String Matching

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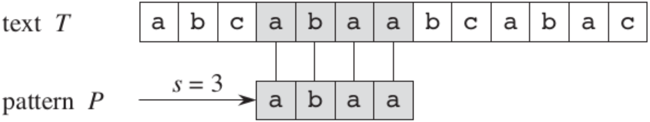


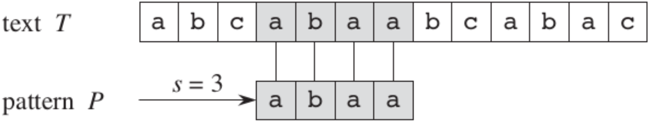
String Problems

- There is an entire area of study dealing with string related problems
- Examples of string related problems:
  - Given a text and a pattern, find all exact or approximate occurrences of the pattern in the text (classic text search)
  - Given a string, find the largest string that occurs at least k times
  - Given two strings find the edit distance between them, with various operations available, such as deletions, additions and substitutions.
  - Given two strings, find the largest common substring
  - Given a set of strings, find the "better" tree that can describe and connect them (phylogeny tree)
  - Given a set of strings, find the shortest superstring that contains all the strings (one of the core problems of DNA sequencing)
  - ...
- Here we will just give a brief glimpse on the whole field and in particular we will focus on the **string matching problem**

The String Matching Problem

Let's formalize the string matching problem:

- **Text**: array $T[1..n]$ of length n
- **Pattern**: array $P[1..m]$ of length $m \leq n$
- The characters of T and P are characters drawn from an alphabet Σ
  - For example, we could have $\Sigma = \{0, 1\}$ or $\Sigma = \{a, b, ..., z\}$
- A pattern P occurs with shift s in text T (or occurs beginning at position $s + 1$) if $T[s + i] = P[i]$ for $1 \leq i \leq m$



String Matching Problem

Given a text T and a pattern P, find all valid shifts of P in T, or output that no occurrence can be found.

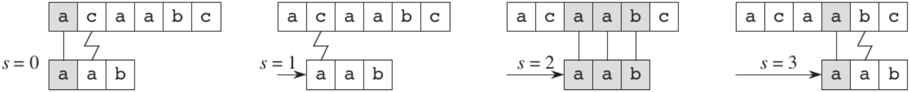
- One common variation is to find only one (ex: the first) possible shift

Naive String Matching

- Here is an (obvious) brute force algorithm for finding all valid shifts:

```
NAIVE-STRING-MATCHER($T$, $P$)
1  $n = T.length$
2  $m = P.length$
3  for $s = 0$ to $n - m$
4      if $P[1\,..\,m] == T[s+1\,..\,s+m]$
5          print "Pattern occurs with shift" $s$
```

- This algorithm tries explicitely every possible shift *s*
- Line 4 implies a loop to check if all characters match or exits if there is a mismatch



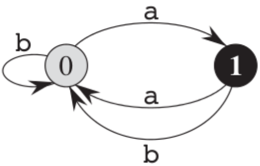
Naive String Matching

- What is the time complexity of the naive algorithm?
- $\mathcal{O}((n-m)m)$, which is $\mathcal{O}(mn)$ assuming m is "relatively small" $(m < n/2)$ compared to n.
- The worst case is something like searching for *aaa...aaab* in a text consisting solely of *a*'s.
- If the text is random, this algorithm would be "not too bad" (if exiting as soon as a mismatch is found) but real text (ex: english or DNA) is really not completely random.
- This solution can also be acceptable if m is "really" small

Deterministic Finite Automaton

- How can we do better?
- Once we are at a certain shift, what information can we use about the previous shifts we tested?
- One possible (high-level) idea is to build a **deterministic finite automaton** (DFA) to represent what we know about the pattern and in what state of the search we are.

	input	
state	a	b
0	1	0
1	0	0



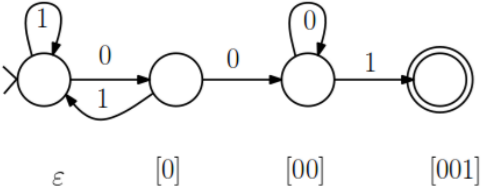
An example DFA that matches strings of $\Sigma = \{a, b\}$ finishing with an odd number of a's

Deterministic Finite Automaton

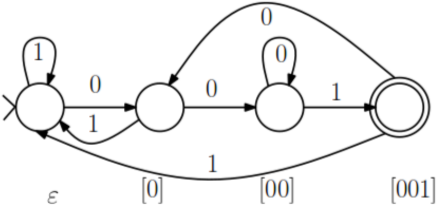
- Imagine a DFA with $m+1$ states, arranged in a "line"
- The *i*-th state represents that we are now at position *i* of the pattern, that is, we matched the first *i* characters.
- Now, if we match the next character, we move to state $i+1$ (matched $i+1$ characters). If not, we can skip to another (previous) state.
- Which state should we go once we have a miss? If we go back to the initial state, then we are no better than the naive algorithm! We should go to the furthest state we know its possible.

Deterministic Finite Automaton

Imagine $P = 001$. We could use the following DFA:

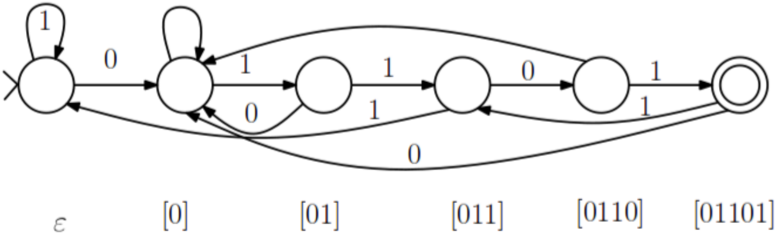


This would only find the first occurrence of P. What to change so that it finds all occurrences?



Deterministic Finite Automaton

What if the pattern is for instance $P = 01101$?



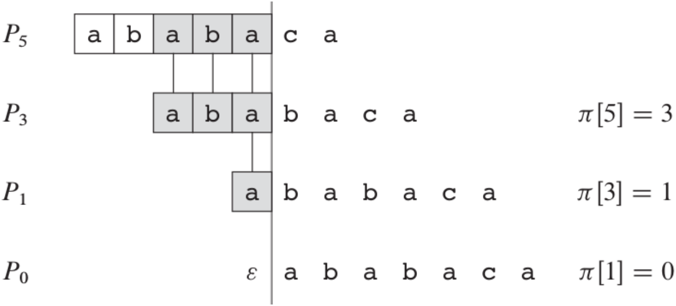
Deterministic Finite Automaton

- What is the complexity of matching after having a DFA like this?
- The matching is linear on the size of the text! $\mathcal{O}(n)$
- We must however take in account the time to build the respective DFA. If it takes $f(m)$, than the total time is $f(m) + \mathcal{O}(n)$.
- We will now show how the **Knuth-Morris-Pratt (KMP) algorithm** can build the "equivalent" of this DFA in time linear on the size of the pattern! $\mathcal{O}(m)$

Knuth-Morris-Pratt Algorithm

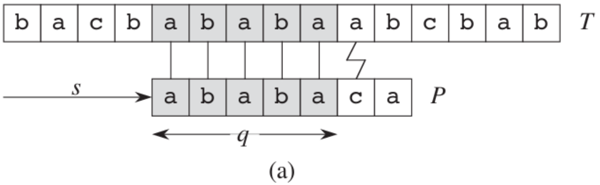
- Let $\pi[i]$ be the largest integer smaller than i such that $P[1..\pi[i]]$ (longest prefix) is a suffix of $P[1..i]$.

i	1	2	3	4	5	6	7
$P[i]$	a	b	a	b	a	c	a
$\pi[i]$	0	0	1	2	3	0	1

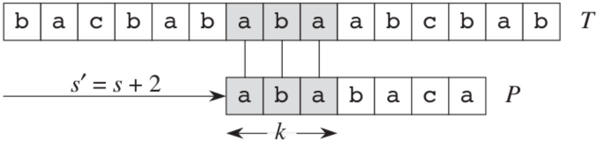


Knuth-Morris-Pratt Algorithm

- How can we use the information in $\pi[]$ to our matching?
- When he have a mismatch at position $i + 1$... we rollback to $\pi[i]$!
  - This is the next possible "largest" partial match



(a)



(b)

Knuth-Morris-Pratt Algorithm

- Let us look at the KMP main algorithm:

```
KMP-MATCHER($T$, $P$)
$n = T.length$
$m = P.length$
$\pi$ = COMPUTE-PREFIX-FUNCTION($P$)
$q = 0$                                  // number of characters matched
for $i = 1$ to $n$                        // scan the text from left to right
    while $q > 0$ and $P[q+1] \neq T[i]$
        $q = \pi[q]$                      // next character does not match
    if $P[q+1] == T[i]$
        $q = q + 1$                       // next character matches
    if $q == m$                           // is all of $P$ matched?
        print "Pattern occurs with shift" $i - m$
        $q = \pi[q]$                      // look for the next match
```

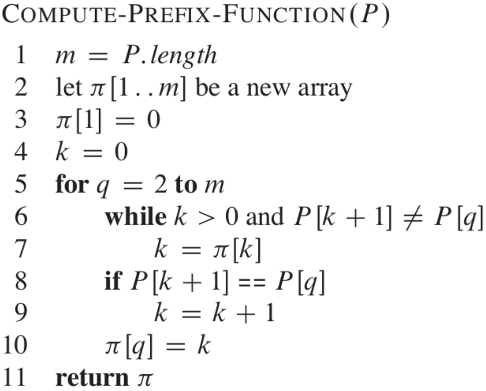
- What is the **temporal complexity** of this algorithm?

Knuth-Morris-Pratt Algorithm

- Let's for now ignore the time taken in computing π.
- The loop on line 5 takes time n. But what about the loop on line 6?
- The main "insight" is that we can never go back more than what we have already advanced. If we advance k characters in the text, than the call to line 7 can only make q go back k characters
- In other words, q is only increased in line 9 (at most once per each iteration of the cycle of line 5). Since when it is decreased it can never be negative (by the definition of π), this means it will have at most n decrements.
- This means that the while loop will never have more than n iterations!
- In an amortized sense (aggregate method), the time needed for the entire procedure is **linear on the size of the text**: $\mathcal{O}(n)$

Knuth-Morris-Pratt Algorithm

- What about computing π?
- It is basically comparing the pattern against itself!



```
COMPUTE-PREFIX-FUNCTION($P$)
$m = P.length$
let $\pi[1..m]$ be a new array
$\pi[1] = 0$
$k = 0$
for $q = 2$ to $m$
    while $k > 0$ and $P[k+1] \neq P[q]$
        $k = \pi[k]$
    if $P[k+1] == P[q]$
        $k = k + 1$
    $\pi[q] = k$
return $\pi$
```

- What is the **temporal complexity** of this part?

Knuth-Morris-Pratt Algorithm

- Using a similar rationale to what we did before, the time is **linear on the size of the pattern**: $\mathcal{O}(m)$
- The entire KMP algorithm then takes $\mathcal{O}(n + m)$
    - Pre-processing: $\mathcal{O}(m)$
    - Matching: $\mathcal{O}(n)$

Rabin-Karp Algorithm

- Let's now look at a completely different approach
- Imagine that we have an **hash function** h that maps each possible string to an integer.
- We could then proceed as follows:
  - Start by computing $h(P)$
  - For every possible shift s, compute $h_i = h(T[s+1...s+m])$
  - If $h_i \neq h(P)$ then we know we do not have a match
  - If $h_i = h(P)$ we could have a match, and we loop to see if its really a match on that position
- The efficiency of this procedure depends mainly on two things:
  - How good is the hash function (how well does it separate strings), because some invalid shifts may not be filtered out
  - How many valid occurrences exist, because for each of these shifts we will really make a loop of at most m

Rabin-Karp Algorithm

- Let's actually create a procedure using these core ideas
- We will start be defining a suitable **rolling hash function**.
- Suppose each character is assigned an integer. For ease of explanation, we will show examples only with digits (0..9) and a decimal base, but if we have $k = |\Sigma|$ characters, we could use base k.
- A pattern of a k-sized alphabet can be seen as a number on base k. With our simple scheme for digits, the pattern `"12345"` could then be viewed as the number $12,345$. Let's call this function *value*.

Rabin-Karp Algorithm

- We can compute the value of the pattern in time $\mathcal{O}(m)$:

$$value(P) = P[m] + 10(P[m-1] + 10(P[m-2] + ... + 10(P[2] + 10P[1])...))$$

Example: $value(\texttt{"324"}) = 4 + 10(2 + 10 \times 3) = 324$

- Similarly, if $T_i = T[i+1...i+m]$, we can compute $value(T_i)$ in $\mathcal{O}(m)$

- After we compute T_0, do we really need m operations to compute T_1? No! We can do it in constant time:

$$value(T_{s+1}) = 10(T_s - 10^{m-1}T[s+1]) + T[s+m+1]$$

Example: $value(\texttt{"5678"}) = 5,678$
$value(\texttt{"6789"}) = 10(5,678 - 10^3 \times 5) + 9 = 6,789$

- This means we can compute all T_i's in time linear to the size of the text!

Rabin-Karp Algorithm

- If we ignore the fact that our *value*() could get really large, we would have an $\mathcal{O}(n)$ algorithm for doing string matching
- The problem is that we cannot assume that the m characters of P will give origin to arithmetic operations that take constant time.
- How can we solve this problem? Consider that we know that:

$$(a \times b) \mod c = ((a \mod c) \times (b \mod c)) \mod c$$
$$(a + b) \mod c = ((a \mod c) + (b \mod c)) \mod c$$

- What we can do is always apply **mod q** operation on our results! In that way the value will always stay between 0 and $q-1$!

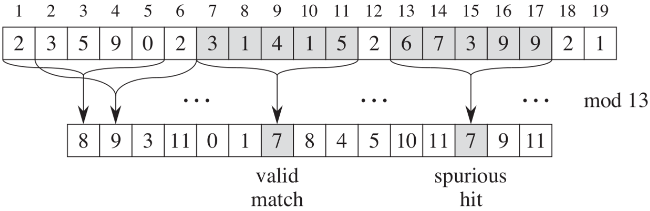
$$value(T_{s+1}) = (10(T_s - 10^{m-1}T[s+1]) + T[s+m+1]) \mod q$$

Rabin-Karp Algorithm

- The solution with **mod q** is not perfect, however...
  - $value(T_s) \mod q = value(P) \mod q$ does not imply $T_s = P$
  - However, $value(T_s) \mod q \neq value(P) \mod q$, implies that $T_s \neq P$

- If the values are equal **mod q** we still have to test to see if we have a match or not. On case it is not a match we have a **spurious hit**.

- Example: imagine we are looking for $31,415$ and use $q = 13$
  We have that $31,415 \mod 13 = 7$.



Rabin-Karp Algorithm

- Our *value*() function is in reality just a fast *heuristic* for ruling out invalid shifts.
- If q is high enough, we *hope* that the spurious hits will be rare

```
RABIN-KARP-MATCHER($T, P, d, q$)
$n = T.length$
$m = P.length$
$h = d^{m-1} \bmod q$
$p = 0$
$t_0 = 0$
for $i = 1$ to $m$                    // preprocessing
    $p = (dp + P[i]) \bmod q$
    $t_0 = (dt_0 + T[i]) \bmod q$
for $s = 0$ to $n - m$                // matching
    if $p == t_s$
        if $P[1..m] == T[s+1..s+m]$
            print "Pattern occurs with shift" $s$
    if $s < n - m$
        $t_{s+1} = (d(t_s - T[s+1]h) + T[s+m+1]) \bmod q$
```

Rabin-Karp Algorithm

- How to analyze the running time?
- What would the **worst case be**? Imagine a string always with the same characters, and a pattern also with the same characters. In that case we will always have a hit and will always be making the verification.
- In many applications, however, the valid shifts are rare. In those cases this may be a good choice.
- If we have only c occurrences, than the expected time will be $\mathcal{O}(n + cm)$, plus the time for the spurious hits.

Rabin-Karp Algorithm

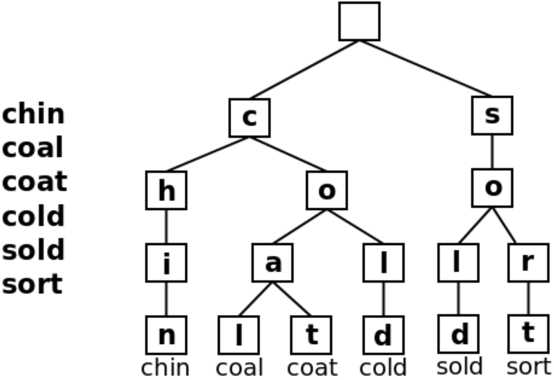
- How often do spurious hit occur? How good is our hash function? This is not going to be explored today, but choosing a (large) prime not close to a power of two is a good choice.
- If we are able to really spread the possible values, and the text is "random", than the number of expected spurious hits is $\mathcal{O}(n/q)$ (the chance that an arbitrary substring has the same value of P is $1/q$).
- If v is the number of valid shifts, then the running time is $\mathcal{O}(n + m(v + n/q))$.
- If v is $\mathcal{O}(1)$ and $q > m$ then the total expected running time is $\mathcal{O}(n)$!

Trie

- From algorithms revolving around the pattern, we will now focus on data-structures centered on the text (or set of words) being searched
- A **trie** (also known as **prefix tree**) is a data structure representing a set of words (that can have values associated with it)
    - The root represents the empty string
    - Descendants share the same prefix

Trie

An example trie with 6 words



Note that the letters can be thought of as the edges and not the nodes

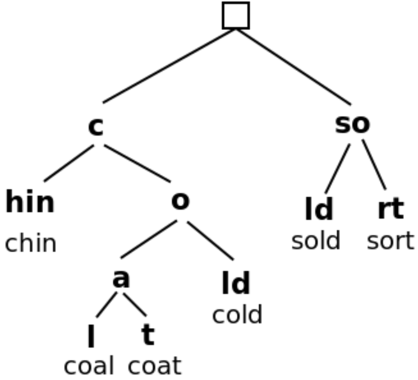
Trie

- We can **check** if a string of size n is stored in the trie in $\mathcal{O}(n)$ time
- We can **insert** a new word of size n in $\mathcal{O}(n)$ time
- We can **remove** a word of size n in $\mathcal{O}(n)$ time
- We exemplified with words, but tries **can store other types of data** (ex: numbers, or any data that we can separate in individual pieces)
- For **space efficiency** we can compact the tree: if a node has only one child, merge it with that child. This type of tree is called a **compressed prefix tree**, which is sometimes called **radix tree**

Compressed Prefix Tree

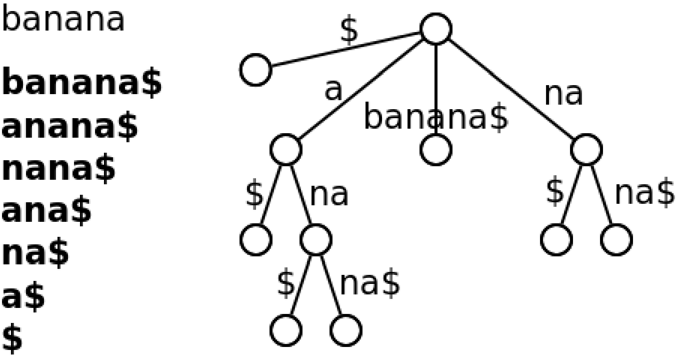
An example compressed prefix tree with 6 words

chin
coal
coat
cold
sold
sort



Suffix Tree

- A trie is not efficient in searching for substrings.
- For that we need a different data structure: a **suffix tree**. It is essentially a compressed trie of all suffixes of a given word.



$ is being used for marking the end of a word

Suffix Tree

- We can check if a string of size n is a **substring** in $\mathcal{O}(n)$ time
- A suffix tree of a word of size n can be **created** in $\mathcal{O}(n)$ time, but the algorithms have an high constant factor and are not trivial to implement (ex: Ukkonen's algorithm)
- We can put more than one word in the suffix tree: a **generalized suffix tree** is just is a suffix tree of a set of words.

Suffix Tree

There are many possible applications besides the "obvious" substring matching. Here are some examples:

- **Longest repeated substring** on a single word? Just find node with the highest depth through which two different suffixes passed by.
- **Longest common substring** of two words? Just put both on a suffix tree and find the node with the highest depth through which both strings passed by.
- **Most frequent** k**-gram**? (substring of size k) For all nodes with depth k, find which has more leafs descending from it.
- **Shortest unique substring**? Find the lowest depth node with only one leaf descending from it
- ...

Suffix Arrays

- The biggest problem with suffix trees is it **high memory usage**.
- A much more space efficient alternative with the same kind of applications is the **suffix array**: a sorted array of all suffixes

Suffix Arrays

An example

Consider **S="banana"**

i	1	2	3	4	5	6	7
s[i]	b	a	n	a	n	a	$

Suffixes of S

Suffix	i
banana$	1
anana$	2
nana$	3
ana$	4
na$	5
a$	6
$	7

Sorted Suffixes

Suffix	i
$	7
a$	6
ana$	4
anana$	2
banana$	1
na$	5
nana$	3

The suffix array *A* contains the starting positions of these sorted suffixes:

i	1	2	3	4	5	6	7
A[i]	7	6	4	2	1	5	3

Suffix Arrays

Substring matching

- How to search if a string P is a substring of a text T?
- You can use **binary search** on the suffix array of T!
- Without auxiliary data structures each comparison takes $\mathcal{O}(|P|)$ and you need to make $\mathcal{O}(log|T|)$ comparisons, leading to an $\mathcal{O}(|P| \times log|T|)$ algorithm.

Suffix Arrays vs Suffix Trees

- Suffix arrays can be constructed by performing a depth-first traversal (DFS) of a suffix tree. The suffix array corresponds to the leaf-labels given in the order in which these are visited during the traversal, if edges are visited in the lexicographical order of their first character.
- A suffix tree can be constructed in linear time by using a combination of suffix arrays and **LCP array**
- In fact, **every suffix tree algorithm can be systematically replaced by an algorithm with suffix arrays** by using auxiliary information (such as the LCP array), having an **"equivalent" time complexity** (just a bit slower).

Suffix Arrays

LCP Array

- What is the **LCP array**? LCP = Longest Common Prefix
  It stores the lengths of the longest common prefixes between pairs of consecutive suffixes in the sorted suffix array.

Consider **S="banana"**

i	1	2	3	4	5	6	7
s[i]	b	a	n	a	n	a	$

Suffix Array *A*:

i	1	2	3	4	5	6	7
A[i]	7	6	4	2	1	5	3

LCP Array *H*

i	1	2	3	4	5	6	7
A[i]	_	0	1	3	0	0	2

Example: $H[4] = 3$ because ana and anana have a common prefix of size 3

Suffix Arrays
LCP Array

- How can we use the LCP array?
- Imagine again you want to check if a string P is a substring of T.
- You can use binary search on the suffix array of T
- Without anything else we can use binary search in $\mathcal{O}(|P| \times log|T|)$
- With LCP and derivatives you can turn this into $\mathcal{O}(|P| + log|T|)$
- Consider an LCP-LR array that tells you the longest common prefix of any given suffixes (not necessarily consecutive).
- We can use LCP-LR to only check the "new characters". How?

Suffix Arrays and Binary Search

- During the binary search we consider a range $[L, R]$ and its central point M. We then decide whether to continue with the left half $[L, M]$ or the right half $[M, R]$.
- For that decision, we compare P to the string at position M. If $P == M$, we are done. If not, we have compared the first k chars of P and then decided whether P is lexicographically smaller or larger than M. Let's assume the outcome is that P is larger than M.
- In the next step we will therefore consider $[M, R]$ and a new central point M' in the middle:

```
          M ...... M' ...... R
          |
   we know:
      lcp(P,M)==k
```

Suffix Arrays and Binary Search

```
          M ...... M' ...... R
          |
       lcp(P,M)==k
```

- The "trick" now is that LCP-LR is precomputed such that a $\mathcal{O}(1)$ lookup gives the longest common prefix of M and M', lcp(M,M').
- We know already that *M* itself has a prefix of *k* chars common with P: $lcp(P, M) = k$. Now there are 3 possibilities:
  - $\mathrm{k} < \mathrm{lcp(M, M')}$. This means the (k+1)-th char of M' is the same as *M*. Since *P* is lexicographically larger than *M*, it must be lexicogr. larger than M', too. We continue in the right half $[M', R]$
  - $\mathrm{k} > \mathrm{lcp(M, M')}$. the common prefix of *P* and M' would be $< k$, and M' would be lexicographically larger than *P*, so, without actually making the comparison, we continue in the left half $[M, M']$
  - $\mathrm{k} == \mathrm{lcp(M, M')}$. *M* and M' have the same first *k* chars as *P*. It suffices to compare *P* to M' starting from the $(k + 1)$-th char.

Suffix Arrays and Binary Search

- In the end every character of P is compared to any character of T only **once**!
- We get our desired $\mathcal{O}(|P| + log|T|)$ complexity!
- But how to build the LCP-LR array?
  - Only certain ranges may appear during a binary search
  - In fact, every entry of the suffix array is the central point of exactly one possible range
  - So there are $|T|$ distinct ranges, and it suffices to compute $lcp(L, M)$ and $lcp(M, R)$ for those ranges
  - In the end we have $2 \times |T|$ values to pre-compute
  - There is a "straightforward" recursive algorithm to compute the $2 \times |T|$ values of LCP-LR in $\mathcal{O}(|T|)$ from the standard LCP array.