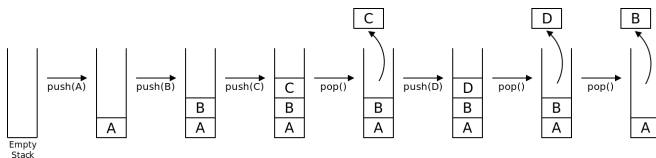


Linked Lists, Deques, Stacks, Queues and Applications

L.EIC

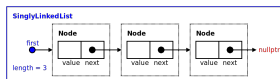
Algoritmos e Estruturas de Dados

2026/2027



P Ribeiro, AP Tomás

Singly Linked-List



```
{}
```

Low level...first = 0x0, length = 0

insert the numbers 1 to 2 at the end

```
{1,2}
```

Low level...first = 0x600003eb0020, length = 2

value = 1 next = 0x600003eb0030

value = 2 next = 0x0

insert the numbers 3 to 4 at the front

```
{4,3,1,2}
```

Low level...first = 0x600003eb0050, length = 4

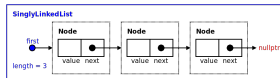
value = 4 next = 0x600003eb0040

value = 3 next = 0x600003eb0020

value = 1 next = 0x600003eb0030

value = 2 next = 0x0

Singly Linked-List



remove the first element

{3,1,2}

Low level...first = 0x600003eb0040, length = 3

value = 3 next = 0x600003eb0020

value = 1 next = 0x600003eb0030

value = 2 next = 0x0

remove the last element

{3,1}

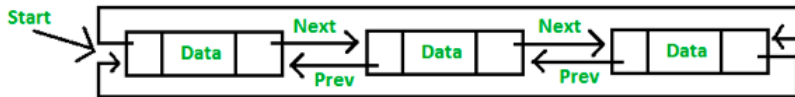
Low level...first = 0x600003eb0040, length = 2

value = 3 next = 0x600003eb0020

value = 1 next = 0x0

Circular Doubly Linked-List

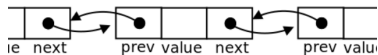
- A **circular doubly linked-list** is a linked list where each node is connected to both its previous and next nodes, and **the last node links back to the first node**.



- The implementation we will describe is based on the one given for Lab Class 05 for `template <class T> class SinglyLinkedList`. For the circular doubly linked-list:
 - ▶ Each node has three attributes: `next`, `prev` and `value`.
 - ▶ The head node is given by a pointer called `first`.

Circular Doubly Linked-List

Node



```
template <class T> class Node {  
private:  
    T value;    // value in the node  
    Node<T> *next;    // pointer to next node  
    Node<T> *prev;    // pointer to previous node  
public:  
    Node(const T & v, Node<T> *n=nullptr, Node<T> *p=nullptr):  
        value(v), next(n), prev(p) {}  
    T & getValue() { return value; }  
    Node<T> *getNext() { return next; }  
    Node<T> *getPrev() { return prev; }  
    void setValue(const T & v) { value=v; }  
    void setNext(Node<T> *n) { next = n; }  
    void setPrev(Node<T> *p) { prev = p; }  
};
```

Circular Doubly Linked-List

Implementation

```
template <class T> class CircularDoublyLinkedList {
private:
    Node<T> *first;      // head (front) of the list
    int length;          // number of nodes
public:
    // Construtor (creates empty list)
    CircularDoublyLinkedList(): first(nullptr), length(0) {}

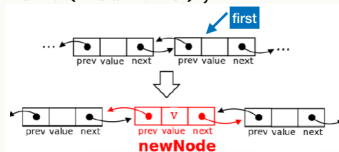
    // Destrutor
    ~CircularDoublyLinkedList() {
        while(!isEmpty()) {
            assert(first != nullptr && "message...");
            removeFirst();
        }
        assert(first == nullptr && "message...");
    }
}; // ...continues
```

"message..." stands for a message to be written if the condition fails

Circular Doubly Linked-List

AddFirst

```
template <class T> class CircularDoublyLinkedList {
...
void addFirst(const T & v) {    // adds v to the front
    Node<T> *newNode;    // to be a pointer to the new node
    if (first == nullptr) {
        newNode = new Node<T>(v,nullptr,nullptr);
        newNode -> setNext(newNode);
        newNode -> setPrev(newNode);    // a loop to itself
    } else {
        newNode = new Node<T>(v,first, first -> getPrev());
        (first -> getPrev()) -> setNext(newNode);
        first -> setPrev(newNode);
    }
    first = newNode;
    length++;
}
```



Creates a new node **newNode** with value **v**. It will be the new head (**first**). Links it to the previous first and last nodes. When the list is empty, links **newNode** to itself.

Circular Doubly Linked-List

AddFirst, getFirst, size, isEmpty

```
template <class T> class CircularDoublyLinkedList {
    ...
    // Adds v to the end of the list
    void addLast(const T & v) {
        addFirst(v);
        if (length > 1) first = first -> getNext();
    }

    // Returns the reference to the first value
    T & getFirst() {
        assert(!isEmpty() && "empty list has no first node");
        return first->getValue();
    }

    // Returns the length of the list
    int size() { return length; }

    // Returns true iff the list is empty
    bool isEmpty() { return (length == 0); }
```

Circular Doubly Linked-List

removeFirst

```
template <class T> class CircularDoublyLinkedList {  
... // Pops the first node (does nothing if list is empty)  
void removeFirst() {  
    if (isEmpty()) return;  
    Node<T> *victim = first;  
    if (length > 1) {  
        first = first -> getNext();  
        first -> setPrev(victim -> getPrev());  
        victim -> getPrev() -> setNext(first);  
    } else first = nullptr;  
    delete victim;  
    length--;  
}
```



The `victim` is the `first` node. If the list has other nodes, we link `victim->next` and `victim -> prev` and change `first` accordingly. Otherwise, we set `first` to `nullptr` since the list will be empty in the end. The victim is deleted ("freeing" the memory).

Circular Doubly Linked-List

removeChain

```
template <class T> class CircularDoublyLinkedList {
    ...// deletes the chain defined by start and end (deletes both also)
    void removeChain(Node<T> *start, Node<T> *end) {
        Node<T> *nextEnd = end -> getNext();
        Node<T> *prevStart = start -> getPrev();
        Node<T> *victim;
        do {
            victim = start;
            start = start -> getNext();
            delete victim;
            length--;
        } while (victim != end);
        if (length == 0) first = nullptr; // final list is empty
        else { // links initial nodes start -> prev and end -> next
            nextEnd -> setPrev(prevStart);
            prevStart -> setNext(nextEnd);
            first = nextEnd; // first becomes the initial end -> next
        }
    }
}
```

Circular Doubly Linked-List

toString

```
// Convert a list to a string
std::string toString() {
    if (isEmpty()) return "{}";
    Node<T> *curr = first;
    std::stringstream sstr;
    sstr << "{" << curr->getValue();
    while ( (curr = curr -> getNext()) != first) {
        sstr << "," << curr -> getValue();
    }
    sstr << "}";
    return sstr.str();
}
```

Requires `<sstream>` to be included. The values will be separated by a comma "," in the string. There is no comma in the end.

Circular Doubly Linked-List

getStart, getLast, setFirst

```
// Returns start node
Node<T> *getStart(){
    assert(!isEmpty() && "empty list has no start node");
    return first;
}

// Returns last node (node before the first)
Node<T> *getLast(){
    assert(!isEmpty() && "empty list no last node");
    return first -> getPrev();
}

// Changes the head to be newfirst
void setFirst(Node<T> *newfirst) {
    first = newfirst;    //does not check if newfirst exists
}
```

“Point in convex polygon?” by binary search

Full description: [https://icpcarchive.github.io/Southwestern_Europe_Regional_Contest_\(SWERC\).html](https://icpcarchive.github.io/Southwestern_Europe_Regional_Contest_(SWERC).html) (SWERC 2025)

Problem J - Saint John Festival:

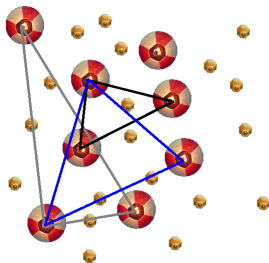
Given two sets of points $\mathcal{A}$ and $\mathcal{B}$ in the **plane**, such that $|\mathcal{B}| \gg |\mathcal{A}|$, how many points in $\mathcal{B}$ are in the interior or on the boundary of triangles defined by any 3 points in $\mathcal{A}$?

Charatheodory's Theorem:

The union of all triangles with vertices in $\mathcal{A}$ is the **convex hull** $\mathcal{CH}(\mathcal{A})$ of $\mathcal{A}$.

Geometry

- Convex hull: $\mathcal{CH}(\mathcal{A})$ of $\mathcal{A}$?
- Check $p \in \mathcal{CH}(\mathcal{A})$?
Point p in **convex polygon**?

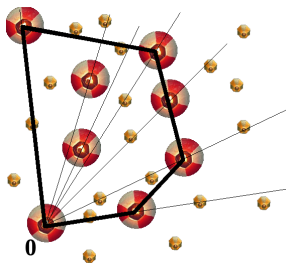


Convex Hull in 2D

Application of Circular Doubly Linked-Lists

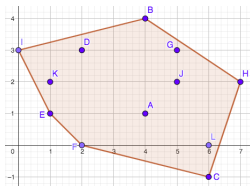
The convex hull problem

Given a set P of n points in the plane, defined by their cartesian coordinates, compute the list of vertices of its convex hull $\mathcal{CH}(P)$, in counterclockwise order.



"St. Saint John Festival"

The **convex hull** of $P \subseteq \mathbb{R}^2$ is the **smallest convex polygon** that contains all points of P .



We are going to see two algorithms for computing the convex hull in $\mathcal{O}(n \log n)$ time: the **incremental algorithm**, by Edelsbrunner, and **Graham's scan**.

Applications of Convex Hulls

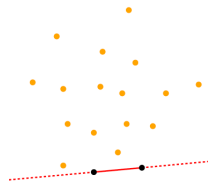
- There are many problems that are **comparatively easy to solve for convex sets** and much harder in general. E.g., POINT IN CONVEX POLYGON can be solved in $\mathcal{O}(\log n)$.
- Convex hulls in 2D and 3D have many applications. They can be used for instance in:
 - ▶ computer visualization
 - ▶ ray tracing
 - ▶ path finding
 - ▶ computing accessibility maps
 - ▶ visual pattern matching
 - ▶ diameter computation
 - ▶ cluster analysis
 - ▶ mesh generation
 - ▶ ...

Convex hull in 2D

Some geometrical properties

"Good solutions to algorithmic problems of a geometric nature are mostly based on two ingredients. One is a thorough understanding of the geometric properties of the problem, the other is a proper application of algorithmic techniques and data structures." (in M. Berg, et al (2008), [Computational Geometry](#))

Given $P = \{p_1, p_2, \dots, p_n\}$, the **extreme points** of convex hull $\mathcal{CH}(P)$ are points of P . A segment $p_i p_j$ is an **extreme segment** iff all the points p_k , with $k \neq i, j$, lie in the same halfplane defined by the line $p_i p_j$.



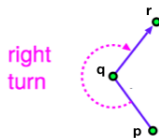
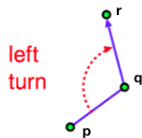
Convex hull in 2D

Some geometrical properties

- If we walk around the boundary of a simple polygon in counterclockwise order (CCW), **its interior is always to the left**.

In a **convex polygon**, no interior angle is greater than 180 degrees: if $v_0, v_1, \dots, v_{n-1}$ represents the sequence of its vertices in **CCW order**, then $(v_i, v_{(i+1)\%n}, v_{(i+2)\%n})$ defines a **left-turn**, at $v_{(i+1)\%n}$, for all i .

- Orientation tests (left-turn, right-turn, collinear) using the **cross product**:

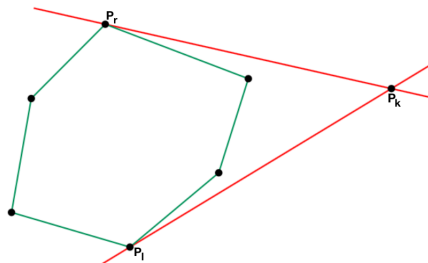


$$\vec{pq} \times \vec{pr} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_q - x_p & y_q - y_p & 0 \\ x_r - x_p & y_r - y_p & 0 \end{vmatrix}$$

For the usual coordinate system, (p, q, r) is a **left-turn** iff the z-component $(x_q - x_p)(y_r - y_p) - (x_r - x_p)(y_q - y_p)$ of the cross product $\vec{pq} \times \vec{pr}$ is **positive** (recall the "right-hand rule"). It is a **right-turn** if that component is negative, and the three points are **collinear** when it is zero.

Incremental Algorithm for Convex hull in 2D

```
Sort the points by increasing x-coord (tie-break: largest y first)
l = InitialPolygon(p1,p2,p3)    // l must be sorted in CCW
For k = 4 to n do:
    - Find the points p_l and p_r that define the supporting lines
      from p_k to the polygon
    - Replace the chain p_l,...,p_r in l with the chain p_l,p_k,p_r
Return l
```



In the implementation, we refer to p_r as **upper** and p_l as **lower**.

Incremental Algorithm for Convex hull in 2D

```
Polygon *convexhull(vector<PointI> &v) {
    sort(v.begin(),v.end(),compare); // sorted by abscissa
    Polygon *pol = initialize(v);    // sorting v[0],v[1],v[2] CCW
    Vertex *lastv, *lower, *upper;
    for(int i=3; i < v.size(); i++) {
        lastv = pol -> getStart(); // previous vertex added
        upper = upper_tangent(lastv,v[i]);
        lower = lower_tangent(lastv,v[i]);
        if (lower -> getNext() != upper)
            pol -> removeChain(lower->getNext(),upper->getPrev());
        pol -> setFirst(upper);
        pol -> addFirst(v[i]);
    }
    return pol; // printPolygon(*pol);
}
```

with **Polygon**, **Vertex**, and **PointI** defined as:

```
typedef CircularDoublyLinkedList<Point<int> > Polygon;
typedef Node<Point<int> > Vertex;
typedef Point<int> PointI;
```

Incremental Algorithm for Convex hull in 2D

For simplification, we will assume that there are **no three collinear points** in P (a kind of *general position assumption*). The algorithm can be adjusted to deal with degenerated (i.e., special) cases too, which occur in practice!

For "St. Saint John Festival", we cannot assume general position.

```
bool compare(PointI &p, PointI &q) { // for sorting
    if (p.getX() < q.getX()) return true;
    if (p.getX() > q.getX()) return false;
    return p.getY() > q.getY();
}
```

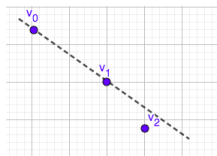
```
bool left_turn(PointI &p0, PointI &p1, PointI &p2) {
    int x0 = p0.getX(), x1 = p1.getX(), x2 = p2.getX();
    int y0 = p0.getY(), y1 = p1.getY(), y2 = p2.getY();
    return (x1-x0)*(y2-y0)-(x2-x0)*(y1-y0) > 0;
    // for a cartesian system with right-handed orientation
}
```

Point

```
template <class P> class Point {
private:
    P x;
    P y;
public:
    //constructors and destructor
    Point(){};
    Point(P a, P b): x(a), y(b){}
    ~Point(){};
    // getters and setters
    P getX(){ return x; };
    void setX(P v) { x = v;}
    P getY(){ return y; }
    void setY(P v) { y = v;}
    string toString(){ // to convert to a string format
        stringstream sstr;
        sstr << "(" << x << "," << y << ")";
        return sstr.str();
    };
};
```

Incremental Algorithm for Convex hull in 2D

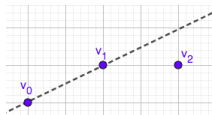
```
Polygon *initialize(vector<PointI> &v) { // to ensure CCW
    Polygon *pol = new Polygon();
    pol -> addFirst(v[0]);
    pol -> addFirst(v[1]);
    if (left_turn(v[0], v[1], v[2]))
        pol -> setFirst(pol -> getLast());
    pol -> addFirst(v[2]);
    return pol;
}
```



Right-turn

keep first at v_1

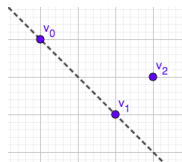
v_0 , v_2 , v_1



Right-turn

keep first at v_1

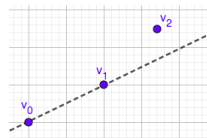
v_0 , v_2 , v_1



Left-turn

move first to v_0

v_1 , v_2 , v_0



Left-turn

move first to v_0

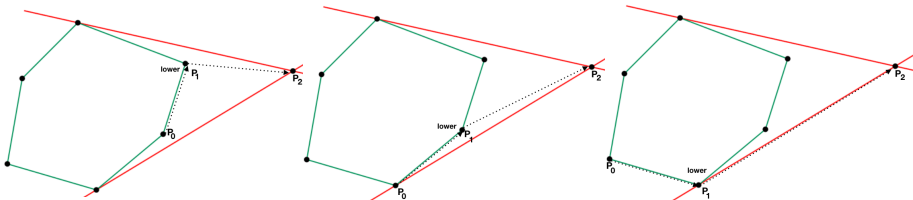
v_1 , v_2 , v_0

In all cases, v_2 will be the first element after insertion.

Incremental Algorithm for Convex hull in 2D

```
Vertex *lower_tangent(Vertex *lower, PointI & p2){  
    PointI p1 = lower -> getValue();  
    PointI p0 = lower -> getPrev() -> getValue();  
    while (!left_turn(p0,p1,p2)) {  
        lower = lower -> getPrev();  
        p1 = lower -> getValue();  
        p0 = lower -> getPrev() -> getValue();  
    }  
    return lower;  
}
```

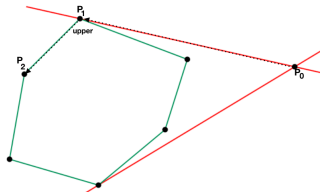
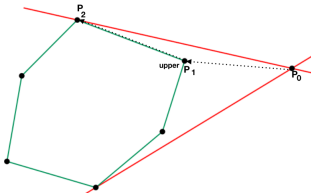
At start, **lower** points to the rightmost node in the convex hull computed so far (always the last point added before the new **p2**).



Incremental Algorithm for Convex hull in 2D

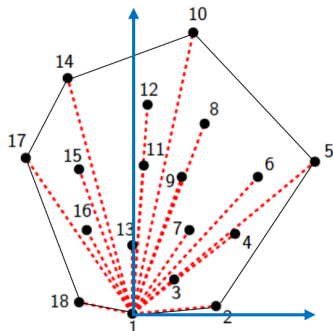
```
Vertex *upper_tangent(Vertex *upper, PointI & p0){
    PointI p1 = upper -> getValue();
    PointI p2 = upper -> getNext() -> getValue();
    while (!left_turn(p0,p1,p2)) {
        upper = upper -> getNext();
        p1 = upper -> getValue();
        p2 = upper -> getNext() -> getValue();
    }
    return upper;
}
```

At start, **upper** points to the rightmost node in the convex hull computed so far (always the last point added before the new **p0**).



Graham's Algorithm for Convex Hull in 2D

Graham scan uses a technique called *rotational sweep*, processing points in the order of the polar angles they form with a reference vertex.



The property Graham scan explores

In a counterclockwise walk around the boundary of a **convex polygon** P , starting from the bottom-most vertex p_1 , the vertices of P are sorted in increasing order of polar angle w.r.t. p_1 .

The **polar angle** of a point q with respect to a point p_1 is the anti-clockwise angle between a horizontal line and the line through p_1 and q .

- When there are two or more points with minimum y -component, the bottom-most vertex p_1 can be the leftmost among them (or the rightmost).
- For efficiency and numerical robustness, we do **not compute polar angles**. Instead, we use **orientation tests**: $p_j < p_k$ if (p_1, p_j, p_k) is a left-turn.

Graham's Algorithm for Convex Hull in 2D

Find the bottom-most point p_1 in P , push it into a stack S
 Let $p_2, p_3, \dots, p_n$ be the other points angularly sorted w.r.t. p_1
 Push p_2 into S

$k = 3$

While $k \leq n$ do:

$p = \text{top}(S)$

$p_- = \text{previous}(\text{top}(S))$

 If (p_-, p, p_k) is a left turn:

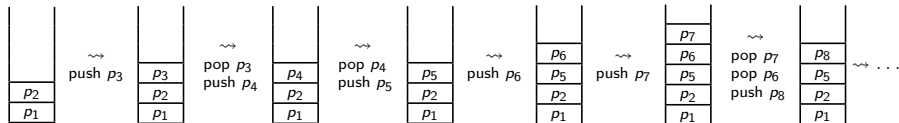
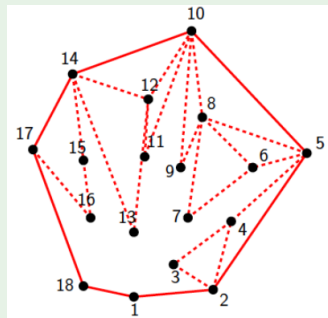
 Push p_k into S

$k = k + 1$

 else:

 Pop p from S

Return S



For an animation, check <https://dccg.upc.edu/wp-content/uploads/2020/06/GeoC-Convex-hulls-in-2D.pdf>

The Stack ADT

A **stack** is a collection whose elements are added to and removed from one end, called the **top of the stack**. It is a "last in first out" (**LIFO**) data structure.

- **Basic stack operations** besides the **creation/destruction**:

- ▶ **push(item)**: Add an element to the top
- ▶ **pop()**: Remove the top element
- ▶ **top()**: Examine the top element
- ▶ **size()**: Return the number of elements
- ▶ **empty()**: Test whether the stack is empty



- Usually, no support for **random access** or **iterators** or **sorting**.

If we need that, we'd better check whether a stack is the best data structure for our application, before we start implementing a specialized stack version.

Implementation of MyStack

Using a linked-list as container

```
#include <list>
using namespace std;

template <typename T> class MyStack {
private:
    list<T> l;
public:
    MyStack(){};          // creates empty list
    ~MyStack(){};
    void push(T d) { l.push_front(d); }
    void pop() { l.pop_front(); }
    bool empty() { return l.size() == 0; }
    T top() { return l.front(); }
    int size() { return l.size(); }
    void print() {        // not a basic stack operation
        for (auto x: l)  cout << x << endl;
    }
};
```

Implementation of MyStack

Using an array as container

```
#include <vector>
#include <cassert>
using namespace std;

template <typename T> class MyStack {
private:
    vector<T> v;
    int nmax, n;
public:
    MyStack(int c=100): v(vector<T>(c)), nmax(c), n(0) {};
    void push(T d) { assert(n < nmax && "full"); v[n++] = d; }
    void pop() { assert(n > 0 && "empty"); n--; }
    bool empty() { return n == 0; }
    T top() { assert(n > 0 && "empty"); return v[n-1]; }
    int size() { return n; }
    // the following are not basic operations in Stack ADT
    void print() { for(int i=0;i<n;i++)cout << v[i] << endl; }
    T atPos(int k) { assert(k>=0 && k<n); return v[n-1-k]; }
};
```

STL stack template

- C++ STL **std::stack** is implemented as a **container adaptor**, with **std::deque** as default container, if no container class is specified.

```
template <class T, class Container = deque<T> > class stack;
```

Examples:

- ▶ `std::stack<int> s0;`
- ▶ `std::stack<int, std::list<int> > s1;`

- The class template acts as a wrapper to the underlying container – **only a specific set of functions is provided** (Documentation for [std::stack](#)).

Provides no support for **random access** or **iterators**.

- In **Graham scan**, for instance, we need **previous(top(S))**.
 - ▶ With **C++ STL stack**, we could remove the top to get access to **previous(top(S))** and then push the top again (if useful).
 - ▶ (With our **array-based MyStack**, **s.atPos(1)** yields **previous(top(S))**; **s.atPos(0)** is **s.top()**; **previous(previous(top(S)))** is **s.top(2), ...**)

Back to Convex hull

Complexity of Graham scan and the incremental algorithm

Graham's scan and Edelsbrunner's incremental algorithm find the convex hull of n points in 2D in $\mathcal{O}(n \log n)$ time and $\mathcal{O}(n)$ extra space. The **sorting step dominates their time complexity**. For the remaining steps, the time is $\Theta(n)$.

● Graham scan:

- ▶ Each iteration of the **while loop** costs $\Theta(1)$ because each pop, push, top, and left_turn operation takes $\Theta(1)$.
- ▶ There are less than $2n$ iterations, because either p_k is push into the stack or the top, i.e., a previous point, is removed from the stack.
- ▶ Thus, although the analysis of some particular point p_k may require $\mathcal{O}(\text{size}(S) - 1)$ iterations, the **total time for the loop is $\Theta(n)$** and not $\Omega(n^2)$. We have **$\mathcal{O}(1)$ amortized time complexity per point**.

● Incremental algorithm:

- ▶ the overall time for finding all upper and lower tangents, removing chains, and linking points, is $\Theta(n)$, because each point is added just once. Removed chains won't be analysed again.

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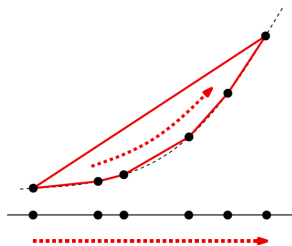
Time complexity lower bound

Any convex hull algorithm that uses line-side tests to find the hull requires $\Omega(n \log n)$ line-side tests in the worst case (in a decision tree model).

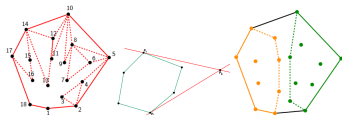
Idea of the proof (by reduction from sorting): Given a set of n distinct integers, $V = \{x_1, x_2, \dots, x_n\}$, if we compute the $\mathcal{CH}(P)$ with

$$P = \{(x_k, x_k^2) \mid 1 \leq k \leq n\}$$

we can list V in increasing order by starting at the leftmost point of $\mathcal{CH}(P)$. Since sorting requires $\Omega(n \log n)$ in the worst case, so does the convex hull problem in 2D. $\square$



Graham scan, the incremental algorithm, and a divide-and-conquer convex hull algorithm are asymptotically optimal algorithms for the convex hull problem, in the worst case.



The Queue ADT

A **queue** is a collection whose elements are added at one end (called the **tail** or **end** of the queue) and removed from the other end (called the **front** or **head** the queue). It is a "**first in first out**" (**FIFO**) data structure.

- **Basic queue operations** besides the **creation/destruction**:

- ▶ **push(item)**: Add an element at the end
- ▶ **pop()**: Remove the first element
- ▶ **front()**: Access the first element
- ▶ **back()**: Access the last element
- ▶ **empty()**: Test whether the queue is empty
- ▶ **size()**: Return the size of the queue

- Usually, no support for **random access** or **iterators** or **sorting**.

If we need that, we'd better check whether a queue is the best data structure, before we start implementing a specialized queue version.



Implementation of MyQueue

Using a linked-list as container

```
#include <list>
using namespace std;

template <typename T> class MyQueue {
private:
    list<T> l;
public:
    MyQueue(){};          // the list is empty
    void push(T d) { l.push_back(d); }
    void pop() { l.pop_front(); }
    bool empty() { return l.size() == 0; }
    T front() { return l.front(); }
    T back() { return l.back(); }
    int size() { return l.size(); }
    void print() { for (auto x: l) cout << x << endl; } // !!
};
```

In some applications, it would be interesting to be able to **remove any element** from the queue, e.g., if elements can **abandon the queue**. This ADT does not support that.

Implementation of MyQueue

Using a circular array as container

```
#include <vector>
#include <cassert>
using namespace std;

template <typename T> class MyQueue {
private:
    vector<T> v;
    int nmax, start, end;
public:
    MyQueue(int c=100):
        v(vector<T>(c)), nmax(c), start(-1), end(0) {};
    void push(T d) {
        assert(start != end && "queue is full");
        if (empty()) start = end;
        v[end] = d;
        end = (end+1) % nmax;
    }
};
```

// continues in next slide

Implementation of MyQueue

Using a circular array as container

```
void pop() {
    assert(!empty() && "queue is empty");
    start = (start+1) % nmax;
    if (start == end) { // it had just one element
        start = -1; end = 0;
    }
}

int size() {
    if (empty()) return 0;
    return (end+nmax-start)%nmax;
}

bool empty() { return start == -1; }
T front() { assert(!empty()); return v[start]; }
T back() { assert(!empty()); return v[(end+nmax-1)%nmax]; }
};
```

Time complexity of: push, pop, size, empty, front, and back is $O(1)$.

STL queue template

- C++ STL **std::queue** is implemented as a **container adaptor**, with **std::deque** as default container, if no container class is specified.

```
template <class T, class Container = deque<T> > class queue;
```

Examples:

- ▶ `std::queue<int> q0;`
- ▶ `std::queue<int, std::list<int> > q1;`

- The class template acts as a wrapper to the underlying container – **only a specific set of functions is provided** (Documentation for [std::queue](#)).

Provides no support for **random access** or **iterators**.

- If we need those features, we'd better use, e.g., a **deque** directly or **another data structure**.
 - ▶ A **deque** (**double-ended queue**) is an indexed sequence container that allows fast insertion and deletion at both its beginning and its end.
 - ▶ Link to STL documentation: [std::deque](#)