How fast is fast?

Order of magnitude analysis

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Definition of big-O

Let $f(n)$ and $g(n)$ be two functions. We say that

$$f(n) = \mathrm{O}(g(n))$$

if and only if there exist constants C and N such that

$$f(n) \leq Cg(n) \text{ for all } n \geq N.$$

If an algorithm takes $f(n)$ seconds to run on an input of size n, then it is said to be $\mathrm{O}(g(n))$.

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- It is only an upper bound: $n = \mathrm{O}(n^2)$, although one usually gives the tightest possible bound.
- It is a worst case bound: linear search is $\mathrm{O}(n)$, even though it might only take one step.

Rule of thumb

To estimate the running time of your program

- Substitute the data set size into the big-O formula
- Divide by a scale factor for the machine (10–100 million per second)
- Estimating the scale factor is a bit of an art.

Examples

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for i := 1 to N do
    for j := 1 to N − i do
        if a[j] > a[j + 1] then
            swap(a[j], a[j + 1]);
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K := 0;
for i := 1 to N do
    while (K < i) and (bad[i][i − K]) do K := K + 1;
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Divide and conquer

```
left  := 1;
right  := N;
while ( left  <  right )  do
begin
      middle  := ( left  +  right )  div 2;
      if  a[middle] < target  then  left  :=  middle + 1
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Answer: $O(\log n)$.

Divide and conquer

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procedure recurse(left, right : integer);
begin
    for i := left to right do do_something_constant(i);
    middle := ( left + right ) div 2;
    if ( left < middle) then recurse(left , middle);
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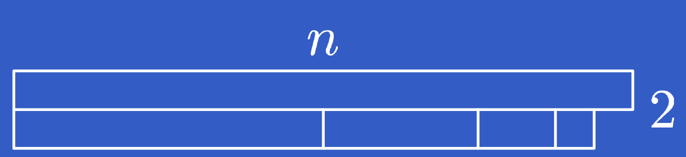

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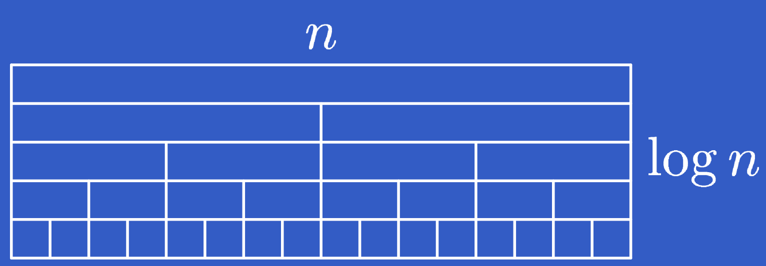

Average case

It is occasionally the case that the average case performance is much better than the worst case performance.

- Binary search tree insertion: $\mathrm{O}(\log n)$ on average, but $\mathrm{O}(n)$ in the worst case.
- Quicksort is $\mathrm{O}(n \cdot \log n)$ on average, but $\mathrm{O}(n^2)$ in the worst case.

However, this assumes a random distribution.

How to solve a problem

1. Look at the constraints
2. Cook up some algorithms and evaluate
3. Estimate your score from the constraints
4. Pick the **simplest** algorithm you can
5. Remember to leave time to code and debug

Other lessons

- Don't optimise unless you need to.
- Don't bother optimising the fast bits.
- Don't bother optimising the outer loops.
- Don't ignore an optimisation just because it doesn't affect the big O — it can still make a big difference.
- $O(\log n)$ is much closer to $O(1)$ than to $O(n)$

Questions

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