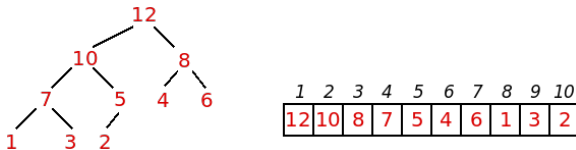


Priority Queues and Heaps

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Algorithms and Data Structures

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Priority Queues - Motivation

- Hospital emergency departments often operate with the **Manchester Triage System**, which allows classifying the severity of each patient's situation, assigning them one of the following colors:

Category	Classification	Time to be seen
1	IMMEDIATE	STRAIGHT AWAY
2	VERY URGENT	WITHIN 10 MIN
3	URGENT	WITHIN 60 MIN
4	STANDARD	WITHIN 120 MIN
5	NON-URGENT	WITHIN 240 MIN

- The order in which patients are attended depends on their **priority**. For example, a **red** patient who arrives later is always attended to before any **green** or **blue** patient, even if they have been waiting in the emergency room for a long time.

Priority Queues - Definition

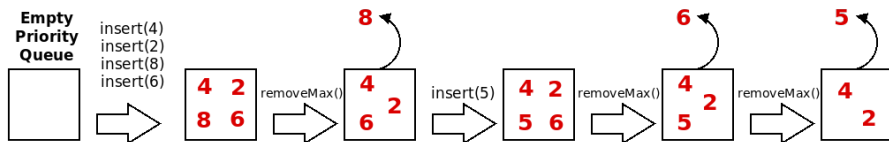
- The ADTs (*abstract data structures*) we know depend only on the **order of arrival** and do not (directly) adjust to a process like this:
 - ▶ A **stack** is always LIFO (Last In, First Out)
 - ▶ A **queue** is always FIFO (First In, First Out)
 - ▶ A **deque** only allows access to the first or last element
- We need an ADT that takes into account **priorities**.
 - ▶ If there were always only the 5 priorities of the triage system, we could use queues. But what happens if there are more?
 - ▶ And in a more general case where the number of different priorities is not limited? (e.g., the priority is any number, a *double*)
- A **priority queue** (**Priority Queue**) is an ADT for storing a collection of elements that supports three main operations:
 - ▶ **insert(x)** which adds an element x to the collection
 - ▶ **peek()** which returns (without removing) the **higher priority** element
 - ▶ **remove()** which returns and removes the **higher priority** element

Priority Queues - Applications

- Priority queues are useful in many other scenarios. Here are some **examples** where they can be applied:
 - ▶ A queue (e.g., at the post office) with **priority service** (e.g., for pregnant women)
 - ▶ A router with **priority traffic** (e.g., VoIP calls)
 - ▶ Simulation of **discrete events**: imagine several events starting at different times. We can use priority queues to determine the next event to occur (start time as priority)
- ▶ There are many **algorithms** that use priority queues as a basic building block. Some examples:
 - ★ **Dijkstra's Algorithm** (shortest paths): to determine the next closest unprocessed node
 - ★ **Prim's Algorithm** (for minimum spanning trees): to determine the next closest node to the tree that has not yet been added
 - ★ **A* Algorithm** (*best-first* search): to determine the next node to visit with the best heuristic value

Priority Queues - What is higher priority?

- To think about implementation, we need to **define** what it means to have "**higher priority**"
- Without loss of generality, we will assume we are working with **comparable** elements and that the higher priority one is the **largest**.
 - ▶ For example, if we have the integers $\{4, 8, 5\}$, the largest is 8.
 - ▶ In a case like triage, it would be enough to associate larger numbers to more prioritized colors (e.g., red=5, orange=4, yellow=3, ..)
 - ▶ If the smallest rather than the largest was useful, simply adjust the priorities correspondingly (e.g., store the negatives of the numbers so the "largest" is originally the smallest)
- Thus, the operation **remove()** can also be thought of as **removeMax()** (and we will call the operation **peek()** **max()**)



Priority Queues - Implementations

- How can we implement a priority queue using the data structures we already know?
 - ▶ **Unordered List:** an array or linked list without any order. Insertion is easy, but returning the maximum requires a linear search
 - ▶ **Ordered List:** an array or linked list in ascending order. Returning the maximum is easy (at the end), but insertion requires maintaining order
 - ▶ **Binary Search Tree:** insertion and removal (rightmost node) are associated with the tree height

	insert	max	removeMax
Unordered List	$\mathcal{O}(1)$	$\mathcal{O}(n)$	$\mathcal{O}(n)$
Ordered List	$\mathcal{O}(n)$	$\mathcal{O}(1)$	$\mathcal{O}(1)$
Binary Search Tree (if balanced)	$\mathcal{O}(\log n)$	$\mathcal{O}(\log n)$	$\mathcal{O}(\log n)$

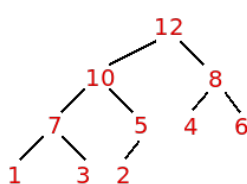
Note: if the search tree is unbalanced, insertion and removal can cost $\mathcal{O}(n)$. We could also create a dedicated variable to store the maximum and respond to `max()` in $\mathcal{O}(1)$.

Heaps - Invariant

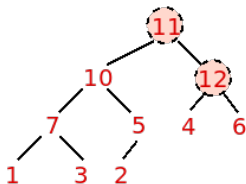
- Let us look at another **specialized** and very **efficient** solution.
- A **heap** is a tree that obeys the following restriction:

(max)Heap Invariant

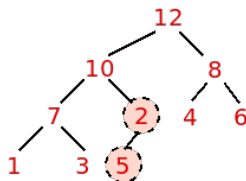
The parent of any node always has higher priority than the child. In a **maxHeap**, the parent is always *larger* than its children.



✓ It is a Heap



× It is not a Heap



× It is not a Heap

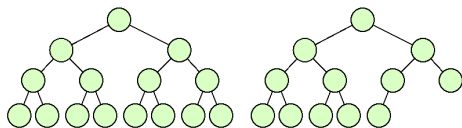
Note: in a **minHeap**, a node would always be smaller than its children.

Heaps - Height $\mathcal{O}(\log n)$

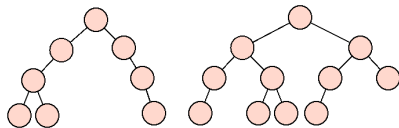
- To ensure the efficiency of associated operations, a heap must also be a **complete binary tree**:

Complete Tree

A tree where all levels (except potentially the last) are fully filled with nodes, and all nodes are as far left as possible.



2 examples of complete trees

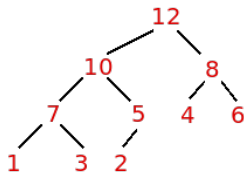


2 examples of incomplete trees

- In a complete tree with n nodes, the height is $\mathcal{O}(\log n)$.
 - ▶ It is a highly balanced tree, and we've discussed this before. Intuitively, think that to *increase the height by 1*, it is necessary to *double* the number of elements.

Heaps - Mapping to an Array

- The easiest and most compact way to implement a heap is to use an **array** that *implicitly* represents the tree.
 - ▶ The elements appear in the array in *level order* (from top to bottom, left to right).
 - ▶ If we place the root at position 1, then:
 - ★ The children of the node at pos. i are at pos. $i \times 2$ and $i \times 2 + 1$.
 - ★ The parent of a node i is at position $i/2$ (integer division).
- Let's look at an example:



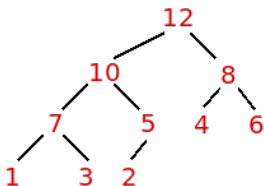
1	2	3	4	5	6	7	8	9	10
12	10	8	7	5	4	6	1	3	2

E.g.: children of pos. 3 (node 8) are at pos. $3 \times 2 = 6$ (node 4) and $3 \times 2 + 1 = 7$ (node 6). The parent of pos. 3 is the node at pos. $3/2 = 1$ (node 12).

- Since the tree is complete, this means the array has consecutively filled positions.

Heaps - Operation max()

- Since each node is larger than its children, the largest node of all is guaranteed to be at the root of the heap (the first element of the array):



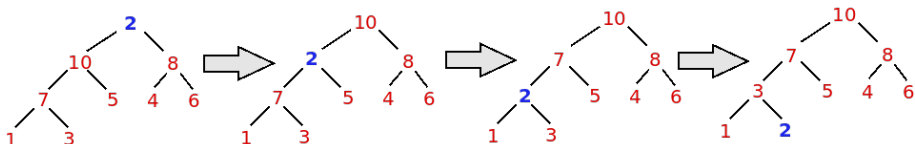
1	2	3	4	5	6	7	8	9	10
12	10	8	7	5	4	6	1	3	2

- **max()**: time $\mathcal{O}(1)$

Heaps - Operation removeMax()

- After removing the root, it is necessary to restore the heap properties. To do this:
 - Take the last element of the array and place it in the root's position (the tree remains complete).
 - The element "sinks" (**downHeap**), swapping with the largest of its children, until the heap invariant is restored.
 - At most, this traverses the entire height of the tree, which is $\mathcal{O}(\log n)$.

An example for the previous heap after removing the maximum (12) and placing the last element (2) in its position (the root):



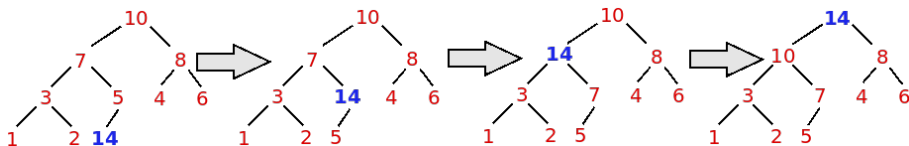
- removeMax():** time $\mathcal{O}(\log n)$

Heaps - Operation insert(x)

- To **insert** an element:

- ▶ Place it immediately after the last occupied position, the first free position in the array (the tree remains complete).
- ▶ The element "rises" (**upHeap**), swapping with its parent, until the heap invariant is restored.
- ▶ At most, this traverses the height of the tree, which is $\mathcal{O}(\log n)$.

An example for the insertion of 14 into the heap from the previous slide:



- **insert(x):** time $\mathcal{O}(\log n)$

Heaps - Visualization

- You can visualize insertion and removal in heaps (try the indicated URL):

<https://visualgo.net/en/heap>

VISUALGO.NET /en /heap **BINARY HEAP** Exploration Mode **LOGIN**

The Binary Max Heap has been created from array A:
12,64,58,12,95,79,13,59,7,12,10.

```
leave the input array A as it is
for (i = A.length/2; i >= 1; --i)
    shiftDown(i)

// priority_queue.cpp/py/java/ml
```

1.0x 1M 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 41 42 43 44 45 46 47 48 49 50 51 52 53 54 55 56 57 58 59 60 61 62 63 64 65 66 67 68 69 70 71 72 73 74 75 76 77 78 79 80 81 82 83 84 85 86 87 88 89 90 91 92 93 94 95 96 97 98 99 100 101 102 103 104 105 106 107 108 109 110 111 112 113 114 115 116 117 118 119 120 121 122 123 124 125 126 127 128 129 130 131 132 133 134 135 136 137 138 139 140 141 142 143 144 145 146 147 148 149 150 151 152 153 154 155 156 157 158 159 160 161 162 163 164 165 166 167 168 169 170 171 172 173 174 175 176 177 178 179 180 181 182 183 184 185 186 187 188 189 190 191 192 193 194 195 196 197 198 199 200 201 202 203 204 205 206 207 208 209 210 211 212 213 214 215 216 217 218 219 220 221 222 223 224 225 226 227 228 229 230 231 232 233 234 235 236 237 238 239 240 241 242 243 244 245 246 247 248 249 250 251 252 253 254 255 256 257 258 259 260 261 262 263 264 265 266 267 268 269 270 271 272 273 274 275 276 277 278 279 280 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2025 2026 2027 2028 2029 2030 2031 2032 2033 2034 2035 2036 2037 2038 2039 2040 2041 2042 2043 2044 2045 2046 2047 2048 2049 2050 2051 2052 2053 2054 2055 2056 2057 2058 2059 2060 2061 2062 2063 2064 2065 2066 2067 2068 2069 2070 2071 2072 2073 2074 2075 2076 2077 2078 2079 2080 2081 2082 2083 2084 2085 2086 2087 2088 2089 2090 2091 2092 2093 2094 2095 2096 2097 2098 2099 2100 2101 2102 2103 2104 2105 2106 2107 2108 2109 2110 2111 2112 2113 2114 2115 2116 2117 2118 2119 2120 2121 2122 2123 2124 2125 2126 2127 2128 2129 2130 2131 2132 2133 2134 2135 2136 2137 2138 2139 2140 2141 2142 2143 2144 2145 2146 2147 2148 2149 2150 2151 2152 2153 2154 2155 2156 2157 2158 2159 2160 2161 2162 2163 2164 2165 2166 2167 2168 2169 2170 2171 2172 2173 2174 2175 2176 2177 2178 2179 2180 2181 2182 2183 2184 2185 2186 2187 2188 2189 2190 2191 2192 2193 2194 2195 2196 2197 2198 2199 2200 2201 2202 2203 2204 2205 2206 2207 2208 2209 2210 2211 2212 2213 2214 2215 2216 2217 2218 2219 2220 2221 2222 2223 2224 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Building a Heap

- How can we **build a heap** from an initially general unsorted array with n elements?
- A first answer could be: **insert all elements into a new heap**
 - ▶ This would cost n insertions.
Since each one costs $\mathcal{O}(\log n)$ the total cost would be $\mathcal{O}(n \log n)$
- But what if we want reuse the same initial array and just "*heapify*" it?
We could now have two options:
 - 1 Start at the beginning of the array (top of the heap) and move forward towards the end. At each iteration, call **upHeap** on each item.
At each step, the items before the current item in the array form a valid heap, and moving the next item up places it into a valid position in the heap. After moving up each node, all items satisfy the heap property.
 - 2 Start at the end of the array and move backwards towards the front. At each iteration, call **downHeap** on each item.
Can you see why it works? At each step, what can we say about the items after the current item?
- Both approaches produce a valid heap. **Which one is faster?**

Building a Heap

- Both **upHeap** and **downHeap** depend on the height of the tree
 - At most they will go up or down $\mathcal{O}(\log n)$ times
- A first examination of the cost would therefore give us the bounds:
 - Calling n times **upHeap** costs $\mathcal{O}(n \log n)$
 - Calling n times **downHeap** costs $\mathcal{O}(n \log n)$
- While correct (\mathcal{O} provides an upper bound), one of these bounds is **not asymptotically tight** (and in fact its cost is $\mathcal{O}(n)$)
- Why? We will apply the operation to all nodes in the array, but:
 - For **upHeap** the cost is proportional to the distance to the top (expensive for nodes at the bottom of the tree)
 - For **downHeap** the cost is proportional to the distance to the bottom (expensive for nodes at the top of the tree)
- Both operations are $\mathcal{O}(\log n)$ in the worst case but **only one node is at the top whereas half the nodes lie in the bottom layer.**

It should therefore be no surprise that if we have to apply an operation to every node, we would **prefer downHeap over upHeap**

Building a Heap

- Let's see this more visually on a heap with 7 nodes:

- Calling **upHeap** on all nodes will do it from 1 to 7



At most this will cost:

- ★ 1 will **never** move up (max cost = 0)
- ★ 2 and 3 will move up **at most once** (max cost = $1+1 = 2$)
- ★ 4, 5, 6 and 7 move up **at most twice** (max cost = $2+2+2+2 = 8$)
- ★ **Total cost** = $0+2+8 = 10$

- Calling **downHeap** on all nodes will do it from 7 to 1



At most this will cost:

- ★ 4, 5, 6 and 7 will **never** move down (max cost = 0)
- ★ 2 and 3 will move down **at most once** (max cost = $1+1 = 2$)
- ★ 1 will move down **at most twice** (max cost = 2)
- ★ **Total cost** = $0+2+2 = 4$ (**less than upHeap**)

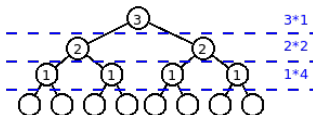
Building a Heap

- Our algorithm to **build an heap** for any array of size n could be:

```
// No need to call on bottom level (start on second to last level)
for (int i=n/2; i>=1; i--)
    downHeap(i)
```

You can visualize it: <https://visualgo.net/en/heap> [create(A) - $O(N)$]

- This has **linear cost** on n , that is, $O(n)$
- Let's sketch a proof. For simplicity of analysis let's consider a **complete tree**, that is, an array with $n = 2^{h+1} - 1$ items
 - ▶ Level 0 has 1 node, level 1 has 2 nodes, ..., level h has 2^h nodes
 - ▶ Second to last level: 2^{h-1} nodes cost at most 1
 - ▶ Third to last level: 2^{h-2} nodes cost at most 2
 - ▶ ...
 - ▶ First level: 1 node costs at most h



Building a Heap

- Let's call the total cost $T(n) = T(2^{h+1} - 1)$. Then:

$$T(n) \leq \sum_{i=1}^h i \times 2^{h-i} = \sum_{i=1}^h i \times \frac{2^h}{2^i} = 2^h \sum_{i=1}^h \frac{i}{2^i} \leq 2^h \sum_{i=1}^{\infty} \frac{i}{2^i}$$

- Let's focus on the summation of $S = \sum_{i=1}^{\infty} \frac{i}{2^i}$

$$(1) \ S = 1/2 + 2/4 + 3/8 + 4/16 + \dots$$

$$(2) \ S/2 = 1/4 + 2/8 + 3/16 + 4/32 + \dots$$

$$(1)-(2) \ S/2 = 1/2 + 1/4 + 1/8 + 1/16 + \dots$$

This is a geometric series $a + ar + ar^2 + ar^3 + \dots$ with $a = 1/2$ and $r = 1/2$

When $|r| < 1$ we know that this sum converges to $\frac{a}{1-r}$, so $S/2 = \frac{1/2}{1-1/2} = 1$

- $S = 2$, so finally:

$$T(n) \leq 2^h \sum_{i=1}^{\infty} \frac{i}{2^i} = 2^h \times 2 = 2^{h+1} = n + 1 \in \mathcal{O}(n) \quad \square$$

HeapSort

- Heaps suggest an **obvious sorting algorithm**. To sort n elements, simply do the following:
 - 1 Create a heap with the n elements
 - 2 Remove the n elements from the heap one by one
- Since the elements are removed in order of priority, they will appear... in **descending order**!
- What is the cost of this process?
 - 1 Building the heap costs $\mathcal{O}(n)$
 - 2 Removing one by one costs $\mathcal{O}(n \log n)$ [downHeap from root]
- The cost will be dominated by (2) and the sort will cost $\mathcal{O}(n \log n)$
- This algorithm (in its essence) is known as **HeapSort**.
- You can visualize it here:
<https://www.cs.usfca.edu/~galles/visualization/HeapSort.html>

Heaps - Implementation

- Let's see a simple implementation of heaps in C++
- We will implement a generic `maxHeap<T>` supporting any type using the natural comparator with 4 methods:
 - ▶ `bool isEmpty()`: is the heap empty?
 - ▶ `T max()`: return max value
 - ▶ `T removeMax()`: return and remove max value
 - ▶ `bool insert(T value)`: insert a new value

Heaps - Implementation

- The attributes, constructor and easy methods:

```
template <class T> class MaxHeap {
private:
    std::vector<T> data; // Items between 1 and size
    int size;           // Number of items
    T none;              // What to return when there are no items

public:
    // Capacity cap and value N used for indicating no element
    MaxHeap(int cap, T n) : data(cap+1), size(0), none(n) {}

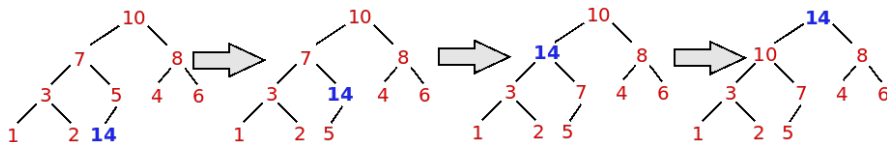
    // Empty heap?
    bool isEmpty() {
        return (size==0);
    }

    // Return (without removing) maximum element
    T max() {
        if (isEmpty()) return none;
        return data[1];
    }
    //..
}
```

Heaps - Implementation

- **Insert** an element into the heap: place it at the end of the array and call *upHeap* until the element reaches its position:

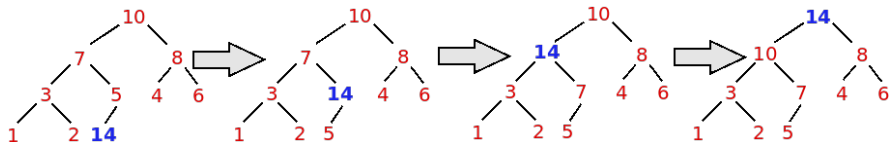
```
// Insert an element in the heap (true on success)
bool insert(T value) {
    if (size+1 >= (int)data.size()) return false;
    size++;
    data[size] = value;
    upHeap(size);
    return true;
}
```



Heaps - Implementation

- **upHeap**: move the element up while it is larger than the parent

```
// Make an element go up until its position
void upHeap(int i) {
    // While the element is greater than parent and not on root
    while (i>1 && data[i] > data[i/2]) {
        std::swap(data[i], data[i/2]);
        i = i/2;
    }
}
```

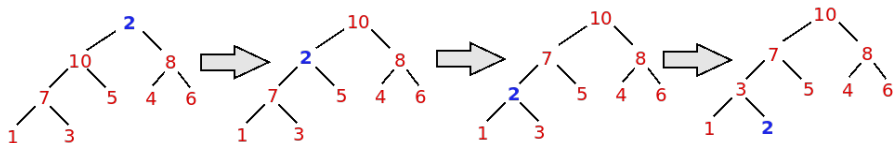


- `std::swap` is an existing C++ function

Heaps - Implementation

- **Remove** an element from the heap: return the root; place the last element at the root and call *downHeap* until the element reaches its position:

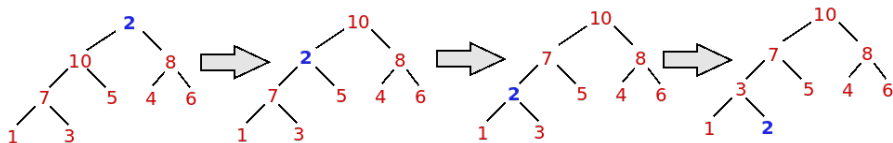
```
// Remove and return maximum
T removeMax() {
    if (isEmpty()) return none;
    T max = data[1];
    data[1] = data[size];
    size--;
    downHeap(1);
    return max;
}
```



Heaps - Implementation

- **downHeap**: move the element down while one of the children is larger (and swap with the larger child)

```
// Make an element go down until its position
void downHeap(int i) {
    while (2*i <= size) { // While still on the limits of the heap
        int j = i*2;
        // Choose largest child (position i*2 or i*2+1)
        if (j < size && data[j+1] > data[j]) j++;
        // If node is already larger or equal than largest child, stop
        if (data[i] >= data[j]) break;
        // Otherwise, swap with largest child
        std::swap(data[i], data[j]);
        i = j;
    }
}
```



Heaps - Usage Example

- Let's now see some usage examples.

```
#include <iostream>
#include "maxHeap.h"

int main() {
    // Create a heap with capacity for 10 integers (-1 for no item)
    MaxHeap<int> h(10, -1);

    // Create array for 10 integers
    std::vector<int> v = {10, 4, 3, 12, 9, 1, 7, 11, 5, 8};

    // Insert on heap all elements of array
    for (auto x : v) h.insert(x);

    // Remove elements one by one and print
    for (int i=0; i<10; i++) std::cout << h.removeMax() << " ";
    std::cout << std::endl;

    return 0;
}
```

12 11 10 9 8 7 5 4 3 1

Heaps - Usage Example

- Any comparable type can be used, not just integers.
- Here is an example with strings:

```
// Create a heap h (for strings)
MaxHeap<std::string> h(5, "---");

// Create an array of 5 strings
auto v = {"heap", "tree", "stack", "queue", "deque"};

// Insert on heap all elements of array
for (auto s : v) h.insert(s);

// Remove elements one by one and print
for (int i=0; i<5; i++) std::cout << h.removeMax() << " ";
std::cout << std::endl;
```

tree stack queue heap deque

Priority Queues in C++ Standard

- C++ standard directly implement priority queues as max heaps
- The name of the container is `priority_queue`
- To use it you should use `#include <queue>`

```
// Create a priority_queue as a heap
std::priority_queue<int> h;

// Create array for 10 integers
std::vector<int> v = {10, 4, 3, 12, 9, 1, 7, 11, 5, 8};

// Insert on heap all elements of array
for (auto x : v) h.push(x);

// Remove elements one by one and print
for (int i=0; i<10; i++) {
    std::cout << h.top() << " "; // returns, but does not remove
    h.pop();                      // removes, but does not return
}
std::cout << std::endl;
```

12 11 10 9 8 7 5 4 3 1

Priority Queues in C++ Standard

- Any comparable type can be used, not just integers.
- Here is an example with strings:

```
// Create a priority_queue as a heap h (for strings)
std::priority_queue<std::string> h;

// Create an array of 5 strings
auto v = {"heap", "tree", "stack", "queue", "deque"};

// Insert on heap all elements of array
for (auto s : v) h.push(s);

// Remove elements one by one and print
for (int i=0; i<5; i++) {
    std::cout << h.top() << " ";
    h.pop();
}
std::cout << std::endl;
```

tree stack queue heap deque

Priority Queues in C++ Standard

- We can for instance use a custom comparator
- The template is `priority_queue<type, container, comparator>`
(code here with `using namespace std` for spacing reasons)

```
// Custom comparator (using lambda function)
auto comp = [](string a, string b) {return b.size() > a.size();};

// Create a heap h (for strings)
priority_queue<string, vector<string>, decltype(comp)> h(comp);

// Create an array of 5 strings
auto v = {"aa", "bbbbbb", "ccc", "dddd", "e"};

// Insert on heap all elements of array
for (auto s : v) h.push(s);

// Remove elements one by one and print
for (int i=0; i<5; i++) {cout << h.top() << " "; h.pop();}

cout << endl;
```

bbbbbb dddd ccc aa e

Heaps in C++ Standard

- C++ also allows to directly use a container as a heap
- You can call `make_heap` to make a heap in linear time

```
std::vector<int> v = {10, 4, 3, 12, 9, 1, 7, 11, 5, 8};

// Creates heap from the vector (needs a range as an argument)
std::make_heap(v.begin(), v.end());
for (auto x : v) std::cout << x << " "; std::cout << std::endl;

// Pops the maximum (max is pushed to the end of the range)
// Effectively range that stays a heap decreases in size
std::pop_heap(v.begin(), v.end());
for (auto x : v) std::cout << x << " "; std::cout << std::endl;

// Pushes the last element to its heap position
std::push_heap(v.begin(), v.end());
for (auto x : v) std::cout << x << " "; std::cout << std::endl;
```

```
12 11 7 10 9 1 3 4 5 8
11 10 7 8 9 1 3 4 5 12
12 11 7 8 10 1 3 4 5 9
```

Heaps in C++ Standard

- For instance, with this API, doing an **HeapSort** as described is just:
 - ▶ Call `make_heap` on a container with n items
 - ▶ Call `pop_heap` n times on the container
(each time on a smaller range)

```
std::vector<int> v = {10, 4, 3, 12, 9, 1, 7, 11, 5, 8};

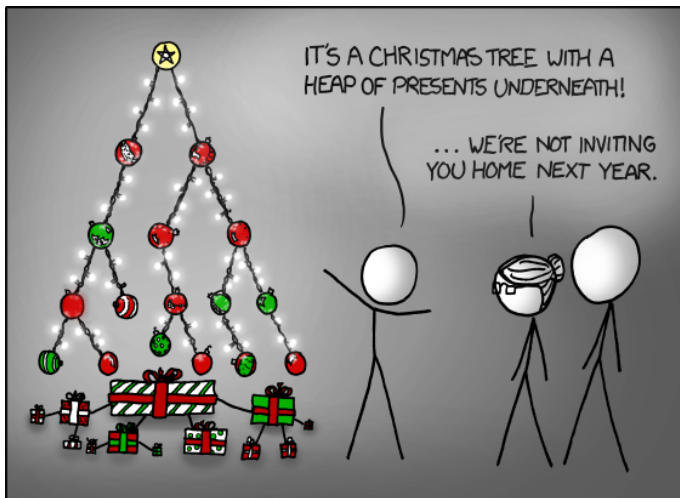
// Creates heap from the vector [in O(n)]
std::make_heap(v.begin(), v.end());

// Pops max on consecutively smaller ranges (in O(n log n))
for (int i=0; i<(int)v.size(); i++)
    std::pop_heap(v.begin(), v.end()-i);

// Show vector at the end
for (auto x : v) std::cout << x << " ";
std::cout << std::endl;
```

1 3 4 5 7 8 9 10 11 12

Heaps



(image from <https://xkcd.com/835/>)

"Not only is that terrible in general, but you just KNOW Billy's going to open the root present first, and then everyone will have to wait while the heap is rebuilt."