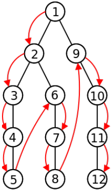
Graphs: Depth-First Search (DFS)

L.EIC

Algorithms and Data Structures

2025/2026



P Ribeiro, AP Tomas

Graph Search

- One of the most important tasks it to **traverse** a graph, that is, **passing through all the nodes** using the **connections between them**

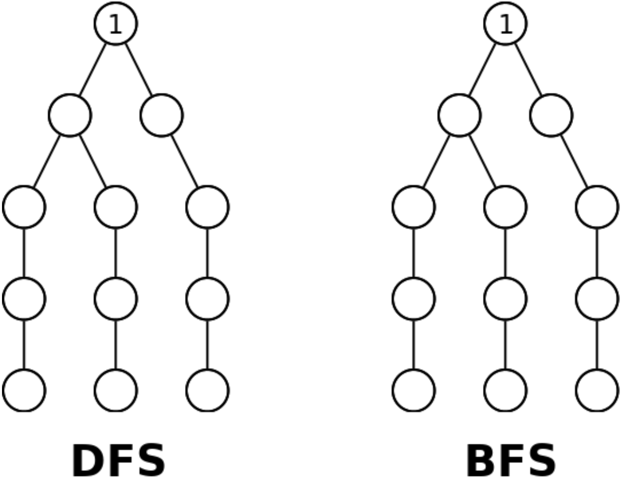
- This is known as a **graph search** (or **graph traversal**)

- There are two fundamental graph search methodologies that vary on **order in which they traverse the nodes**:

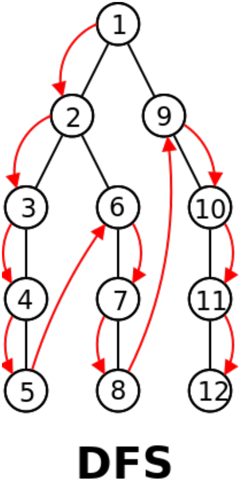
  - **Depth-First Search - DFS**
    Traverse the entire subgraph connected to a neighbor before entering the next neighbor node

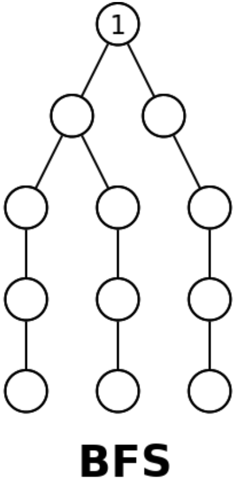
  - **Breadth-First Search - BFS**
    Traverse the nodes by increasing distance of number of links to reach them

Graph Search

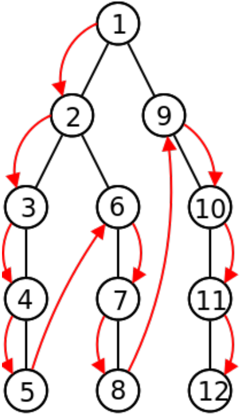


Graph Search

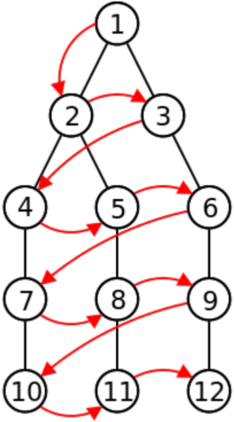




Graph Search



DFS



BFS

Graph Search

- In their essence, DFS and BFS do the "same": **traverse all the nodes**
- When to use one or the other depends on the **order that betters suits the problem** that you are solving
- We will show how to **implement** both and give examples of applications

Representing Graphs in C++

- To implement graph search we first need to **represent a graph**
- There is **no graph data structure** in the C++ standard
  - This is mainly due to the flexibility and different variations of a graph (adj. list/matrix, node labeling, undirected/directed, weighted/unweighted, nodes/edge colors, temporal graphs, etc)
  - A *"one size fits them all"* approach would have too much overhead (a custom graph class can be much more suited for a specific situation)
  - In case you are curious there is Boost Graph Library (BGL)

Representing Graphs in C++

- For the purposes of this class we will (mainly) be using a **simple graph class** to offer some **reusability**:
  - The class should be very lightweight, with only the essentials (add what's needed for a problem, focus more on the algorithms)
  - Support for simple graphs only (no self-loops or parallel edges)
  - Support directed/undirected and weighted/unweighted graphs
  - We will use adjacency lists as the edges representation
  - We will assume nodes are labeled from 1 to $|V|$

(note: *as explained before, we could have a .h with declarations and a .cpp with implementation; here we will use a single .h file to simplify code submissions*)

A simple lightweight Graph class

- A simple lightweight graph class: `graph.h`

```
class Graph {
  struct Edge {
    int dest;   // Destination node
    int weight; // An integer weight
  };

  struct Node {
    std::list<Edge> adj; // The list of outgoing edges (to adjacent nodes)
  };

  int n;                     // Graph size (vertices are numbered from 1 to n)
  bool hasDir;               // false: undirected; true: directed
  std::vector<Node> nodes; // The list of nodes being represented

public:
  // Constructor: nr nodes and direction (default: undirected)
  Graph(int nodes, bool dir = false) { ... }

  // Add edge from source to destination with a certain weight
  void addEdge(int src, int dest, int weight = 1) { ... }
};
```

A simple lightweight Graph class

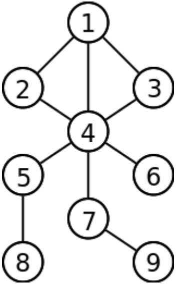
- The two included methods:

```
Graph(int num, bool dir = false) : n(num), hasDir(dir), nodes(num+1) {}

void addEdge(int src, int dest, int weight = 1) {
  if (src<1 || src>n || dest<1 || dest>n) return;
  nodes[src].adj.push_back({dest, weight});
  if (!hasDir) nodes[dest].adj.push_back({src, weight});
}
```

- Example usage:

```
Graph g(9, false); // 9 nodes, undirected graph
g.addEdge(1, 2);   // assuming weight=1 (unweighted)
g.addEdge(1, 3);
g.addEdge(1, 4);
g.addEdge(2, 4);
g.addEdge(3, 4);
g.addEdge(4, 5);
g.addEdge(4, 6);
g.addEdge(4, 7);
g.addEdge(5, 8);
g.addEdge(7, 9);
```



Depth-First Search - DFS

The "backbone" of a DFS:

DFS (recursive version)

dfs(node v**)**:
  mark v as visited
  For all neighbors w of v **do**
    If w has not yet been visited **then**
      dfs(w)

Complexity:

- Temporal:
  - Adjacency List: $\mathcal{O}(|V| + |E|)$
  - Adjacency Matrix: $\mathcal{O}(|V|^2)$
- Spatial: $\mathcal{O}(|V|)$

DFS: Implementation

- Let's see a possible implementation with some *livecoding*

```
// we also need to add the attribute
// 'visited' to a node
void dfs(int v) {
  std::cout << v << " "; // show nodes
  nodes[v].visited = true;
  for (auto e : nodes[v].adj) {
    int w = e.dest;
    if (!nodes[w].visited)
      dfs(w);
  }
}
```

DFS (recursive version)

dfs(node v**)**:
  mark v as visited
  For all neighbors w of v **do**
    If w has not yet been visited **then**
      dfs(w)

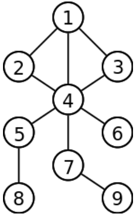
- Example execution (assuming graph g was already created)

```
g.dfs(1); // assuming nodes are unvisited before call
```

```
1 2 4 3 5 8 6 7 9
```

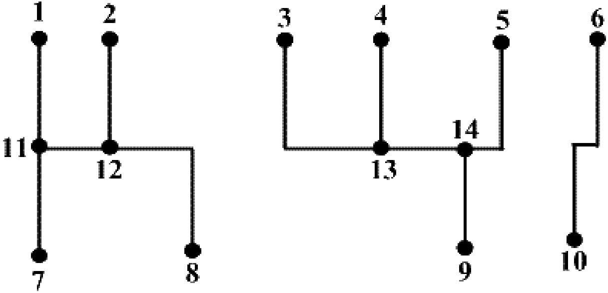
```
g.dfs(9); // assuming nodes are unvisited before call
```

```
9 7 4 1 2 3 5 8 6
```



Example Application: Connected Components

- Find the number of **connected components** of a graph G
- Example: the following graph has **3 connected components**



Example Application: Connected Components

The "backbone" of a program to solve it:

Finding connected components

counter $\leftarrow$ 0
set all nodes as **unvisited**
For all nodes v of the graph **do**
  If v has not yet been visited **then**
    counter$++$
    dfs(v)
return *counter*

Temporal complexity:

- Adjacency List: $\mathcal{O}(|V| + |E|)$
- Adjacency Matrix: $\mathcal{O}(|V|^2)$

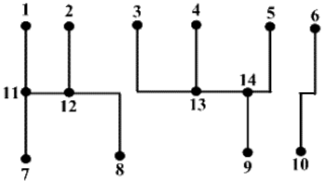
Example Application: Connected Components

- In C++ code (and some *livecoding*):

```
int connectedComponents() {
  int counter = 0;
  for (int v=1; v<=n; v++)
    nodes[v].visited = false;
  for (int v=1; v<=n; v++)
    if (!nodes[v].visited) {
      counter++;
      std::cout << "connected component: ";
      dfs(v);
      cout << endl;
    }
  return counter;
}
```

```
cout << g.connectedComponents() << endl;
```

```
connected component: 1 11 7 12 2 8
connected component: 3 13 4 14 9 5
connected component: 6 10
3
```



Implicit Graphs

- We do not always need to explicitly store the graph
- Example: find the number of "blobs" (connected spots) in a matrix. Two cells are adjacent if they are connected vertically or horizontally.

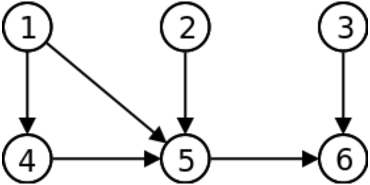
```
#.##..##                         1.22..33
#.....##                         1.....33
...##...   -->  4 blobs  -->     ...44...
...##...                         ...44...
```

- To solve we simply need to do $dfs(x, y)$ to visit the cell (x, y) where the neighbors are $(x+1, y), (x-1, y), (x, y+1)$ and $(x, y-1)$
- Using DFS to "color" the connected components is known as doing a **Flood Fill**.

Topological Sorting

- Given a DAG G (directed acyclic graph), find an order of nodes such that u comes before v if and only if there is no edge (v, u)

- Example: For the graph below a possible topological sorting would be: $1, 2, 3, 4, 5, 6$ (or $1, 4, 2, 5, 3, 6$ - there are other possible valid orders)



A classic example of application is to decide in which order to execute a set of tasks with precedences.

Topological Sorting

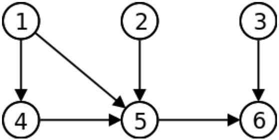
- How to solve this problem with DFS? What is the relationship between topological sorting and the DFS node order?

Topological Sorting - time: $\mathcal{O}(|V| + |E|)$ (list)

order $\leftarrow$ empty
set all nodes as **unvisited**
For all nodes v of the graph **do**
  If v has not yet been visited **then**
    dfs(v)
return *order*

dfs(node v):
  mark v as visited
  For all neighbors w of v **do**
    If w has not yet been visited **then**
      dfs(w)
  add v to the beginning of *order*

Topological Sorting



Example of execution:

- $order = \emptyset$

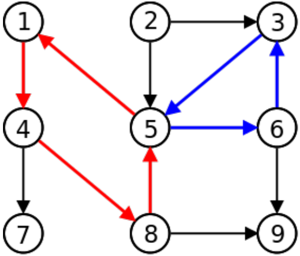
- start dfs(1) $|order = \emptyset$
- start dfs(4) $|order = \emptyset$
- start dfs(5) $|order = \emptyset$
- start dfs(6) $|order = \emptyset$
- end dfs(6) $|order = 6$
- end dfs(5) $|order = 5, 6$
- end dfs(4) $|order = 4, 5, 6$
- end dfs(1) $|order = 1, 4, 5, 6$
- start dfs(2) $|order = 1, 4, 5, 6$
- end dfs(2) $|order = 2, 1, 4, 5, 6$
- start dfs(3) $|order = 2, 1, 4, 5, 6$
- end dfs(3) $|order = 3, 2, 1, 4, 5, 6$

- $order = 3, 2, 1, 4, 5, 6$

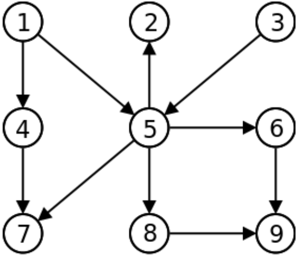
Cycle Detection

- Find if a (directed) graph G is **acyclic**

- Example: the left graph has a cycle; the right graph doesn't



Graph with cycles



Acyclic Graph

Cycle Detection

Let's use 3 "colors":

- **White** - unvisted node
- **Gray** - node being visited (we are exploring its descendants)
- **Black** - node already visited (we visited all its descendants)

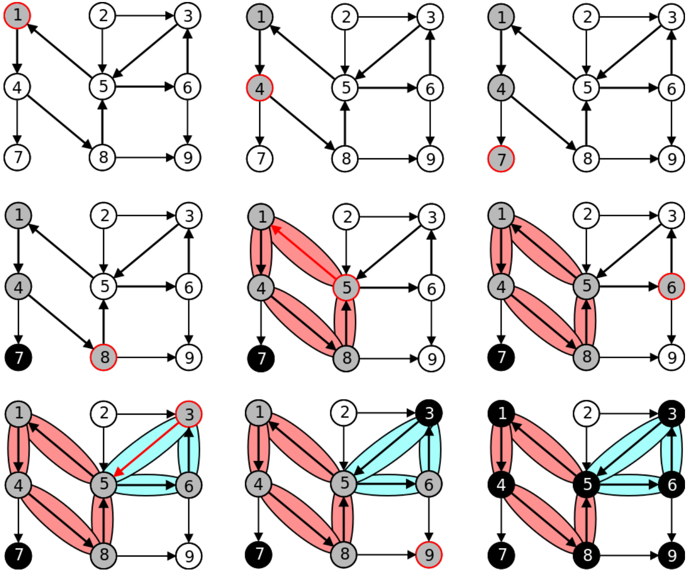
Cycle Detection - $\mathcal{O}(|V| + |E|)$ (list)

color[$v \in V$] $\leftarrow$ white
For all nodes v of the graph **do**
  If $color[v]$ = white **then**
    dfs(v)

dfs(node v):
  color[v] $\leftarrow$ gray
  For all neighbors w of v **do**
    If color[w] = gray **then**
      write("Cycle found!")
    Else if color[w] = white **then**
      dfs(w)
  color[v] $\leftarrow$ black

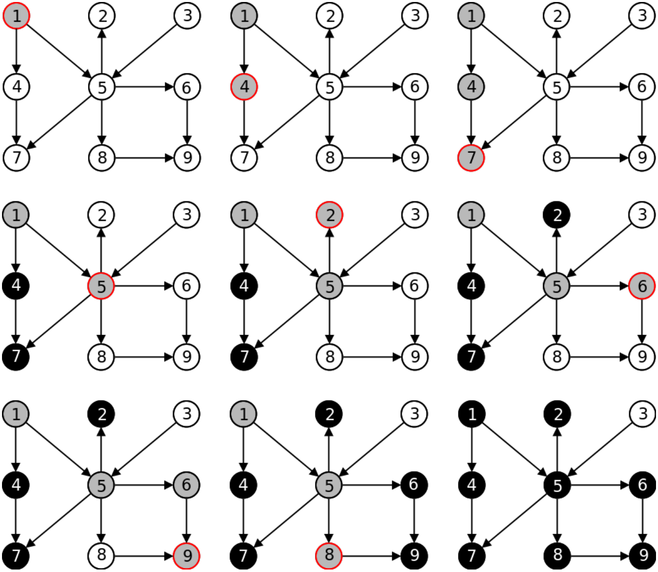
Cycle Detection

Example (starting on node 1) - graph with two cycles



Cycle Detection

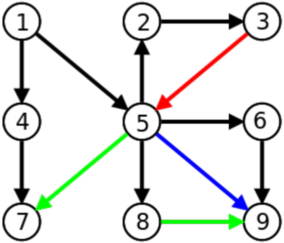
Example (starting on node 1) - acyclic graph

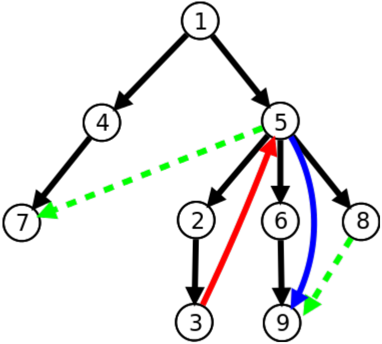


Classifying DFS Edges

Another "angle" of DFS

- A DFS visit separates the edges into 4 categories
  - **Tree Edges** - Edges from the DFS tree
  - **Back Edges** - Edge from a node to one of its tree ancestors
  - **Forward Edges** - Edge from a node to one of its tree descendants
  - **Cross Edges** - All other edges (from one branch to another)





Classifying DFS Edges

Another "angle" of DFS

- Example application: finding cycles is finding... **Back Edges**!
- Knowing the edge types may help to solve problem!
- Note: an undirected graph has only **Tree Edges** and **Back Edges**.

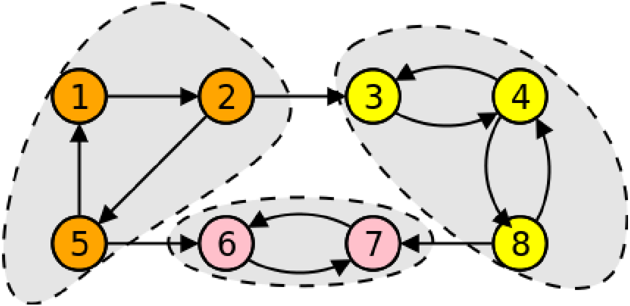
Strongly Connected Components

A more complex DFS application

- Decompose a graph into its **strongly connected components**

A **strongly connected component** (SCC) its a maximal subgraph where there is a (directed) path between each of its nodes.

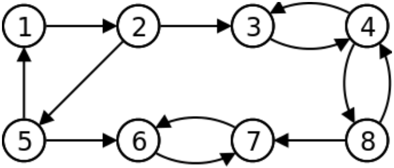
An example graph with 3 SCCs:

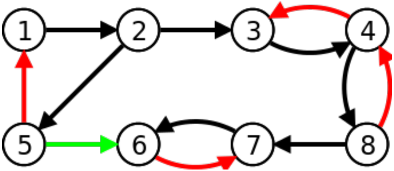


Strongly Connected Components

A more complex DFS application

- How to compute SCCs?
- Let's try to use our knowledge about DFS edge types:

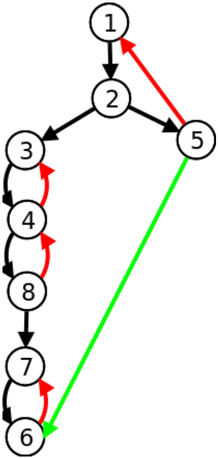




Strongly Connected Components

A more complex DFS application

- Let's look at the generated tree:



- What is the "lowest" ancestor reachable by a node?
  - 1: it's 1
  - 2: it's 1
  - 5: it's 1

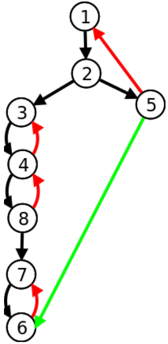
  - 3: it's 3
  - 4: it's 3
  - 8: it's 3

  - 7: it's 7
  - 6: it's 7
- *Et voilà!* Here are our SCCs!

Strongly Connected Components

A more complex DFS application

- Let's add 2 attributes to the nodes in a DFS visit:
  - **num(i)**: order in which i is visited
  - **low(i)**: smallest $num(i)$ reachable by the subtree that starts in i. It's the minimum between:
    - $num(i)$
    - smallest $num(v)$ between all back edges (i, v)
    - smallest $low(v)$ between all tree edges (i, v)



i	$num(i)$	$low(i)$
1	1	1
2	2	1
3	3	3
4	4	3
5	8	1
6	7	6
7	6	6
8	5	4

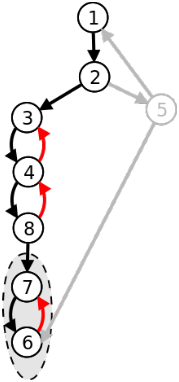
Strongly Connected Components

A more complex DFS application

Main ideas of **Tarjan Algorithm** to find SCCs:

- Make a **DFS** and in each node *i*:
  - Keep pushing the nodes to a **stack** S
  - Compute and store the values of num(i) and low(i).
  - If when finishing the visit of a node *i* we have that num(i) = low(i), then *i* is the "root" of a SCC. In that case, remove all the elements in the stack until reaching *i* and report those elements as belonging to a SCC!

Example of execution: in the moment we leave $dfs(7)$, we find that $num(7) = low(7)$ (7 is the "root" of a SCC)



State of Stack **S**:

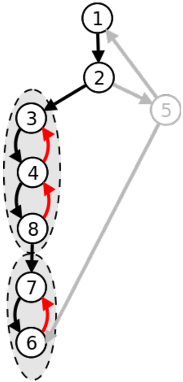
6
7
8
4
3
2
1

Remove elements from stack until reaching **7**; output SCC: {6, 7}

Strongly Connected Components

A more complex DFS application

Example of execution: in the moment we leave *dfs*(3), we find that *num*(3) = *low*(3) (3 is the "root" of a SCC)



State of Stack **S**:

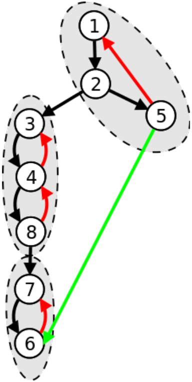
8
4
3
2
1

Remove elements from stack until reaching **3**; output SCC: {8, 4, 3}

Strongly Connected Components

A more complex DFS application

Example of execution: in the moment we leave $dfs(1)$, we find that $num(1) = low(1)$ (1 is the "root" of a SCC)



State of Stack **S**:

5
2
1

Remove elements from stack until reaching **1**; output SCC:: {5, 2, 1}

Strongly Connected Components

Tarjan Algorithm for SCCs

index $\leftarrow 1$; $S \leftarrow \emptyset$
For all nodes v of the graph **do**
  If $num[v]$ is still undefined **then**
    dfs_scc(v)

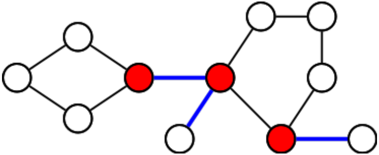
dfs_scc(node v):
  $num[v] \leftarrow low[v] \leftarrow index$; $index \leftarrow index + 1$; S.push(v)
  /* Traverse edges of v */
  For all neighbors w of v **do**
    If $num[w]$ is still undefined **then** /* Tree Edge */
      dfs_scc(w) ; $low[v] \leftarrow min(low[v], low[w])$
    Else if w is in S **then** /* Back Edge */
      $low[v] \leftarrow min(low[v], num[w])$
  /* We know that we are at the root of an SCC */
  If $num[v] = low[v]$ **then**
    Start new SCC C
    Repeat
      $w \leftarrow$ S.pop() ; Add w to C
    Until $w = v$
    Write C

Articulation Points and Bridges

An **articulation point** is a **node** whose removal increases the number of connected components.

A **bridge** is an **edge** whose removal increases the number of connected components.

Example (in red the articulation points; in blue the bridges):



A graph without articulation points is said to be **biconnected**.

Articulation Points

A more complex DFS application

- Finding **articulation points** is a very useful problem
  - For instance, a "robust" graph should not have articulation points that when "attacked" will disconnect them.
- How to compute? A possible (naive) **algorithm**:
  1. Make a DFS and count the number of connected components
  2. Remove a node from the original graph and execute a new DFS, counting again the connnected components. If this number increased, them the node is an articulation point.
  3. Repeat step 2 for all nodes in the graph
- What would be the **complexity** of this method? $\mathcal{O}(|V|(|V| + |E|))$, because we will make $|V|$ calls to DFS, each one taking $|V| + |E|$.
- **It is possible to do much better... using a single DFS**!

Articulation Points

A more complex DFS application

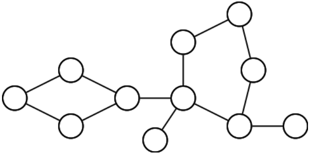
An idea:

- Apply DFS to the graph and obtain the **DFS tree**
- If a **node** v **has a child** w **without any path to an ancestor of** v**, then** v **is an articulation point**! (since removing it would disconnect w from the rest of the graph)
  - This corresponds to verify if $low[w] \geq num[v]$
- The only exception is the **root** of the DFS tree. If it has more than one child in the tree... it is also an articulation point!

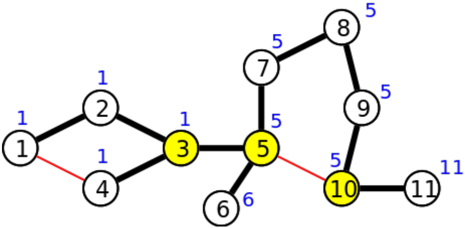
Articulation Points

A more complex DFS application

- An example graph:



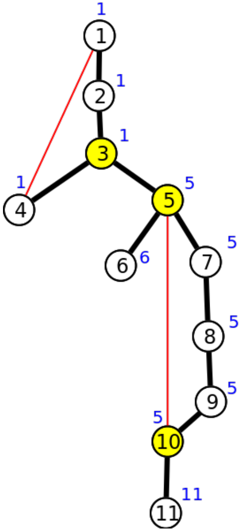
- *num*[*i*] - numbers inside the node
- *low*[*i*] - blue numbers
- articulation points: yellow nodes



(here we are assuming that we cannot lower the *low* value of a node going "back" on tree edges - this is why for instance *low[5]=5* and not 3; the algorithm would still work even if you also assume these edges as "back edges", with *low[5]=3*)

Articulation Points

A more complex DFS application



- 3 is an articulation point:
  $low[5] = 5 \geq num[3] = 3$
- 5 is an articulation point:
  $low[6] = 6 \geq num[5] = 5$
  ou
  $low[7] = 5 \geq num[5] = 5$
- 10 is an articulation point:
  $low[11] = 11 \geq num[10] = 10$
- 1 is not an articulation point:
  it only has one tree edge

Articulation Points

Algorithm very similar to Tarjan, but with different DFS:

Algorithm to find articulation points

dfs_art(node v**)**:
  $num[v] \leftarrow low[v] \leftarrow index$; $index \leftarrow index + 1$; S.push(v)
  For all neighbors w of v **do**
    If $num[w]$ is not yet defined **then** /* Tree Edge */
      dfs_art(w) ; $low[v] \leftarrow min(low[v], low[w])$
      If $low[w] \geq num[v]$ **then**
        write(v + "is an articulation point")
    Else if w is in S **then** /* Back Edge */
      $low[v] \leftarrow min(low[v], num[w])$
  S.pop()

Instead of a stack, we could have used colors (gray means it is in the stack)

Remember that the **root node** of the dfs must be treated differently and is an articulation point if and only if it has more than one child on the dfs tree