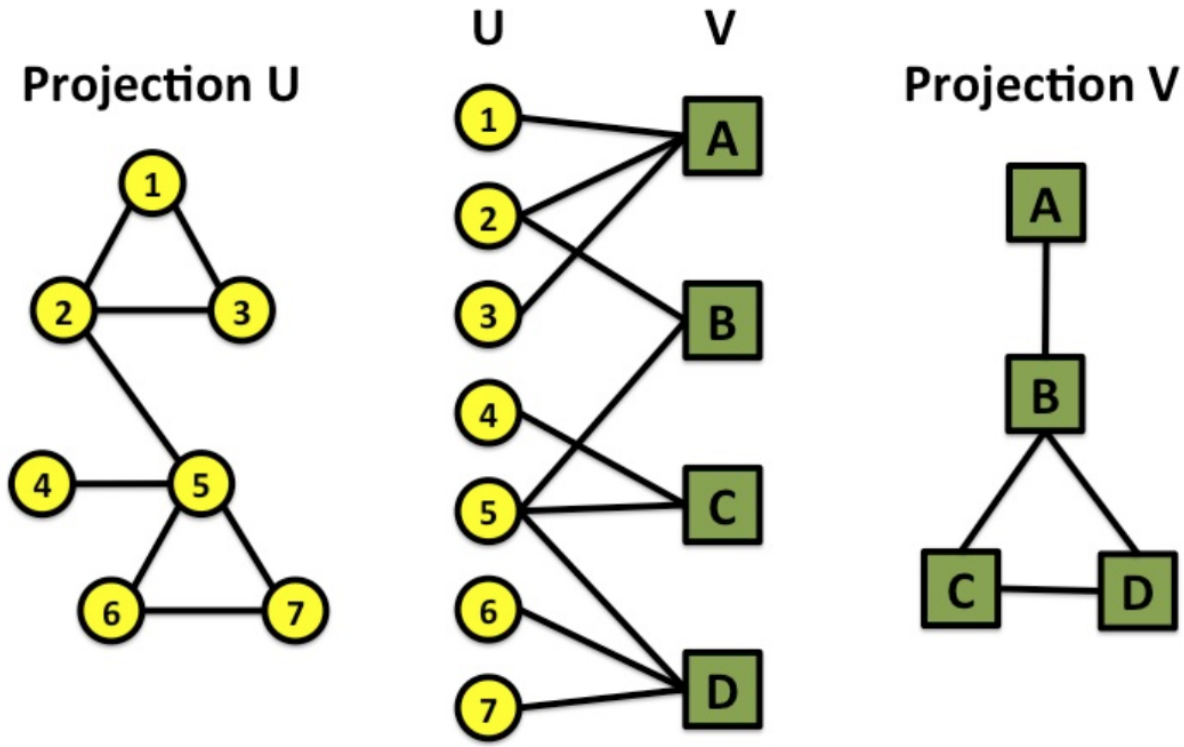
Here are some interesting websites with graphs:

- **Network Repository:** http://networkrepository.com/
- **Konect:** http://konect.cc/
- **SNAP:** https://snap.stanford.edu/data/



Bipartite Graphs

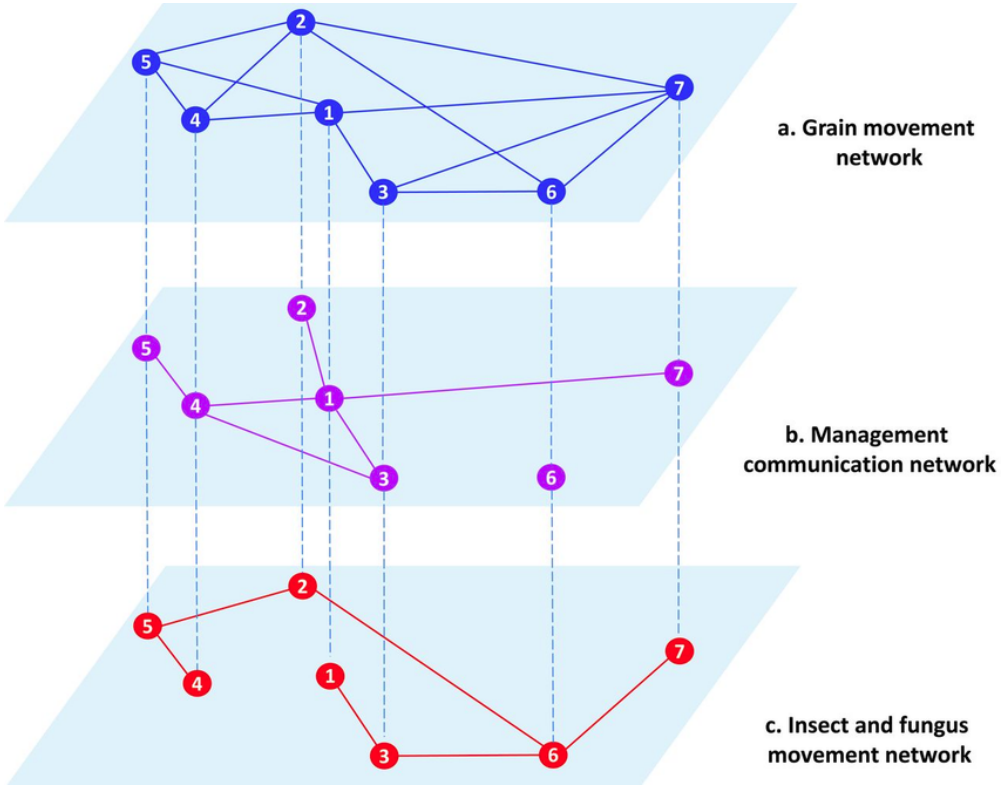
- A **bipartite graph** is a graph whose nodes can be divided into two disjoint sets U and V such that every edge connects a node in U to one in V



- Many (real world) networks come from projections (ex: actors and movies, diseases and genes)

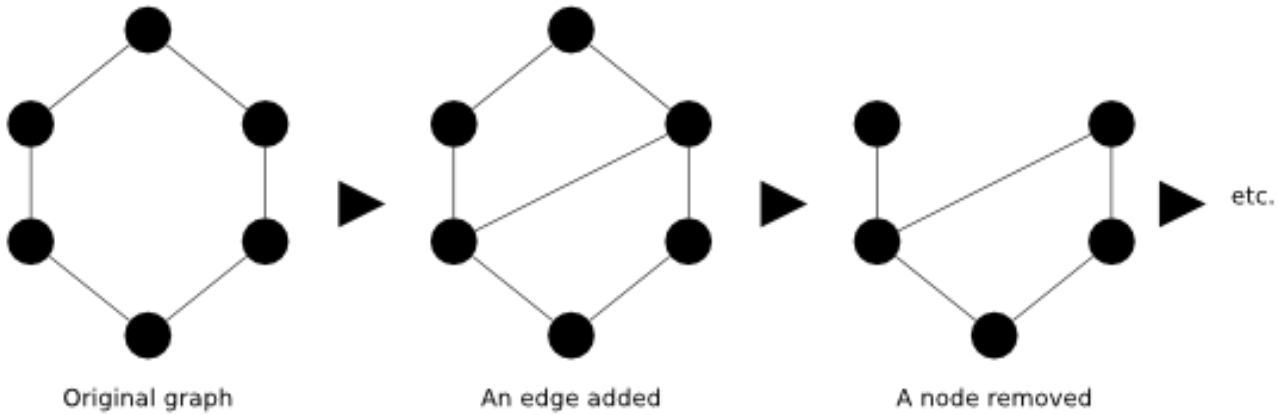
Other Graph Types: Multilayer / Multiplex

- Graphs can have different layers



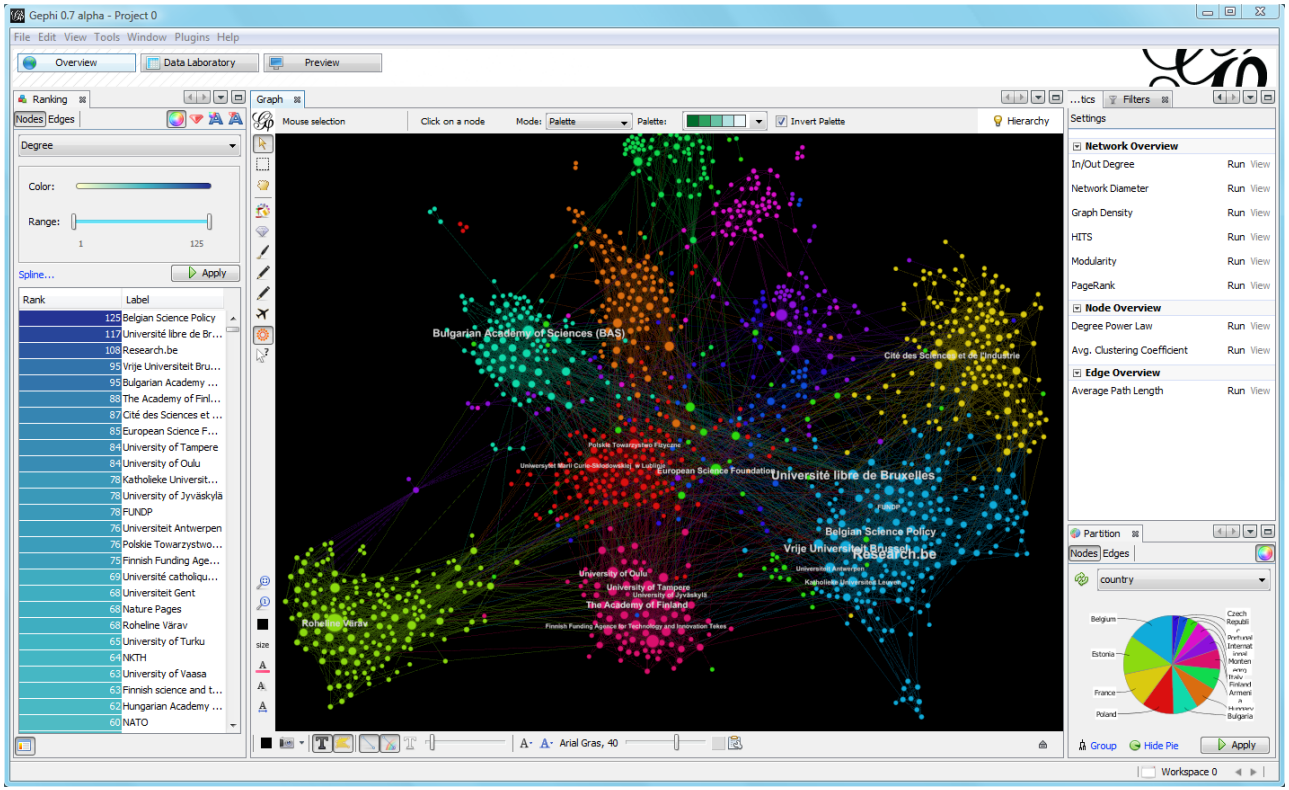
Other Graph Types: Temporal Networks

- Graphs can evolve over time



Extra: Graph Software (not needed for this course)

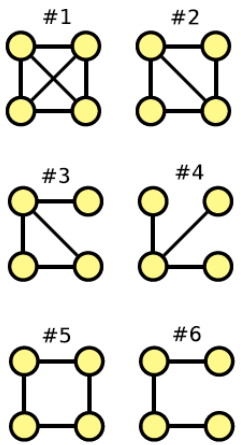
- Software such as **Gephi** or **Cytoscape** would allow you to visualize (and interact) with graphs

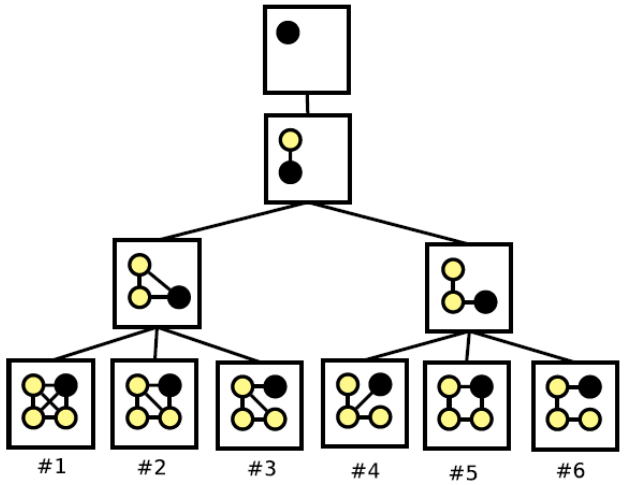


- Software packages such as **networkx** (Python) or **igraph** (C/C++, R, Python, Mathematica) provide graph implementations and many metrics and algorithms (they are *Network Science* packages)

Network Science / Graph Mining

My main research area





PhD Thesis (2011):

Efficient and Scalable Algorithms for Network Motifs Discovery

Publications: [personal website] [google scholar]

(and for instance a **Network Science course** at masters level)