Dynamic Programming II
Partitions, Games, Dags, Search and Digits

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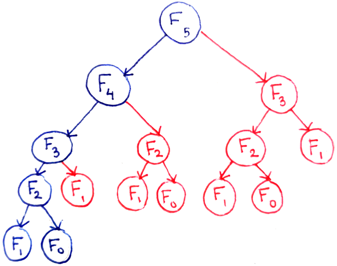

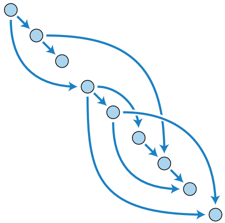

Overview

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    - Minimax Principle for Games
    - DP in DAGs (and Trees)
    - DP with Linear Partitions
    - DP with Bitmasks
    - Digit Dynamic Programming

What is Dynamic Programming?

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- How? Trade-off between **Memory** $\longleftrightarrow$ **Time**
- Dynamic Programming (DP) ... the holy grail in the world of problem-solving techniques?
- Not quite! Although a powerful technique requires **optimal substructure** and **overlapping sub-problems**.

How can I solve all DP problems?

Is there a standard way to approach all DP problems? Not really, but first devising a **recursive solution** helps!

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Let's go over some examples ...

- ▸ [UVA] Bachet's Game
- ▸ [CSES] Removal Game

DP with Linear Partitions

- We want to **partition** an array a of size n into k disjoint consecutive sub-arrays that minimize or maximize a given cost function.

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Figure: Array of size 8 partitioned into 3 sub-arrays.

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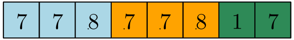

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Follow-up: Can you solve the problem in $\mathcal{O}(n \times \log(n) \times k)$ time complexity?

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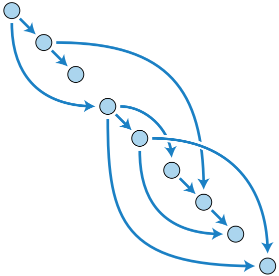

Figure: Sample DAG with 9 vertices.

DP in DAGs (and Trees)

- A Tree is a graph with **no cycles**

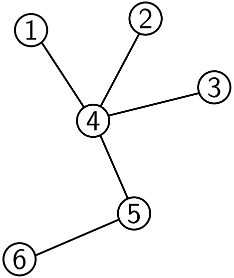

Figure: Tree with 6 vertices.

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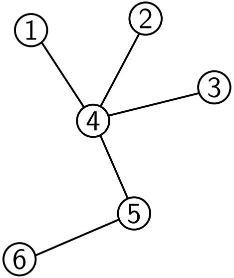

Figure: Tree with 6 vertices.

Examples:

- [AtCoder] Longest Path
- [AtCoder] Independent Set

DP with Bitmasks

- The Travelling Salesman Problem (TSP) asks for the **shortest** path that visits **every node** of a graph **exactly once**
- Some variants include finding the shortest length cycle

DP with Bitmasks

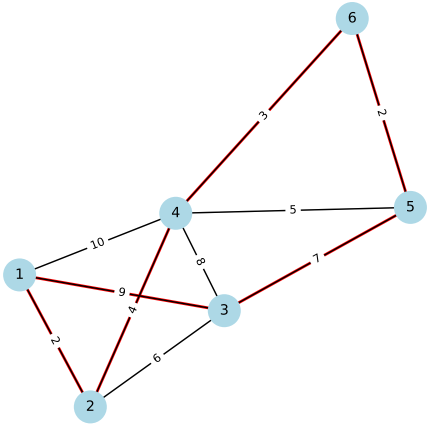

Figure: A sample (cyclic) tour in a graph $\mathcal{G}$ with 6 vertices having cost 27. *Optimal?*

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Examples:

- [ED202] Procurando Pokemons

Digit Dynamic Programming

- How can we **efficiently** count numbers with a given property over a **large** $[L..R]$ range $(1 \leq L \leq R \leq 10^{18})$?
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- ... what is the state?
- Typically:
  - p - position of the digit being filled
  - $flag_{Upper}$ - is the number **lower** than the **upper limit**
  - $flag_{Bigger}$ - is the number **bigger** than the **lower limit**
  - ... other useful problem specific characteristics!
  - The code may be simplified by using the fact that: $count(L, R) = count(0, R) - count(0, L - 1)$
- Examples:
- ▸ [CSES] Counting Numbers
- ▸ [ONI'18] Problema A - Códigos preguiçosos

I Want to Know More!

Here you have a selection of additional resources that you may find useful:

- CP Algorithms: A good reference with high quality explanations
- Codeforces DP problems: A list of all DP tagged problems on Codeforces sorted by (expected) difficulty *(highly recommended!)*
- CSES DP Section: A well-crafted list of classical DP problems *(a good starting point)*
- AtCoder Educational DP contest: Curated list of 26 *essential* DP problems *(some require more advanced techniques that we did not cover in this course: matrix multiplication, convex hull trick ...)*