

Cryptography

Week #7:

Computational Complexity & Hard problems

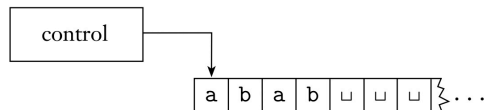
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MSI/MCC/MIERSI - 2022/2023
DCC FCUP

October, 28th 2022

Complexity & Cryptography

Given that the model of secrecy used by modern cryptography is not compatible with “secrecy by obfuscation” its security must rely on the Computational Complexity results.

Turing Machine



- ① A Turing machine can both write on the tape and read from it.
- ② The read–write head can move both to the left and to the right.
- ③ The tape is infinite.
- ④ The special states for rejecting and accepting take effect immediately.

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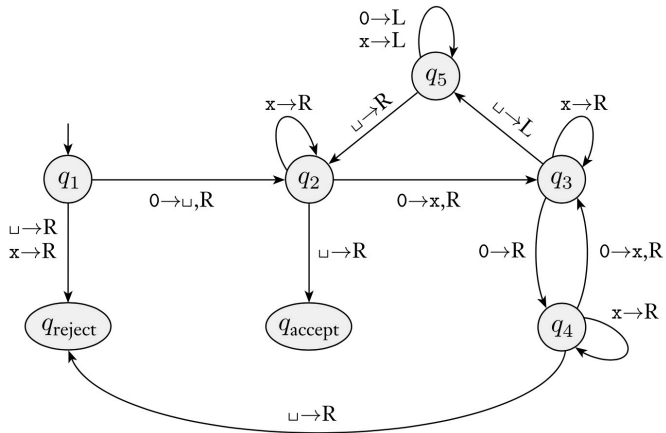
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Turing-decidable language

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Some (actually the vast majority) languages are **not decidable** neither **not recognisable**.

The obvious example is the language of Turing Machines that halt with an empty input. (The “halting problem”).



Turing machine that decides $A = \{0^{2^n} \mid n \geq 0\}$.

Church–Turing thesis

All models of enough expressiveness are equivalent.

Computational Complexity

The computational complexity of a task is the measure of the time necessary to accomplish the task as a function of the size of the input.

This is normally done using a Turing Machine using a single tape as reference, but (as long as they are deterministic) other models do not give results that correspond to different classes of complexity.

We will work on the assumption that the choice of the model of computation used is irrelevant in what complexity classification is concern¹.

¹Kind of a Church-Turing thesis v2.0.

Landau notation

The “big O” notation

$$f(n) = \mathcal{O}(g(n))$$

$$\exists k > 0 \exists n_0 \forall n > n_0 : |f(n)| \leq k g(n)$$

When $f(n) = \mathcal{O}(g(n))$, we say that $g(n)$ is an upper bound for $f(n)$, or more precisely, that $g(n)$ is an asymptotic upper bound for $f(n)$, to emphasise that we are suppressing constant factors.

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- $f_2(n) = 3n \log_2 n + 5n \log_2 \log_2 n + 2 = \mathcal{O}(n \log n)$

Landau Notation

The “small o” notation

$$f(n) = o(g(n))$$

$$\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = 0$$

Complexity Theory

Traditionally Complexity Theory considers only a special kind of programs that takes an input and returns an answer of acceptance or rejection of that input. The set of accepted words by one of these programs is called *the language defined by the program* and we say that the program *decides the language* in the sense that it decides, for each word fed as input, if it belongs to the language or not.

Although this model does not covers all the problems that are necessary to study, most of the problems may be divided in components that follow in this definition of “decision program”. Thus, we may see the study of the complexity of these “decision programs” as the study of lower bounds for complexity of a more general set of programs.

Complexity Classes

TIME

Let $f : \mathbb{N} \rightarrow \mathbb{R}^+$ be a function. Define the time complexity class, $\text{TIME}(t(n))$, to be the collection of all languages that are decidable by an $\mathcal{O}(t(n))$ time Turing machine.

Complexity Classes

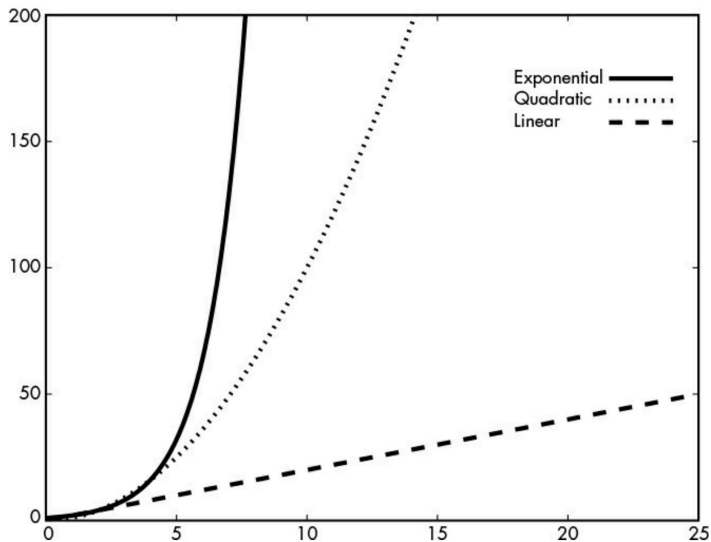
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SPACE

Let $t : \mathbb{N} \rightarrow \mathbb{R}^+$ be a function. Define the space complexity class, $\text{SPACE}(t(n))$, to be the collection of all languages that are decidable by an Turing machine using a $\mathcal{O}(t(n))$ space in their tape.

How important is the complexity in practice?



Complexity Classes

It is easy to see that if a function f is in $\text{TIME}(g(n))$ then it cannot but be in $\text{SPACE}(g(n))$.

$$\text{TIME}(n^2) \subseteq \text{SPACE}(n^2)$$

The most “important” of the complexity classes is P:

The class P

$$P = \bigcup_{k \in \mathbb{N}} \text{TIME}(n^k)$$

Thus P is the class of complexity of all the functions that can be decided in a polynomial time by a Turing Machine.

Examples of languages in P

²This is the “soft-O” notation: $\tilde{O}(f(n)) = O(f(n) \log f(n))$.

Examples of languages in P

RELPRIME Given two integers, decide if their greatest common divider is 1, i.e. if they are coprime. The Euclides' algorithm solves the problem in time $\mathcal{O}(n)$.

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PRIMEP Given an integer decide if it is a prime number. Proven in 2002 (*PRIMES is in P*, Agrawal, Kayal and Saxena, 2002). The AKS algorithm runs in time $\mathcal{O}(n^{12} \log(n^{12})) = \tilde{\mathcal{O}}(n^{12})^2$. Although this a polynomial asymptotic complexity, in practice, and for the size of numbers one wants to use it, the AKS is outperformed by many algorithms of non polynomial complexity.

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Any CFL Any context-free language have a CFG that recognises it and CYK parser can thus operate in time $\mathcal{O}(n^3)$.

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Complexity Classes

In the same manner:

The class PSPACE

$$\text{PSPACE} = \bigcup_{k \in \mathbb{N}} \text{SPACE}(n^k)$$

PSPACE is the class of complexity of the functions that can be decided by a Turing Machine using polynomial space on its tape.

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The second most “important” of the complexity classes is NP. NP does **not** stand for “non-polynomial” but for “**nondeterministic polynomial**”, that is languages that (these conditions are equivalent):

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- can be decided by a nondeterministic Turing Machine in polynomial time;
- can be verified in polynomial time by a (deterministic) Turing Machine.

By “verified” one means that a Turing Machine with input $\langle w, c \rangle$ can decide if w is part of the language using c as additional information (normally called a witness). The time dependence of the machine only takes in account the size of w and not c .

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COMPOSITES Given an integer decide if it belongs to

$\{n \mid \exists p, q, (p, q < n) \wedge (n = pq)\}$. Trivially is in NP because a verifier is straightforward to write using the list of factors as witness. Although many believe that the problem is in P, we still do not know any algorithm that ensure that.

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DISCRETELOG Given p , a and n decide if there is k s.t.

$$a^k \equiv n \pmod{p}.$$

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Let $f : \mathbb{N} \rightarrow \mathbb{R}^+$ be a function. Define the time complexity class, $\text{NTIME}(t(n))$, to be the collection of all languages that are decidable by an $\mathcal{O}(t(n))$ time nondeterministic Turing machine.

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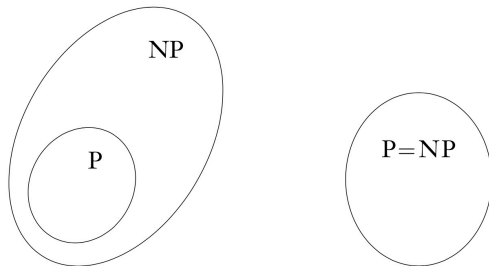
NP

$$\text{NP} = \bigcup_{k \in \mathbb{N}} \text{NTIME}(n^k)$$

P vs NP

P = the class of languages with quick membership decision.

NP = the class of languages with quick membership verification.



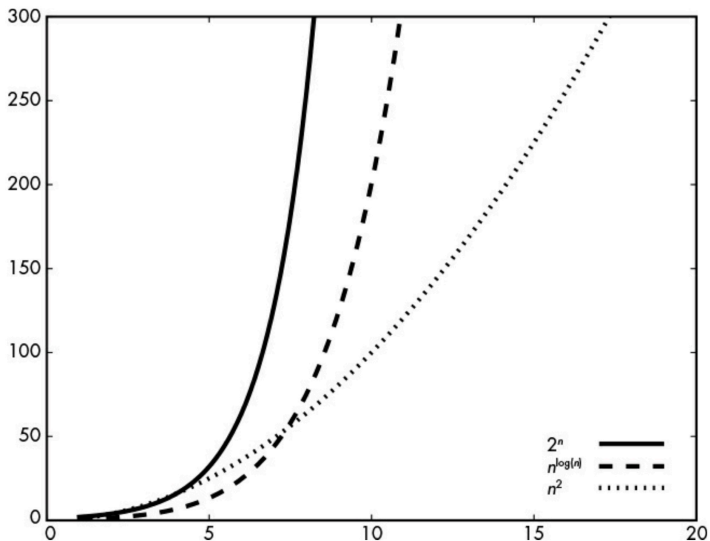
One of these is correct! Which one?

The best deterministic method currently known for deciding languages in NP uses exponential time.

EXPTIME

$$\text{NP} \subseteq \text{EXPTIME} = \bigcup_{k \in \mathbb{N}} \text{TIME} \left(2^{n^k} \right)$$

SUPERPOLYTIME is not EXP TIME



NP-Completeness

Polynomial time computable function

A function $f : \Sigma^* \rightarrow \Sigma^*$ is a **polynomial time computable function** if some polynomial time Turing machine M exists that halts with just $f(w)$ on its tape, when started on any input w .

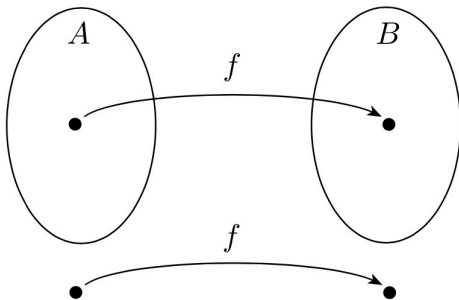
NP-Completeness

Polynomial time reduction

Language A is **polynomial time reducible** to language B , written $A \leq_P B$, if a polynomial time computable function $f : \Sigma^* \rightarrow \Sigma^*$, where

$$\forall w \ w \in A \iff f(w) \in B.$$

The function f is called the **polynomial time reduction of A to B** .



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A language B is NP-complete if it satisfies:

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If B is NP-complete and $B \leq_P C$ for $C \in \text{NP}$, then C is NP-complete.

NP-Completeness

The set of NP-complete languages (problems) is the set of hardest languages (problems) in NP.

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The factoring problem consists of finding the prime numbers p and q given a large number, $N = pq$. The widely used RSA algorithms are based on the fact that factoring a number is difficult. In fact, the hardness of the factoring problem is what makes RSA encryption and signature schemes secure. This problem is indeed **hard**, yet probably **not NP-complete**.

How to factor an integer?

To factor the number n we can try to divide n by every $i \in \{2, \dots, n-1\}$. This will take a time $\mathcal{O}(n)$.

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Still, we can improve.

We can only try to divide n by the primes that are smaller than \sqrt{n} . By the prime number theorem we know that the number of primes below \sqrt{n} is approximately $\frac{\sqrt{n}}{\log \sqrt{n}}$. This makes the task faster. But still we need 2^{120} operations for a 256-bit integer. Not good enough!

The fastest factoring algorithm is the **general number field sieve (GNFS)**, with an average time to operate for a number n of

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Factoring a 1024-bits integer	2^{70} operations
Factoring a 2048-bits integer	2^{90} operations
Factoring a 4096-bits integer	2^{128} operations

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Thus the estimates are **very optimistic** regarding the possible performance of computers and algorithms.

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Factoring may then be slightly easier than NP-complete in theory, but as far as cryptography is concerned, it's hard enough, and even more reliable than NP-complete problems. Indeed, it's **easier** to build cryptosystems on top of the factoring problem than NP-complete problems, because it's hard to know exactly how hard it is to break a cryptosystem based on some NP-complete problems—in other words, how many bits of security you'd get.

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On the other hand... we may have to deal with **quantum computers**...

... and if we are not careful, factoring can become easy!

To factor 17976931348623159077293051907890247336179769789423065
7273430081157739343819933842986982557174198257278917258638193
7092658191860266261806597306650627109955565786394477156084151
8689565284169198292110720231716536912489048151238855803905342
7125099290315449262324709315263256083132540461407052872832790
915388014592 takes just a few seconds because

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The Discrete Logarithm Problem

Consider the multiplicative group \mathbb{Z}_p^* (with p prime). The **DLP** consists, given g and x , in finding y such that $g^y = x$ in \mathbb{Z}_p^* , i.e.

$$g^y \equiv x \pmod{p}.$$

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In this conditions **DLP** seems as hard as the integer factoring problem.

T O BE CONTINUED...